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On a Forced Discontinuous Oscillation

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Abstract

A forced oscillation of a discontinuous self-sustained oscillation which is the limit case of a relaxation oscillation was analysed in the case of small forcing term. The condition that the forced system has the period which is an odd integer multiple of one of the forcing term was obtained.

The result explains the subharmonic synchronization of a relaxation oscillator. Some experimental results were compared with that of the analysis.

I. Introduction

Much interesting researches[1] have been carried, since van der Pol[2], on the forced oscillation with continuous wave form but less has been done on discontinuous one.

The following argument is an analysis of a forced oscillation of a discontinuous self-sustained oscillation which is the limit case[3] of a relaxation oscillation[4].

II. Forced Oscillation by Small Sinusoid

We shall consider a forced oscillation of a discontinuous self-sustained oscillation which is expressed by

(i) differential equation:

$$-\delta(1-x^2)\frac{dx}{d\tau}+x=b\sin(\tau+\varphi) \qquad (2\geq |x|>1) \qquad (1)$$

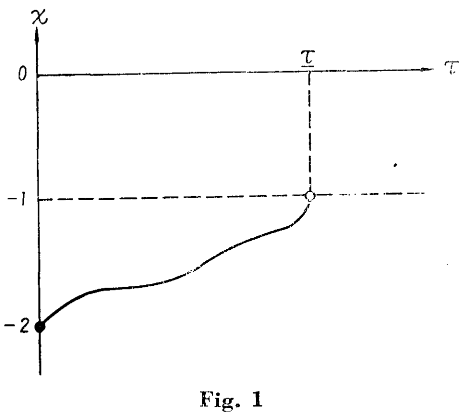

Fig. 1

where δ, b and φ are constant; $\delta>0$, $b>0$ and b is small

(ii) condition of jump:

$$x=\pm 1 \longrightarrow \mp 2$$

which means x jumps from $+1$ to -2 and from -1 to $+2$.

It is clear that the solution of (1) which has the initial condition: $x=-2$ when $\tau=0$, increases monotonically and approaches to $x=-1$ at some τ, as shown in Fig. 1.

Hence we shall consider

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1) For example, M. L. Cartwright: "Contributions to the Theory of Nonlinear Oscillations" Princeton Univ. Press (1950) p. 149

2) B. van der Pol: Phil. Mag. 3 (1927) p. 65

3) J. J. Stoker. "Nonlinear Vibrations" Interscience Publishers Inc. (1950) p. 137

4) N. Minorsky: "Nonlinear Mechanics" Edwards Brothers Inc. (1947) p. 381

$$\frac{d\tau}{dx}=\frac{\delta(1-x^2)}{x-b\sin(\tau+\varphi)} \qquad (2\geq |x| \geq 1) \tag{2}$$

instead of (1).

It is easy to see that Eq. (2) has a solution which satisfies the initial condition: $\tau=0$ when $x=-2$, of the form

$$\tau(x)=\tau_0(x)+b\tau_1(x)+b^2\tau_2(x)+\cdots\cdots\cdots\cdots \tag{3}$$

convergent for $-2\leqq x\leqq -1$ in $b<b_0$, b_0 being some positive constant.

If we substitute (3) into (2) and equate terms containing the same powers of b, we have, after integrations, the following successive relations:

$$\tau_0(x)=\int_{-2}^{x}\frac{\delta(1-\xi^2)}{\xi}d\xi \tag{4}$$

$$\tau_1(x)=\int_{-2}^{x}\frac{\delta(1-\xi^2)\sin\{\tau_0(\xi)+\varphi\}}{\xi^2}d\xi \tag{5}$$

$$\tau_2(x)=\int_{-2}^{x}\frac{\delta(1-\xi^2)[\sin^2\{\tau_0(\xi)+\varphi\}+\tau_1(\xi)\xi\cos\{\tau_0(\xi)+\varphi\}]}{\xi^3}d\xi \tag{6}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

The first relation gives

$$\tau_0(x)=\left\{\alpha-\left(\frac{x^2}{2}-\log|x|\right)\right\}\delta \tag{7}$$

where $\alpha=2-\log 2$, and hence

$$\tau_0(-1)=\left(\frac{3}{2}-\log 2\right)\delta. \tag{8}$$

We write $\tau_0(-1)$ as k, $2k$ being the period of the self-sustained discontinuous oscillation.

The second relation gives

$$\tau_1(x)=\int_{-2}^{x}\frac{\delta(1-\xi^2)\sin\left[\left\{\alpha-\left(\frac{\xi^2}{2}-\log|\xi|\right)\right\}\delta+\varphi\right]}{\xi^2}d\xi \tag{9}$$

and hence

$$\begin{aligned}\tau_1(-1)&=\int_{-2}^{-1}\frac{\delta(1-\xi^2)\sin\left[\left\{\alpha-\left(\frac{\xi^2}{2}-\log|\xi|\right)\right\}\delta+\varphi\right]}{\xi^2}d\xi\\&=I_s(\delta)\cos\varphi+I_c(\delta)\sin\varphi\\&=A(\delta)\sin\{\varphi+\chi(\delta)\}\end{aligned} \tag{10}$$

where

$$A(\delta)=\sqrt{I_s^2(\delta)+I_c^2(\delta)}$$

$$A(\delta)\sin\chi(\delta)=I_c(\delta),\quad A(\delta)\cos\chi(\delta)=I_s(\delta)$$

$$I_s(\delta)=\int_{-2}^{-1}\frac{\delta(1-\xi^2)}{\xi^2}\sin\left[\left\{\alpha-\left(\frac{\xi^2}{2}-\log|\xi|\right)\right\}\delta\right]d\xi \tag{11}$$

$$I_c(\delta)=\int_{-2}^{-1}\frac{\delta(1-\xi^2)}{\xi^2}\cos\left[\left\{\alpha-\left(\frac{\xi^2}{2}-\log|\xi|\right)\right\}\delta\right]d\xi. \tag{12}$$

From (3) (8) and (10), we have

$$\tau(-1)=\tau_0(-1)+b\tau_1(-1)+O(b^2)=k+bA(\delta)\sin\{\varphi+\chi(\delta)\}+O(b^2). \tag{13}$$

We now confine our consideration to the first approximation of b, so that we use $\underline{\tau}$:

$$\underline{\tau}=k+bA(\delta)\sin\{\varphi+\chi(\delta)\} \tag{14}$$

instead of $\tau(-1)$. Hence $\underline{\tau}$ is the time required for the solution to travel from $x=-2$ to $x=-1$ in the first approximation of b.

In the same way, for a solution of (2) which has the initial condition: $\tau(x)=0$ when $x=+2$, we have

$$\tau(+1)=k-bA(\delta)\sin\{\varphi+\chi(\delta)\}+O(b^2).$$

The time required for the solution to travel from $x=+2$ to $x=+1$ is, in the first approximation of b, given by $\overline{\tau}$ where

$$\overline{\tau}=k-bA(\delta)\sin\{\varphi+\chi(\delta)\}. \tag{15}$$

Since the angular velocity of the forcing term is unity, the phase angle of the forcing term at $x=-1$ proceeds $\underline{\tau}$ compared with one at $x=-2$. Similarly the phase angle at $x=+1$ proceeds $\overline{\tau}$ compared with one at $x=+2$.

Now let the initial phase angle be φ_1 at $x=-2$. The phase angle at $x=+2$ (after one jump) is given by

$$\varphi_1'=\varphi_1+k+bA\sin(\varphi_1+\chi) \tag{16}$$

and the phase angle at $x=-2$ (after the next jump) is given by

$$\varphi_2=\varphi_1'+k-bA\sin(\varphi_1'+\chi). \tag{17}$$

Substituting (16) into (17), we get

$$\varphi_2=\varphi_1+2k-2bA\sin\frac{k}{2}\cos\left(\varphi_1+\chi+\frac{k}{2}\right). \tag{18}$$

Hence we shall consider the difference equation:

$$\varphi_{n+1}=\varphi_n+2k-2bA\sin\frac{k}{2}\cos\left(\varphi_n+\chi+\frac{k}{2}\right) \quad (n=1,2,3,\cdots\cdots) \tag{19}$$

where φ_n denote the phase angle of the forcing term at the $2(n-1)$-th jump.

If we put $\theta_n=\varphi_n+\chi+\frac{k}{2}$, $l=2k$, $\beta=2bA\sin\frac{k}{2}$; (19) reduces to

$$\theta_{n+1}=\theta_n+l-\beta\cos\theta_n \quad (n=1,2,3,\cdots\cdots). \tag{20}$$

Now the forced system is said to be synchronized when $\{\theta_n-2\pi nm\}$ tends to θ_0 where θ_0 is a constant and m is a positive integer.

Substituting $\theta_n-2\pi nm=\theta_0$ into (20), we obtain

$$l-2\pi m=\beta\cos\theta_0. \tag{21}$$

Hence the condition that there exists a positive integer m such as

$$|l-2\pi m|\leqq|\beta| \tag{22}$$

is necessary for the system to be synchronized.

Conversely, if there exists such an integer m which satisfies (22), we can find $\cos\theta_0$ which satisfies (21).

Hence from (20) we have

$$\theta_{n+1}-\theta_n=2\pi m+\beta\cos\theta_0-\beta\cos\theta_n.$$

Putting $\phi_n=\theta_n-2\pi nm$ $(n=1,2,3,\cdots\cdots)$, we obtain

$$\phi_{n+1}=\phi_n+\beta(\cos\theta_0-\cos\phi_n) \quad (n=1,2,3,\cdots\cdots). \tag{23}$$

If ϕ_1 belongs to an interval $(2\pi s, 2\pi s+2\pi)$, where s is an integer or zero, i.e. if

$$2\pi s \leqq \phi_1 < 2\pi s+2\pi$$

then setting $\lambda_n=\phi_n-2\pi s$ in (23), we get

$$\lambda_{n+1}=\lambda_n+\beta(\cos\theta_0-\cos\lambda_n) \quad (n=1,2,3,\cdots\cdots) \tag{24}$$

where $0\leqq\lambda_1<2\pi$.

The relation between λ_{n+1} and λ_n is shown in Fig. 2 in the case: $\beta>0$ and $\cos\theta_0>0$. It is evident from the figure that the sequence $\{\lambda_n\}$ tends to λ_0 if $\lambda'_0<\lambda_1<2\pi$, and to λ''_0 if $0\leqq\lambda_1<\lambda'_0$, where $0<\lambda_0\leqq2\pi$, $0\leqq\lambda'_0<2\pi$, $\cos\lambda_0=\cos\lambda'_0=\cos\theta_0$ and $\lambda''_0=\lambda_0-2\pi$.

Hence choosing $\theta_0=\lambda_0+2\pi s$ if $\lambda'_0<\lambda_1<2\pi$, and $\theta_0=\lambda_0+2\pi(s-1)$ if $0\leqq\lambda_1<\lambda'_0$, we can assert that $\{\theta_n-2\pi nm\}$ tends to θ_0.

Similarly it can be seen that $\{\theta_n-2\pi nm\}$ thnds to θ_0 in the other three cases: $\beta>0$, $\cos\theta_0<0$; $\beta<0$, $\cos\theta_0>0$; $\beta<0$, $\cos\theta_0<0$.

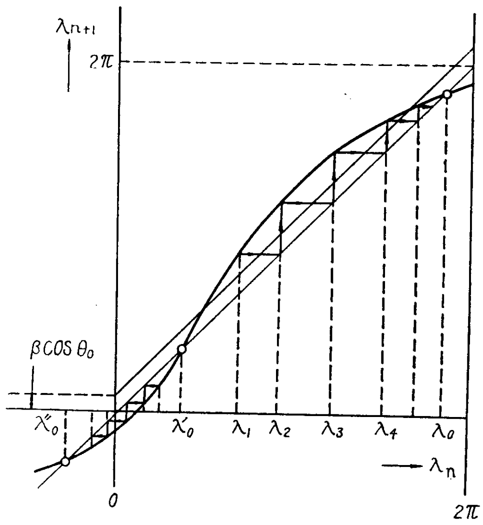

Fig. 2

Hence we may conclude that the condition

$$|l-2\pi m| < |\beta|\ * \tag{25}$$

is necessary and sufficient for the forced system to be synchronized.

Now if the period of the self-sustained oscillation is μ times as many as that of the forcing term, then

$$k=\mu\pi. \tag{26}$$

Substituting (26) into (25) we obtain

$$|\mu-m| < \frac{bA(\delta)}{\pi}|\sin\frac{\mu\pi}{2}| \tag{27}$$

implying that μ must be near to a positive integer m for the system to be synchronized.

Moreover setting $\mu=m(1+\zeta_m)$ and substituting into (27) we have

$$|\zeta_m| < \frac{bA(\delta)}{m\pi}|\sin\frac{m\pi}{2}(1+\zeta_m)| \tag{28}$$

implying that $|\zeta_m|$ must be small.

If m is an even integer, we get from (28)

$$|\zeta_m| < \frac{bA}{m\pi}\,\frac{m\pi}{2}\,|\zeta_m|$$

*The equality should be omitted since it corresponds to semi-stable limit.

and this is impossible since b is small.

Hence m must be an odd integer.

If the condition (25) is satisfied with an odd integer m, the period of the synchronized system is evidently $2\pi m$, that is, exactly m times as many as that of the forcing term. We will call m as the order of subharmonic synchronization.

As far as the first approximation of b is concerned, we may use m instead of μ in the right hand side of (28) since $\zeta_m = O(b)$.

Hence (28) may be written as

$$|\zeta_m| < \zeta(m) \tag{29}$$

where $\zeta(m) = \frac{b}{m\pi} A(\beta)$, $\beta = \frac{m\pi}{\frac{3}{2} - \log 2}$ and $\zeta(m)$ represents the width of the synchronous range of the m-th order subharmonic synchronization.

After all we have the following conclusion.

As far as the first approximation of the small forcing term is concerned, the subharmonic synchronization of the m-th order occurs under the condition

$$|\zeta_m| < \zeta(m)$$

where $\zeta(m)$ represents the synchronous range of the m-th order and

$$\zeta_m = \frac{\mu}{m} - 1 \, , \quad \zeta(m) = \frac{b}{m\pi} A(\beta) \, , \quad A(\beta) = \sqrt{I_s^2(\beta) + I_c^2(\beta)}$$

$$I_s(\beta) = \int_2^1 \beta \frac{(1-x^2)}{x^2} \sin\left[\left\{\alpha - \left(\frac{x^2}{2} - \log x\right)\right\}\beta\right] \alpha x$$

$$I_c(\beta) = \int_2^1 \beta \frac{(1-x^2)}{x^2} \cos\left[\left\{\alpha - \left(\frac{x^2}{2} - \log x\right)\right\}\beta\right] \alpha x \tag{30}$$

$$\beta = \frac{m\pi}{\alpha - \frac{1}{2}}, \quad \alpha = 2 - \log 2 \, ,$$

μ is the ratio of the period of the self-sustained discontinuous oscillation to that of the forcing term and m is the odd integer nearest to μ.

III. Forced Oscillation by Distorted Small Sinusoid

Next we shall consider the case where the forcing term has higher harmonics.

The forced system is expressed by

(i) differential equation:

$$-\delta(1-x^2)\frac{dx}{d\tau} + x = \sum_{s=1}^{N} b^{(s)} \sin(s\tau + \varphi^{(s)}) \quad (2 \geq |x| \geq 1) \tag{31}$$

where $b^{(s)}$'s are small

(ii) condition of jump:

$x = \pm 1 \longrightarrow \mp 2.$

By the same method described in II, we have, corresponding to (20), the following simultaneous difference equations:

$$\theta^{(r)}_{n+1} = \theta^{(r)}_{n} + r\left\{l - \sum_{s=1}^{N} \beta^{(s)} \cos\theta^{(s)}_{n}\right\} \quad (r = 1, 2, \cdots\cdots N \, ; \; n = 1, 2, 3, \cdots\cdots) \tag{32}$$

where

$$\theta_n^{(r)} = \varphi_n^{(r)} + \chi(r\delta) + \frac{rk}{2}$$

$$\beta^{(r)} = b^{(r)} \frac{2A(r\delta)}{r} \sin \frac{rk}{2} \quad (r=1,2,3,\cdots\cdots N).$$

In this case, the forced system is said to be synchronized when every $\left\{\theta_n^{(r)} - 2\pi n m^{(r)}\right\}$ tends to $\theta_0^{(r)}$ respectively where $m^{(r)}$'s are positive integers.

It is clear from (32) that

$$\theta_{n+1}^{(r)} - \theta_n^{(r)} = r\left\{\theta_{n+1}^{(1)} - \theta_n^{(1)}\right\}, \tag{33}$$

and hence

$$\theta_{n+1}^{(r)} - \theta_1^{(r)} = r\left\{\theta_{n+1}^{(1)} - \theta_1^{(1)}\right\}. \tag{34}$$

It follows from (34) that the simultaneous difference equations can be reduced to the following single one:

$$\theta_{n+1}^{(1)} - \theta_n^{(1)} = l - \sum_{s=1}^{N} \beta^{(s)} \cos\left\{s\theta_n^{(1)} + \left(\theta_1^{(s)} - s\theta_1^{(1)}\right)\right\} \quad (n=1,2,3,\cdots\cdots). \tag{35}$$

Therefore the condition that every $\left\{\theta_n^{(r)} - 2\pi n m^{(r)}\right\}$ tends respectively to $\theta_0^{(r)}$ is also reduced to the single condition that $\left\{\theta_n^{(1)} - 2\pi n m^{(1)}\right\}$ tends to $\theta_0^{(1)}$ where $m^{(1)}$ is an integer.

We can obtain the condition of synchronization by the similar way but it will be rather troublesome to get explicit result.

IV. Application to Practical Circuits

By numerical computation of (30) we know that

$\zeta(1) = 0.39b$

$\zeta(3) = 0.15b$

$\zeta(5) = 0.09b$

and $\zeta(m)$ is approximately equal to $\frac{0.48b}{m}$ when m (odd integer) is large.

These results will explain the synchronization of the van der Pol Oscillator[5], a negative transconductance relaxation oscillator.

In this case the assumption that the nonlinear characteristic of the vacuum-tube is expressed by a cubic polynomial can be satisfied fairly well by suitable choice of the position of the working point.

When the van der Pol oscillator is synchronized as shown in Fig. 3, the amplitude of the forcing term is approximately proportional to m (the order of subharmonic synchronization). Therefore the width of the synchronous range is approximately

5) W. A. Edson: "Vacuum-tube Oscillators" John Wiley & Sons Inc. (1953) p.272

independent of m.

But the above mentioned assumption on nonlinear characteristic is not satisfied well in the case of the synchronization of the Multivibrator as shown in Fig. 4.

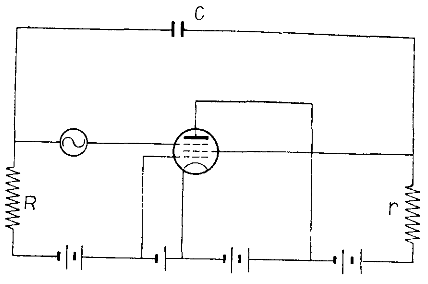

Fig. 3

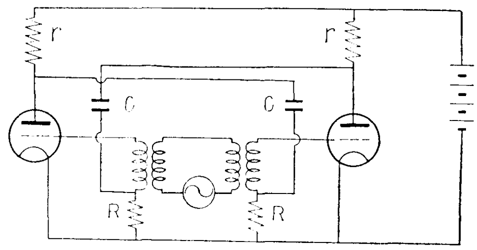

Fig. 4

Appendix

The above mentioned method of analysis can be applied to the synchronization of relaxation oscillator with glow-discharge tube as shown in Fig. 5.

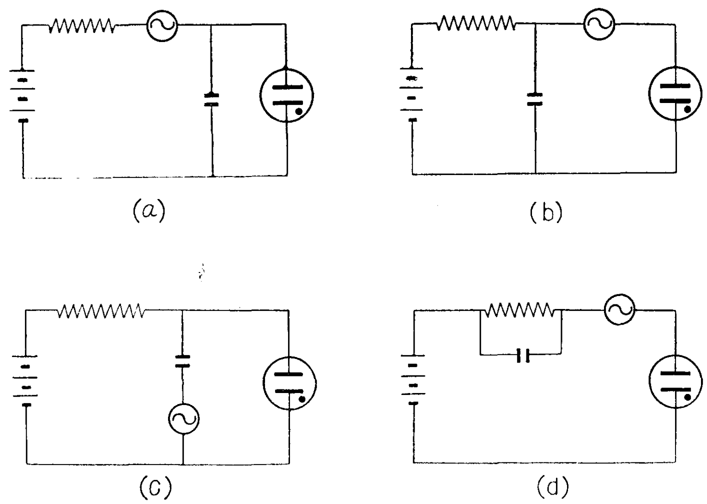

Fig. 5

Under the assumption that the time required to discharge can be neglected compared with the period of the self-sustained oscillation, the results are respectively as follows.

(a): $\zeta(m)=\dfrac{1}{\sqrt{4\pi^2m^2+\log^2h}}\,e$

(b): $\zeta(m)=\dfrac{1}{\log h}\,e$

(c): $\zeta(m)=\dfrac{2\pi m}{\log h\sqrt{4\pi^2m^2+\log^2h}}\,e$

(d): $\zeta(m)=\frac{1}{\log h}e$

where

$e=e_0\left(\frac{1}{u_2}-\frac{1}{u_1}\right)$, e_0: synchronizing A.C. voltage, $h=u_1/u_2$,

u_1: (D.C. supply voltage)−(deionizing potential)

u_2: (D.C. supply voltage)−(firing potential).

Some of these results are compared with experimental data in Fig. 6.

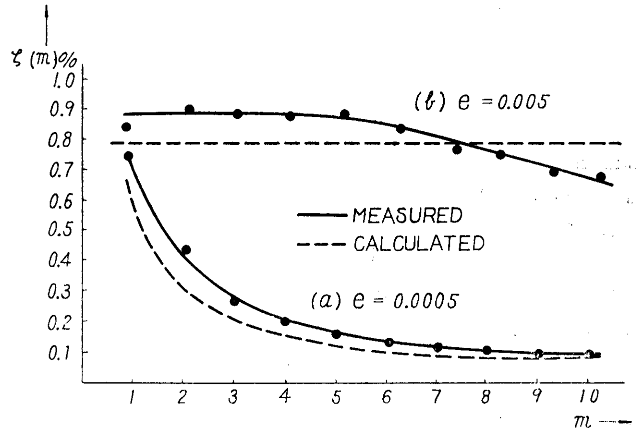

Fig. 6

Acknowledgement

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