Title	Analysis of nonlinear electric circuits having periodic solutions (part I)
Sub Title	
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Publisher	慶應義塾大学藤原記念工学部
Publication year	1954
Jtitle	Proceedings of the Fujihara Memorial Faculty of Engineering Keio University Vol.7, No.25 (1954.) ,p.29(1)- 35(7)
JaLC DOI	
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Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00070025-0001

Analysis of Nonlinear Electric Circuits having Periodic Solutions (Part I)

Hiroichi FUJITA*

Abstract

The differential equations and their solutions of some nonlinear electric circuits are obtained in this paper. The order of these differential equations are not restricted, i. e. we can solve not only the second order but also the first or more than the third order differential equations.

However it is necessary for this method, the circuits contain only one nonlinear element and the solutions are quasi sinusoidal functions.

I. Introduction

Although the general behavior of differential equation solutions for nonlinear electric circuits cannot readily be known, the difficulty can be avoided in the case of a circuit having a periodic solution by paying attention to the fundamental component of the solution and by neglecting the effects of higher harmonics. Van der Pol's method is essentially this type of treatment applied to a differential equation of the second order.

This paper is concerned with the extension of his mehtod to the case of a nonlinear differential equation of an arbitrary order. The author has worked out a method of solving equations for a circuit involving a single nonlinear element provided they have a periodic, steady state solution of nearly sinusoidal form.

A method of formulating circuit equations suitable to the proposed method of solution is also shown. Among the difficulties yet to be solved is the impossibility of formulating equations for a circuit involving two or more nonlinear elements.

Another headache with the new method lies in knowing how to distinguish whether or not the waveform of the solutions resembling close enough to a sinusoid so that the proposed method of solution can safely be applied.

II. The circuit equations

In the theories of linear electric circuits highly refined representations are used, such as impedance, admittance, vectors or integral transformations. But in the nonlinear circuits, they are not adequate because of their original properties. So for the linear parts of nonlinear networks, we use differential operators p^n and impedance or admittance which are the function of differential operators p^n (n is integer and p^n is d^n/dt^n).

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In this section how to make the circuit equation which are suitable for our method will be shown.

II-A Equation for passive network containing one nonlinear element.

For passive network containing only one nonlinear element, we may cosider closed circuit in which nonlinear element and linear two-terminal network are connected in series. v is the voltage across the nonlinear element and i is the current through it. Nonlinearities are generally expressed

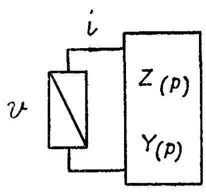

Fig. 1

$$i=f(v) \quad (2\text{-}1)$$

$$v=g(i) \quad (2\text{-}2)$$

When operational impedance and admittance of linear circuit is $Z(p)$ and $Y(p)$ then the circuit equations are

$$v=Z(p)f(v) \quad (2\text{-}3)$$

or

$$i=Y(p)g(i) \quad (2\text{-}4)$$

II-B Note

We may make the equation for the current

$$i=f\{Z(p)i\}$$

and for the voltage

$$v=g\{Y(p)v\}$$

However, these are not suitable for our method. It should be noticed that nonlinear function must not contain operators p^n. We must choose the equations in which impedance or admittance operate to the nonlinear functions. The reason will be state later.

II-C Equation for passive network containing multiple nonlinear elements but equivalently (**II**-A).

One of these types occurs when several nonlinear elements are connected in series. So they have common current i. We denote the nonlinear characteristics as

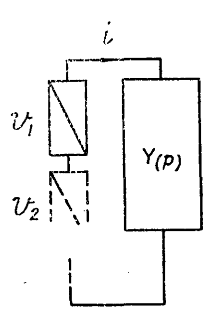

Fig. 2

$$v_1=f_1(i)$$
$$\cdots\cdots \quad (2\text{-}5)$$
$$v_n=f_n(i)$$

From the same argument as (**II**-A)

$$i=Y(p)v=Y(p)\sum v_n \quad (2\text{-}6)$$

The circuit equation is

$$\sum_{s=1}^{n} Y(p)f_s(i)=i \quad (2\text{-}7)$$

The second type is of parallel nonlinear elements and in the same way as the series case, we obtain the equation

$$\sum_{s=1}^{n} Z(p) f_s(v) = v \qquad (2\text{-}8)$$

where nonlinear characteristics are represented in the forms

$$v = f_s(i_s) \qquad (2\text{-}9)$$

II-D Active network containing one nonlinear element.

When the network contains one nonlinear element and one electromotive force, we consider the linear part of the network as four-terminal network whose input is the two terminals of the nonlinear element and output is those of electromotive force. In the input voltage v_1 and current i_1, there is nonlinear equation

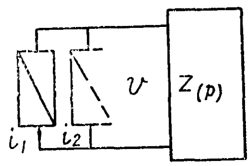

Fig. 3

$$v_1 = f(i_1) \qquad (2\text{-}10)$$

Output voltage v_2 is the known value V. We use four constants A, B, C and D which are represented by differential operator p^n.

Then

$$\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} f(i_1) \\ i_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} V \\ i_2 \end{pmatrix} \qquad (2\text{-}11)$$

$$f(i_1) = AV + Bi_2 \qquad (2\text{-}12)$$

$$i_1 = CV + Di_2$$

$$Df(i_1) - Bi_1 = (AD - BC)V = V \qquad (2\text{-}13)$$

(2-13) is the circuit equation.

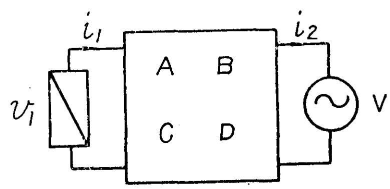

Fig. 4

II-E Equation for the circuit containing multiple nonlinear elements.

The equation for these circuits can not be solved as noted in Note **II**-B. For a simple example, we examine a circuit containing two nonlinear elements which are represented

$$v_1 = f_1(i_1) \qquad (2\text{-}14)$$

$$v_2 = f_2(i_2) \qquad (2\text{-}15)$$

Then using four-termnal constants A, B, C and D

$$\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} A B \\ C D \end{pmatrix} \begin{pmatrix} v_2 \\ i_2 \end{pmatrix} \qquad (2\text{-}16)$$

$$\begin{pmatrix} f_1(i_1) \\ i_1 \end{pmatrix} = \begin{pmatrix} A B \\ C D \end{pmatrix} \begin{pmatrix} f(i_2) \\ i_2 \end{pmatrix} \qquad (2\text{-}17)$$

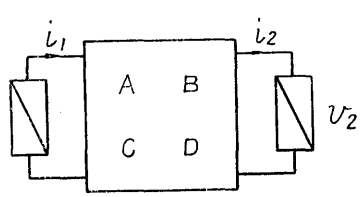

Fig. 5

If i_1 is eliminated then $\quad f_1\{Cf_2(i_2) + Di_2\} = Af_2(i_2) + Bi_2 \qquad (2\text{-}18)$

The right of (2-18) is the case of Note **II**-B, hence it can not be solved.

II-F Equations for the circuits contining iron core.

The nonlinearity of ferromagnetic core is very important in the electrical engineering. Their nonlinearity is expressed by the following form in general

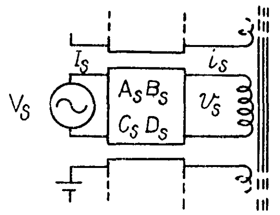

Fig. 6

$$H=f(\phi) \quad \text{or} \quad \phi=g(H) \tag{2-19}$$

where ϕ is flux of core and H is magnetomotive force.

H is linearly dependent on the currents of their winding circuits, but ϕ is not represented explicitly in $Z(p)$ $Y(p)$ or A, B, C and D. Therefore the preceding method is not serviciable in this case. When the magnetic core has winding 1 2 . . . n and the voltages are v_1 . . . . v_n

$$v_1=10^{-8}N_1(d\phi/dt) \; . \; . \; . \quad v_n=10^{-8}N_n(d\phi/dt) \tag{2-20}$$

Denoting i_1 i_2 . . . i_n the currents in the windings

$$lH=\sum_{s=1}^{n} i_s N_s \tag{2-21}$$

where l is the length of magnetic path. If the windings are connected to the linear networks which have the known electromotive force V_s,

$$\begin{pmatrix} V_s \\ I_s \end{pmatrix} = \begin{pmatrix} A_s B_s \\ C_s D_s \end{pmatrix} \begin{pmatrix} v_s \\ i_s \end{pmatrix} \tag{2-22}$$

where A_s, B_s, C_s and D_s are the four-terminal constants of the linear network.

Then

$$V_s=A_s 10^{-8} N_s d\phi/dt+B_s i_s \tag{2-23}$$

Multiply both sides with $N_s B_s^{-1}$ and sum up to s and obtain

$$\sum N_s B_s^{-1} V_s = \sum 10^{-8} N_s^2 B_s^{-1} A_s d\phi dt + lH$$

$$= \sum 10^{-8} N_s^2 B_s^{-1} A_s d\phi/dt + lf(\phi) \tag{2-24}$$

(2-24) is the equation for these circuits.

II-G Equation for the circuit containing three-terminal nonlinear element.

Another important nonlinear circuit are the circuits containing the vacuum tube or transister which have three terminals. In these cases we can not always make suitable equations, but in special cases, we can. One example is the case of a pentode vacuum tube. For the pentode, the plate current i_1 may be considered a function of grid voltage v_1 only. If we take four-terminal linear circuit (see Fig. 7)

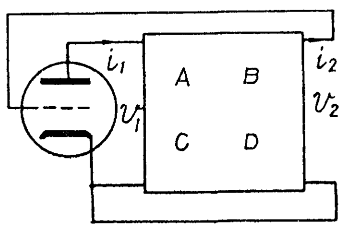

Fig. 7

$$\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} A\,B \\ C\,D \end{pmatrix} \begin{pmatrix} v_2 \\ i_2 \end{pmatrix} \tag{2-25}$$

i_1 is the grid current and it is usually negligible so

$$\begin{pmatrix} v_1 \\ f(v_2) \end{pmatrix} = \begin{pmatrix} A\,B \\ C\,D \end{pmatrix} \begin{pmatrix} v_2 \\ 0 \end{pmatrix} \tag{2-26}$$

Then

$$f(v_2)=Cv_2 \tag{2-27}$$

(2-27) is the equation for the circuit containing pentode vacuum tube.

III. Solution of the Equations

In this section how to obtain solutions of circuit equation will be studied.

The first case is forced oscillation in which there are no D.C. component. One of these types is van der Pol's forced oscillation. The second type is forced oscillations whose solution contain D. C. component generated by nonlinear element. Its popular example is the rectifier. These two types have finite period equal to the forcing oscillation's period. The third type is free oscilations where period is not known explicity.

III-A Solution of forced oscillation having no D.C. component generated by nonlinear element.

If ω is angular frequency of forcing oscillation, put $\omega t=\tau$, then the period of oscillations is 2π. From here on we use independent variable t in this meaning of normalized time. Denote x as dependent variable. On $x-px$ plane the solution describes a near circle. So we suppose

$$x=K(t)\cos(t+\varphi(t)) \qquad (3\text{-}1)$$

$$px=-K(t)\sin(t+\varphi(t)) \qquad (3\text{-}2)$$

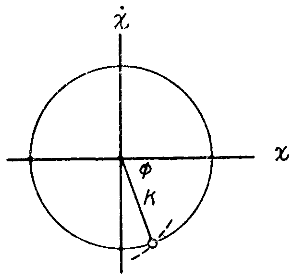

Fig. 8

$K(t)$ $\varphi(t)$ are the function of time (see Fig. 8). These representations are similar to those of van del Pol or Andronow & Witt.

In $px-p^2x$ plane the solution also describe a near circle. Then

$$px=H(t)\cos\{t+\phi(t)\} \qquad (3\text{-}3)$$

$$p^2x=-H(t)\sin\{t+\phi(t)\} \qquad (3\text{-}4)$$

From (3-2) and (3-3)

$$-H(t)\sin\{t+\phi(t)\}=K(t)\cos\{t+\varphi(t)\}$$

then

$$-H=K \qquad \phi-\varphi=\pi/2 \qquad (3\text{-}5)$$

Consequently, we may put the following relations to the circuit equation

$$p^s x=Re(j^s K e^{j(t+\varphi)}) \qquad (3\text{-}6)$$

When the highest order of differential operators is more than the second.

$$dp^{m-2}x/dt=p^{m-1}x \qquad (3\text{-}7)$$

$$ap^{m-1}x/dt=f(p^{m-1}x, p^{m-2}x \;\ldots\; px, x, t) \qquad (3\text{-}8)$$

Then, we get the type of equations (3-9) (3-10) from (3-7) (3-8)

$$\cos(t+\varphi)dK/dt-K\sin(t+\varphi)d\varphi/dt=0 \qquad (3\text{-}9)$$

$$-\sin(t+\varphi)dK/dt-K\cos(t+\varphi)d\varphi/dt=F(K,\varphi,t) \qquad (3\text{-}10)$$

If the circuit equation has the only first order operaters, we operate one more p to the circuit equation to make the second order. Then we obtain the same relation as (3-9) (3-10). It must be remembered that to operate one more p, drops D.C. component of the solution. The nonlinear equation which generates D.C. component will be given in next section. If the equation is not suitable as seen in Note (II-B) $p^m x$ can not be expressed explicitly, so (3-10) can not be obtaind easily. From (3-9) (3-10), we can calculate dK/dt and $d\varphi/dt$

$$dK/dt=D_1/\Delta \tag{3-11}$$

$$d\varphi/dt=D_2/\Delta \tag{3-12}$$

where

$$\Delta=\begin{vmatrix}\cos(t+\varphi) & -K\sin(t+\varphi)\\ -\sin(t+\varphi) & -K\cos(t+\varphi)\end{vmatrix}=-K$$

$$D_1=\begin{vmatrix}0 & -K\sin(t+\varphi)\\ F(K,\varphi,t) & -K\cos(t+\varphi)\end{vmatrix}$$

$$D_2=\begin{vmatrix}\cos(t+\varphi) & 0\\ -\sin(t+\varphi) & F(K,\varphi,t)\end{vmatrix}$$

Then we arrange the right of (3-11)(3-12) in the form of Fourier series. If the circuits are in the state periodic motion, K and φ have no secular term which tend to positive or negative infinity when $t=\infty$. Hence the first terms of these which correspond to constant terms in usual Fourier series are zero. Then

$$\left[\frac{dK}{dt}\right]_{D.C}=0 \tag{3-13}$$

$$\left[\frac{d\varphi}{dt}\right]_{D.C}=0 \tag{3-14}$$

which do not contain time t and from which we can obtatin the values of K and φ.

III-B Solution of forced oscillation having D.C. component generating by nonlinear element.

In the case of rectifier, we must take account of D.C. component in x. Then substitute

$$x=K_0+K\cos(t+\varphi)$$

in the circuit equation. The higher orders are the same as in (3-2).

Now we must decide K. In the circuit shown in Fig. 9 the nonlinear characteristics are

$$i=f(v)$$

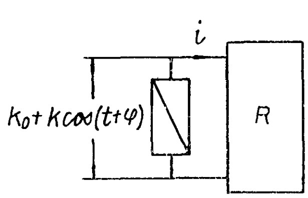

Fig. 9

when the resistance of linear circuit is R, (which is not always the real part of the impedance.).

$$i_0R=K_0$$

where i_0 is the D.C. component of i.

Hence

$$[f\{K_0+\cos(t+\varphi)\}]_{D.C}=K_0 \tag{3-15}$$

Then we can decide K, K_0, and φ from the three equations (3-13) (3-14) and (3-15).

III-C Solution for the self-exiting oscillations

When there are no forcing oscillations, the period is not known directly. So we can not normalize the time. However, using unkown constant angular frequency ω, we transform the time t to τ

$$\omega t=\tau$$

where ω must be decided later from the following condition. The condition is that the solution describe circles on the $p^s x-p^{s+1}$ planes.

When the condition is fulfilled, the same argument as the above, is applicable. In spite of the new unknown ω, the number of the unknown do not increase, because phase φ loses its meaning in the self-oscillation, and φ can be decided arbitrarily.

IV. Conclusion

IV-A As explained in this papar, we can now obtain the solution of equation for the circuits which contain either only one nonlinear element or multiple element acting as one and which have steady state periodic solution nearly equal to sinusoidal function. But two difficult problems have been left, one is how to make the equation for the circuit containing multiple nonlinear elements which is very necessary in electrical engineering. Another is how to decide if a wave form resembles sine wave close enough to be appliable to our method or not. But in many cases we can decide empirically from the type of the circuit.

IV-B Stability

In nonlinear problem, the stability of solution is very important. The usual method to know the stability is as follows. Substitute $K_0+\xi$ $\varphi_0+\eta$ for K and φ in (3-10) (3-11) and ξ, η are small quantities. When K_0 and φ_0 are the solution of (3-12) (3-13), dK_0/dt $d\varphi_0/dt$ vanish. Then neglecting the heigher orders of ξ, η

$$\begin{aligned} d\xi/dt&=P_1(t)\xi+Q_1(t)\eta \\ d\eta/dt&=P_2(t)\xi+Q_2(t)\eta \end{aligned} \qquad \textbf{(4-1)}$$

The convergence of solutions of these linear differential equations means stable solution. Though it is not always easy to solve (4-1), there are so many studies on this problem that I shall refrain from stating anything here. Mathieus or Hill's equations are examples of these with close relations.

IV-C By the above study it has become possible to solve nonlinear circuit equation in the steady state much more generally than formerly. However to generalize the soution of phenomena of a nonlinear circuit seems almost impossible for the present.