

Title	Macroscopic theory of non-linear circuits
Sub Title	
Author	末崎, 輝雄(Suezaki, Teruo)
Publisher	慶應義塾大学藤原記念工学部
Publication year	1954
Jtitle	Proceedings of the Fujihara Memorial Faculty of Engineering Keio University Vol.7, No.24 (1954.) ,p.10(10)- 28(28)
JaLC DOI	
Abstract	<p>The author has developed the theory of electric circuits considering a non-linear character of a source.</p> <p>From the macroscopic point of view, the author has obtained the relations which must be satisfied by the active and reactive power of the systems.</p> <p>In this paper the oscillation problems, such as the hard and soft oscillations, locked oscillations and fractional harmonic synchronization, have been discussed as the examples from the above relations.</p>
Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00070024-0010

慶應義塾大学学術情報リポジトリ(KOARA)に掲載されているコンテンツの著作権は、それぞれの著作者、学会または出版社/発行者に帰属し、その権利は著作権法によって保護されています。引用にあたっては、著作権法を遵守してご利用ください。

The copyrights of content available on the Keio Associated Repository of Academic resources (KOARA) belong to the respective authors, academic societies, or publishers/issuers, and these rights are protected by the Japanese Copyright Act. When quoting the content, please follow the Japanese copyright act.

Macroscopic Theory of Non-Linear Circuits

(Received Dec. 8, 1954)

Teruo SUEZAKI*

Abstract

The author has developed the theory of electric circuits considering a non-linear character of a source.

From the macroscopic point of view, the author has obtained the relations which must be satisfied by the active and reactive power of the systems.

In this paper the oscillation problems, such as the hard and soft oscillations, locked oscillations and fractional harmonic synchronization, have been discussed as the examples from the above relations.

I. Preliminaries

Sinusoidal electromotive force and current, flowing in a circuit, are represented by the vector \mathbf{E} and \mathbf{I} , and their fundamental relation is Kirchhoff's law, namely:

$$\mathbf{E} = \mathbf{Z}\mathbf{I} \quad (1)$$

where \mathbf{Z} is a linear impedance of the complete circuit.

From the above relation, multiplying the conjugate vector \mathbf{I}^* , the following relation is readily obtained:

$$-\mathbf{E}\mathbf{I}^* + \mathbf{Z}\mathbf{I}\mathbf{I}^* = 0 \quad (2)$$

where $*$ denotes a conjugate vector, that is

$$\mathbf{I} = I e^{-j\theta}, \quad \mathbf{I}^* = I e^{j\theta} \quad (3)$$

First term of the left hand side of (2) denotes the generated vector power and the second term denotes the absorbed vector power in the complete circuit and (2) shows the equilibrium of the total power.

These are the familiar basis of the treatment of linear electric circuits, in which we have assumed that the source of power has a linear infinite character.

However, in many electrical devices, notably thermionic valve, current is not directly proportional to the voltage. Such kind of characters may be called a non-linear character.

There are a number of cases where the source has a non-linear character and supplies alternating current which is not far from the sinusoidal waveform.

Non-linear oscillation of a valve oscillator is the most interesting and important in the whole cases mentioned above.

Even under these conditions where the distortion is slight mathematical treatment

*木崎輝雄: Professor at Keio University

is very difficult and we must be satisfied only with the approximate solution.

This paper endeavors to show a more straight forward and facilitated treatment of non-linear oscillations.

In these problems, it is convenient to express the current and the voltage by the rotating vector. For example, the current flowing in a circuit fed from a source which has a non-linear character may best be expressed as

$$\mathbf{I}(t) = I(t) e^{-j\theta(t)}. \quad (4)$$

Both I and θ are considered as functions of time t and their rate of change is assumed to be small enough. In this treatment we must introduce a fictitious electromotive force additionally,

$$-2L \frac{d}{dt} \mathbf{I}(t) \quad (5)$$

where L is the resultant inductance in a closed circuit which has a circulating current $\mathbf{I}(t)$.

Appearance of this fictitious electromotive force (5) has been postulated at the beginning, but it will be seen in the next section that the discussion on the equilibrium of power and its stability, which give the physical reality to the steady state of the oscillation, is greatly facilitated by introducing the term (5).

Let us show some examples taking the simple linear oscillatory circuits.

We will consider the current flowing in the simple linear series tuning circuit fed from a source which has a linear infinite character. Circuit and notations of its elements are shown in Fig. 1.

By virtue of the preceding remarks the following relation will be obtained :

$$\mathbf{E} - 2L \frac{d}{dt} \mathbf{I}(t) = \left\{ R + j \left(\omega L - \frac{1}{\omega C} \right) \right\} \mathbf{I}(t). \quad (6)$$

This is the fundamental relation in the treatment that will be developed in this paper.

Multiplying the above relation by \mathbf{I}^* , the following relation will be obtained ;

$$-2L \mathbf{I}^*(t) \frac{d}{dt} \mathbf{I}(t) = -\mathbf{E} \mathbf{I}^*(t) + \left\{ R + j \left(\omega L - \frac{1}{\omega C} \right) \right\} \mathbf{I}(t) \mathbf{I}^*(t), \quad (7)$$

and using the expression (4), the next two relations may be obtained readily :

$$\begin{aligned} -2LI \frac{dI}{dt} &= -EI \cos \theta + RI^2, \\ 2LI^2 \frac{d\theta}{dt} &= -EI \sin \theta + \left(\omega L - \frac{1}{\omega C} \right) I^2. \end{aligned} \quad (8)$$

Steady state of the system will be correlated with

$$\frac{dI}{dt} = 0, \quad \frac{d\theta}{dt} = 0.$$

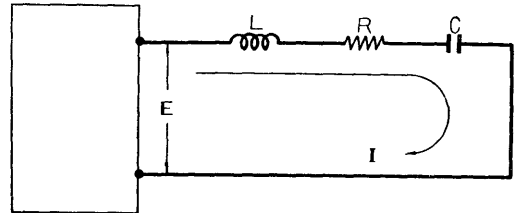


Fig. 1. A Series tuning circuit

Under these conditions I and θ become constant and the right hand side of the first equation (8) expresses the equilibrium of the total active power and the second denotes the equilibrium of reactive power.

In the transient state if we reject E for the sake of simplicity, following relation will be obtained at once, namely

$$I = I_0 e^{-\frac{R}{2L}t}, \quad (9)$$

that is, the amplitude of the oscillation will diminish with the lapse of time and if the rate of the change of amplitude is small enough the frequency of this damped oscillation will coincide with the natural frequency of the circuit, which is given by the second equation of (8).

Thus, the results obtained by the treatment in which the fictitious e.m.f. is introduced will be quite evident. The relation (7) may be called a "vector power relation".

Next, let us consider the case where a parallel tuning circuit is connected to an alternating current source and is excited by another sinusoidal electromotive force of the same frequency in the tank circuit.

Circuit is shown in Fig. 2 and all the circuit elements have a linear character.

Current I_1 circulates in the tank circuit, I is fed from the source and E is another e. m. f. as shown in Fig. 2.

The fundamental relation concerning the circulating current I_1 will be expressed

$$\text{as,} \quad E - 2L \frac{d}{dt} I_1(t) = \left\{ R + j \left(\omega L - \frac{1}{\omega C} \right) \right\} I_1(t) - (R + j\omega L) I, \quad (10)$$

$$\text{and} \quad I_1(t) = j\omega CV(t). \quad (11)$$

The fundamental relation (10) may be rewritten in terms of the voltage introduced by (11) in the following form:

$$-2C \frac{d}{dt} V(t) = \left\{ \frac{CR}{L} + j \left(\omega C - \frac{1}{\omega L} \right) \right\} V(t) - \frac{E}{j\omega L} - \left(1 + \frac{R}{j\omega L} \right) I \quad (12)$$

$$\text{where} \quad V(t) = V(t) e^{j\theta(t)}, \quad E = E, \quad I = I. \quad (13)$$

It will be easily verified from (12) that $\dot{V}^*(t)$ satisfies the following relation:

$$-2C \frac{d}{dt} \dot{V}^*(t) = \left\{ \frac{CR}{L} - j \left(\omega C - \frac{1}{\omega L} \right) \right\} \dot{V}^*(t) + \frac{E}{j\omega L} - \left(1 - \frac{R}{j\omega L} \right) I \quad (14)$$

$$\text{where} \quad \dot{V}^*(t) = V(t) e^{-j\theta(t)}. \quad (15)$$

From (14), multiplying both sides of this equation with $V(t)$ and through use of the approximation

$$1 - \frac{R}{j\omega L} \neq 1,$$

we can get the expression which may be called the "vector power relation",

$$-2CV(t) \frac{d}{dt} \dot{V}(t) = \left\{ \frac{CR}{L} - j \left(\omega C - \frac{1}{\omega L} \right) \right\} V(t) \dot{V}(t) + \frac{EV(t)}{j\omega L} - V(t)I. \quad (16)$$

Substituting V and \dot{V} from (13) and (15) into (16) and equating the real and imaginary parts of both sides of this equation, we obtain the following relations:

$$\begin{aligned} -2CV \frac{dV}{dt} &= -VI \cos \theta + \frac{EV}{\omega L} \sin \theta + \frac{CR}{L} V^2, \\ 2CV^2 \frac{d\theta}{dt} &= -VI \sin \theta - \frac{EV}{\omega L} \cos \theta - \left(\omega C - \frac{1}{\omega L} \right) V^2. \end{aligned} \quad (17)$$

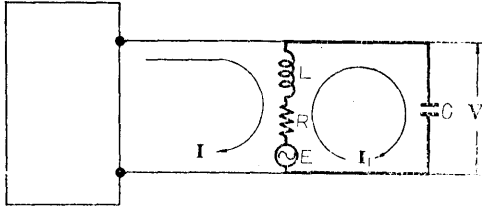


Fig. 2. A parallel tuning circuit with sinusoidal e.m.f.

Steady state of the system will be correlated with

$$\frac{dV}{dt} = 0, \quad \frac{d\theta}{dt} = 0,$$

and under these conditions V and θ are constant.

Then right hand side of the first equation of (17) yields the total active power. In detail, the first term denotes the active power fed from the source, the second the exciting active power

of the external periodic force, and the last the absorbed active power in the circuit.

And each term of the right hand side of the second equation (17) represents the reactive power of the same meaning as mentioned above with the active power.

Before concluding this section something about the active and reactive power must be added.

Absorbed instantaneous power p in a linear electric circuit is expressed by the product of the terminal voltage v and current i flowing into the circuit, that is,

$$p = v i = 2VI (\sin \tau \cos \theta + \cos \tau \sin \theta) \sin \tau \quad (18)$$

where

$$\begin{aligned} i &= \sqrt{2} I \sin \tau, \\ v &= \sqrt{2} V \sin (\tau + \theta). \end{aligned} \quad (19)$$

From these relations absorbed active and reactive power may be expressed respectively:

$$\begin{aligned} P_a &= VI \cos \theta = \frac{1}{2\pi} \int_0^{2\pi} v i d\tau, \\ P_b &= VI \sin \theta = \frac{1}{2\pi} \int_0^{2\pi} v \frac{di}{d\tau} d\tau. \end{aligned} \quad (20)$$

II. Equilibrium of Vector Power and its Stability

In the preceding chapter, the relations which must be satisfied by the active and reactive power have been presented taking as the examples the simple tuning circuits.

By virtue of these relations the discussion of the equilibrium of power which yields the steady states of the oscillation and its stability will be greatly facilitated.

In the preceding examples, on account of the linear character of the source and the circuit elements, there were no question about the stability of the steady state.

But if the source has a non-linear character, the question on the stability about a steady state will become very important and difficult problem.

Now let us express the character of an electric two pole :

$$v = f(i) \quad (21)$$

where v is the terminal voltage of the two pole, i is the current flowing into this two pole and $f(i)$ is a polynomial. Equivalent active and reactive power will be calculated in terms of $i = \sqrt{2} I \sin \tau$ through use of (20) and (21):

$$\begin{aligned} P_a &= \frac{1}{2\pi} \int_0^{2\pi} f(\sqrt{2} I \sin \tau) \sqrt{2} I \sin \tau d\tau, \\ P_b &= \frac{1}{2\pi} \int_0^{2\pi} f(\sqrt{2} I \sin \tau) \sqrt{2} I \cos \tau d\tau. \end{aligned} \quad (22)$$

Next, let us express the character of an electric two pole by

$$i = g(v) \quad (23)$$

where $g(v)$ is a polynomial.

Equivalent active and reactive power will be expressed in terms of $v = \sqrt{2} V \sin \tau$ instead of i , because in a steady state the relation :

$$dp = v di + i dv = 0$$

should be satisfied.

So, we can obtain the following expression of the equivalent active and reactive power from (20) and (23) using $-i dv$ instead of $v di$:

$$\begin{aligned} \bar{P}_a &= \frac{1}{2\pi} \int_0^{2\pi} g(\sqrt{2} V \sin \tau) \sqrt{2} V \sin \tau d\tau, \\ \bar{P}_b &= \frac{-1}{2\pi} \int_0^{2\pi} g(\sqrt{2} V \sin \tau) \sqrt{2} V \cos \tau d\tau. \end{aligned} \quad (24)$$

When the active power that is expressed by P_a in (22) or \bar{P}_a in (24) takes a positive value, it denotes an absorbed active power in an electric two pole.

On the other hand if they take a negative value under any condition, they express a generated active power from the electric two pole, and in these cases P_b and \bar{P}_b denote the generated reactive power. Under these conditions, such an electric two pole may be regarded as a power source.

If the source in the first example in the preceding chapter feeds the power which is expressed by (22) with the conditions mentioned above, the expressions $-EI \cos \theta$ and $-EI \sin \theta$ in (8) must be replaced by P_a and P_b in (22) respectively.

In the same way, if the source in the second example in the preceding chapter feeds the power expressed by (24) the terms $-VI \cos \theta$ and $-VI \sin \theta$ in (17) must be replaced by \bar{P}_a and \bar{P}_b in (24) respectively.

Consequently these two examples may be expressed in more general forms, such as

$$\begin{aligned} -2LI \frac{dI}{dt} &= A(I, \theta), \\ 2LI^2 \frac{d\theta}{dt} &= B(I, \theta), \end{aligned} \quad (25)$$

and

$$\begin{aligned} -2CV \frac{dV}{dt} &= \bar{A}(V, \theta), \\ 2CV^2 \frac{d\theta}{dt} &= \bar{B}(V, \theta), \end{aligned} \quad (26)$$

where $A(I, \theta)$ is the total active power of a system expressed by the current and resistance, $B(I, \theta)$ is the total reactive power expressed by the current and reactance, $\bar{A}(V, \theta)$ denotes the total active power of a system expressed by the voltage and conductance, and $\bar{B}(V, \theta)$ denotes the total reactive power expressed by the voltage and susceptance.

Let us consider the equation (25) further in detail.

The steady state of the oscillation will be expressed by

$$\bar{A}(V_0, \theta_0) = 0, \quad \bar{B}(V_0, \theta_0) = 0. \quad (27)$$

In order to test its stability we will suppose the small variation

$$V_0 + \delta V, \quad \theta_0 + \delta \theta \quad (28)$$

and replace V and θ in (26) by (28), develop the right hand side of (26) in powers of δV and $\delta \theta$, and reject all but the linear term in δV and $\delta \theta$.

Thus we shall obtain the following variational equations:

$$\begin{aligned} -2CV_0 \frac{d}{dt} \delta V &= \left(\frac{\partial \bar{A}}{\partial V} \right)_{0,0} \delta V + \left(\frac{\partial \bar{A}}{\partial \theta} \right)_{0,0} \delta \theta, \\ 2CV_0^2 \frac{d}{dt} \delta \theta &= \left(\frac{\partial \bar{B}}{\partial V} \right)_{0,0} \delta V + \left(\frac{\partial \bar{B}}{\partial \theta} \right)_{0,0} \delta \theta. \end{aligned} \quad (29)$$

It must be considered that the equilibrium will be stable if δV and $\delta \theta$ approach

zero with laps of time t . These conditions are readily obtained from (29)

$$\left(\frac{\partial \bar{A}}{\partial \bar{V}}\right)_{0,0} - \frac{1}{V_0} \left(\frac{\partial \bar{B}}{\partial \theta}\right)_{0,0} > 0, \quad (30)$$

and

$$\left| \begin{array}{cc} \left(\frac{\partial \bar{A}}{\partial \bar{V}}\right)_{0,0} & \frac{1}{V_0} \left(\frac{\partial \bar{A}}{\partial \theta}\right)_{0,0} \\ \left(\frac{\partial \bar{B}}{\partial \bar{V}}\right)_{0,0} & \frac{1}{V_0} \left(\frac{\partial \bar{B}}{\partial \theta}\right)_{0,0} \end{array} \right| < 0. \quad (31)$$

We have assumed that $C > 0$ and $V_0 > 0$ in deriving these conditions.

In the similar way, the equilibrium of power and its stability will be discussed from (25) too.

It will be seen clearly that not only the active power but also the reactive power play important roles in testing the stability of the equilibrium of powers. And as we have seen from these treatment, the discussion on the equilibrium and its stability are greatly facilitated by the introduction of the fictitious electromotive force described in chapter 1.

Stability conditions described in (30) and (31) are the basis of the treatments that will be developed in the following chapters.

III. Self Oscillations

Self oscillations of the grid tuned valve oscillator will be considered as an example of the treatment developed in the preceding chapter.

The essential feature of the oscillator circuit is shown in Fig. 3.

Using the notations which are shown in Fig. 3, fundamental relations will be readily obtained as,

$$-2L \frac{d}{dt} I_1(t) = \left\{ R + j \left(\omega L - \frac{1}{\omega C} \right) \right\} I_1(t) - j\omega M I, \quad (32)$$

$$I_1(t) = j\omega C V(t) \quad (33)$$

Replacing I_1 by V through use of (33) the fundamental relation (32) will become

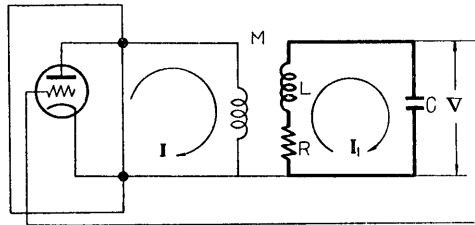


Fig. 3. A feedback Circuit

$$-2C \frac{d}{dt} V(t) = \left\{ \frac{CR}{L} + j \left(\omega C - \frac{1}{\omega L} \right) \right\} V(t) - \frac{M}{L} I, \quad (34)$$

and from (34) we get the conjugate relation,

$$-2C \frac{d}{dt} \bar{V}(t) = \left\{ \frac{CR}{L} - j \left(\omega C - \frac{1}{\omega L} \right) \right\} \bar{V}(t) - \frac{M}{L} \bar{I}. \quad (35)$$

Multiplying V on both sides of this equation the vector power relation will be obtained,

$$-2CV(t) \frac{d}{dt} \mathbf{V}^*(t) = \left\{ \frac{CR}{L} - j \left(\omega C - \frac{1}{\omega L} \right) \right\} V(t) \mathbf{V}^*(t) - \frac{M}{L} V(t) \mathbf{I}^* \quad (36)$$

In the equilibrium of the power, the first term of the right hand side of (36) denotes the absorbed vector power in the circuit and the second denotes the vector power fed from the valve.

Let us assume that the anode current i is expressed by

$$i = g_1 v + g_3 v^3 + g_5 v^5 \quad (37)$$

where g_1 , g_3 and g_5 are constants depending upon the characteristic curve and v is the alternating grid voltage of the valve.

Then the power fed to the circuit may be reexpressed from (24) and (37)

$$\begin{aligned} P_a &= \frac{1}{2\pi} \int_0^{2\pi} -\frac{M}{L} \left\{ g_1 (\sqrt{2} V \sin \tau) + g_3 (\sqrt{2} V \sin \tau)^3 + g_5 (\sqrt{2} V \sin \tau)^5 \right\} \sqrt{2} V \sin \tau d\tau \\ &= -\frac{M}{L} \left(g_1 V^2 + \frac{3}{2} g_3 V^4 + \frac{5}{2} g_5 V^6 \right), \end{aligned} \quad (38)$$

$$P_b = \frac{1}{2\pi} \int_0^{2\pi} \frac{M}{L} \left\{ g_1 (\sqrt{2} V \sin \tau) + g_3 (\sqrt{2} V \sin \tau)^3 + g_5 (\sqrt{2} V \sin \tau)^5 \right\} \sqrt{2} V \cos \tau d\tau = 0.$$

Using this in (36) we get the following relations:

$$\begin{aligned} -2CV \frac{dV}{dt} &= \frac{CR}{L} V^2 - \frac{M}{L} \left(g_1 V^2 + \frac{3}{2} g_3 V^4 + \frac{5}{2} g_5 V^6 \right), \\ 2CV^2 \frac{d\theta}{dt} &= - \left(\omega C - \frac{1}{\omega L} \right) V^2. \end{aligned} \quad (39)$$

If we write,

$$\begin{aligned} \bar{A}(V, \theta) &= \frac{CR}{L} V^2 - \frac{M}{L} \left(g_1 V^2 + \frac{3}{2} g_3 V^4 + \frac{5}{2} g_5 V^6 \right), \\ \bar{B}(V, \theta) &= - \left(\omega C - \frac{1}{\omega L} \right) V^2, \end{aligned} \quad (40)$$

equation (39) takes on the same form as that of (26), and the steady state of the oscillations will be determined from (27) and we shall be able to test their stability from (30), (31).

The amplitude of the oscillation V_0 will be obtained from

$$(CR - g_1 M) V_0^2 - \frac{3}{2} g_3 M V_0^4 - \frac{5}{2} g_5 M V_0^6 = 0 \quad (41)$$

and its frequency from

$$\omega C - 1/\omega L = 0. \quad (42)$$

The stability conditions not only give them physical reality but also particular character of the oscillations, that is, the "hard" and "soft" oscillations.

Let us show, as an example of our treatment, that the oscillations will be "soft" under the following conditions:

$$M > 0, \quad g_1 > 0, \quad g_3 < 0, \quad g_5 > 0. \quad (43)$$

In the relation (40), V appears only in the even power such as V^2 , V^4 and V^6 , and it is convenient to introduce ρ defined by

$$\rho = V^2 \quad (44)$$

and under the condition (42) reactive power $\bar{B}(V, \theta)$ will vanish from the stability conditions.

Thus we obtain the following relation instead of (41):

$$(CR - g_1 M)\rho_0 + aM\rho_0^2 - bM\rho_0^3 = 0 \quad (45)$$

where $a = -3g_3/2$, $b = 5g_5/2$, (46)

and ρ_0 is determined as the roots of equation (45) as

$$\begin{aligned} \rho_{0,0} &= 0, \\ \rho_{0,1,2} &= \frac{a}{2b} \pm \sqrt{\left(\frac{a}{2b}\right)^2 - \frac{1}{bM}(g_1 M - CR)}, \end{aligned} \quad (46')$$

and is stable if,

$$\left(\frac{\partial \bar{A}}{\partial \rho}\right)_0 = \frac{1}{L}(CR - g_1 M + 2aM\rho_0 - 3bM\rho_0^2) > 0,$$

that is,

$$(g_1 M - CR) - 2aM\rho_0 + 3bM\rho_0^2 < 0 \quad (47)$$

and is unstable if

$$(g_1 M - CR) - 2aM\rho_0 + 3bM\rho_0^2 > 0, \quad (48)$$

If we denote the roots of the equation ρ_c , which is obtained by equating the right hand side of (47) to zero, stable domain of the oscillations will be expressed by

$$\rho_{c1} < \rho_0 < \rho_{c2} \quad (49)$$

where

$$\rho_{c.1,2} = \frac{a}{3b} \pm \sqrt{\left(\frac{a}{3b}\right)^2 - \frac{1}{3bM}(g_1M - CR)}.$$

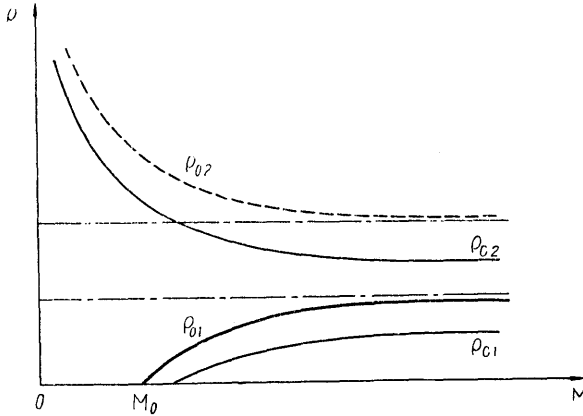


Fig. 4. Characteristics for the soft oscillations

of M will lead to a monotonic increase of amplitude which is shown in Fig. 4.

When M decreases, the amplitude diminishes until M reaches M_0 and then the oscillations will stop.

Appearance and disappearance of the oscillations take place at the same value of M .

Such oscillations are called the soft oscillations.

IV. Synchronism of Self Oscillations

It has been noticed, in Chapter III, that the essential feature of self oscillations is characterized by the stability conditions, but in those cases discussed in Chapter III the reactive power had no important role at all. Now I will show an example in which the reactive power has an important role to test the stability of the equilibrium state.

Let us consider the locked oscillations of valve oscillator.

A plate tuning oscillator is excited by an external sinusoidal electromotive force presented in the tank circuit.

The essential feature of the oscillator circuit and the notation of the elements are shown in Fig. 5.

In the locked state of the oscillations, oscillator frequency coincides with that of the external force, and oscillations that is not far from the sinusoidal may be observed.

Therefore, the locked state of the oscillations may be treated from the equilibrium of vector power.

The fundamental relation will be expressed in the same form as (16).

Under the condition

$$g_1M - CR > 0, \quad (50)$$

it is evident that $\rho_{0,2}$ is unstable, that is, an oscillation will take place. Under this condition, only the $\rho_{0,1}$ which has the negative sign of the root in (46) will be stable.

Therefore, if we increase M larger than the value $M_0 = CR/g_1$ the oscillation of the final amplitude $\rho_{0,1}$ will appear, and further increase

We express the anode current i

$$\begin{aligned} i &= g_1 G v + g_2 G^2 v^2 + g_3 G^3 v^3, \\ G &= D - K, \end{aligned} \quad (51)$$

where $D=1/\mu$, $K=M/L$, μ is the amplification factor of the valve and g_1 , g_3 are the constants depending on the characteristics of the valve. If $g_1 G < 0$ and $g_3 G^3 > 0$ in (51), it expresses the soft working condition of oscillator as we have mentioned in chapter III (43) and it is convenient to express

$$\begin{aligned} g_1 G &= -\alpha, \quad g_2 G^2 = \beta, \quad g_3 G^3 = \gamma. \\ \alpha &> 0, \quad \gamma > 0. \end{aligned} \quad (52)$$

Then the power fed from the valve will be written as follows through use of (24), (51) and (52):

$$\begin{aligned} \bar{P}_a &= -\alpha V^2 + \frac{3}{2} \gamma V^4, \\ \bar{P}_b &= 0, \end{aligned} \quad (53)$$

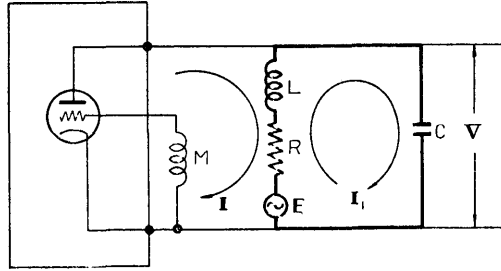


Fig.5. A locked oscillator circuit

Thus from (17), replacing $-VI \cos \theta$ and $-VI \sin \theta$ by \bar{P}_a and \bar{P}_b , we obtain the following fundamental relations:

$$\begin{aligned} -2CV \frac{dV}{dt} &= -\alpha V^2 + \frac{3}{2} \gamma V^4 + \frac{EV}{\omega L} \sin \theta + \frac{CR}{L} V^2, \\ 2CV^2 \frac{d\theta}{dt} &= -\frac{EV}{\omega L} \cos \theta - \left(\omega C - \frac{1}{\omega L} \right) V^2. \end{aligned} \quad (54)$$

Hence, the locked state of the oscillation will be determined from

$$\begin{aligned} \bar{A}(V, \theta) &= \left(\frac{CR}{L} - \alpha \right) V^2 + \frac{3}{2} \gamma V^4 + \frac{EV}{\omega L} \sin \theta = 0, \\ \bar{B}(V, \theta) &= -\left(\omega C - \frac{1}{\omega L} \right) V^2 - \frac{EV}{\omega L} \cos \theta = 0. \end{aligned} \quad (55)$$

By virtue of (55) the response curves of the locked oscillator is expressed by

$$\left\{ \omega_0 (\alpha L - CR) - \frac{3}{2} \omega_0 \gamma L V_0^2 \right\}^2 V_0^2 + \left\{ \frac{\omega_0^2 - \omega^2}{\omega_0^2} \right\}^2 V_0^2 = E^2. \quad (56)$$

In deriving (56) from (55), the following approximation has been used:

$$\omega L \doteq \omega_0 L, \quad \omega_0^2 = 1/LC, \quad (57)$$

V and θ which satisfy the equation (55) are expressed by V_0 and θ_0 respectively.

Next, let us calculate the following quantities from (55).

$$\begin{aligned}
\left(\frac{\partial \bar{A}}{\partial V}\right)_{0,0} &= \left(\frac{CR}{L} - \alpha\right)V_0 + \frac{9}{2}\gamma V_0^3, \\
\left(\frac{1}{V}\frac{\partial \bar{A}}{\partial \theta}\right)_{0,0} &= \frac{E}{\omega L} \cos \theta_0 = \frac{1}{\omega L} \left(\frac{\omega_0^2 - \omega^2}{\omega_0^2}\right)V_0, \\
\left(\frac{\partial \bar{B}}{\partial V}\right)_{0,0} &= \frac{1}{\omega L} \left(\frac{\omega_0^2 - \omega^2}{\omega_0^2}\right)V_0, \\
\left(\frac{1}{V}\frac{\partial \bar{B}}{\partial \theta}\right)_{0,0} &= \frac{E}{\omega L} \sin \theta_0 = -\left(\frac{CR}{L} - \alpha\right)V_0 - \frac{3}{2}\gamma V_0^3.
\end{aligned} \tag{58}$$

Introducing these quantities into (30) and (31), the conditions under which the equilibrium state, (V_0, θ_0) , will be stable, are readily obtained as follows:

$$\begin{aligned}
&\left(\frac{\partial \bar{A}}{\partial V}\right)_{0,0} - \frac{1}{V_0} \left(\frac{\partial \bar{B}}{\partial \theta}\right)_{0,0} \\
&= 2V_0 \left\{ \left(\frac{CR}{L} - \alpha\right) + 3\gamma V_0^2 \right\} > 0,
\end{aligned}$$

also,

$$\begin{aligned}
&\left\{ \left(\frac{\partial \bar{A}}{\partial V}\right) \left(\frac{1}{V}\frac{\partial \bar{B}}{\partial \theta}\right) - \left(\frac{1}{V}\frac{\partial \bar{A}}{\partial \theta}\right) \left(\frac{\partial \bar{B}}{\partial V}\right) \right\}_{0,0} \\
&= -\left\{ \left(\frac{CR}{L} - \alpha\right)V_0 + \frac{9}{2}\gamma V_0^3 \right\} \left\{ \left(\frac{CR}{L} - \alpha\right)V_0 + \frac{3}{2}\gamma V_0^3 \right\} - \left(\frac{1}{\omega L}\right)^2 \left(\frac{\omega_0^2 - \omega^2}{\omega_0^2}\right)^2 V_0^3 < 0.
\end{aligned}$$

Using the approximation (57) and considering $V_0 > 0$ these conditions can be expressed finally as

$$\omega_0(\alpha L - CR) - 3\omega_0\gamma L V_0^2 < 0, \tag{59}$$

$$\left\{ \omega_0(\alpha L - CR) - \frac{9}{2}\omega_0\gamma L V_0^3 \right\} \left\{ \omega_0(\alpha L - CR) - \frac{3}{2}\omega_0\gamma L V_0^3 \right\} + \left(\frac{\omega_0^2 - \omega^2}{\omega_0^2}\right)^2 V_0^3 > 0. \tag{60}$$

Essential features of the locked oscillations are completely described by (56), (59) and (60), that is to say, the response curve of the locked oscillation will be presented by (56) and the synchronous range will be determined from (59) and (60) together with (56).

These relations will lead to the results which have been obtained by B. van der Pol.

V. Subharmonic Synchronization

In the previous example in chapter IV, the synchronization, which may be called the subharmonic synchronization, will be observed again when the external periodic force has almost the same frequency as the integral multiple of the frequency of the oscillator.

Now, in this synchronous state the oscillator frequency is exactly the integral fraction $1/n$ of that of the external force. Therefore, we may call them "frequency demultiplication".

Mandelstam and Papalexli have treated this problem with skilled mathematical method, but it is rather difficult and complicated.

Our treatment will enable us to deal with this problem more straightforwardly. Let us demonstrate taking as an example $1/2$ subharmonic synchronization.

We will express the anode current by (51) for the sake of simplicity, because it disturbs the essential treatment of our method the least.

$$i = -\alpha v + \beta v^2 + \gamma v^3 ,$$

and in this case, we put the anode voltage in the synchronous state as follows ;

$$v = V_1 \sin\left(\frac{\omega}{2} t + \theta\right) + V_2 \sin \omega t . \quad (61)$$

which may be written

$$v = V_1 \sin \tau + V_2 \sin 2(\tau - \theta) . \quad (61')$$

Then the anode current will be expressed by

$$i = -\alpha V_1 \sin \tau + \beta V_1 V_2 \cos(\tau - 2\theta) + \frac{3}{4} \gamma V_1^3 \sin \tau + \frac{3}{2} \gamma V_1 V_2^2 \sin \tau + [\dots] . \quad (62)$$

By virtue of (24) and (62), power fed from the valve at the fundamental frequency will be written thus:

$$\begin{aligned} \bar{P}_a &= \frac{1}{2} \left(-\alpha V_1^2 + \frac{3}{4} \gamma V_1^4 + \frac{3}{2} \gamma V_1^2 V_2^2 + \beta V_1^2 V_2 \sin 2\theta \right) , \\ \bar{P}_b &= -\frac{1}{2} \beta V_1^2 V_2 \cos 2\theta , \end{aligned} \quad (63)$$

where \bar{P}_a denotes the active power and \bar{P}_b denotes the reactive power.

And the power absorbed in the circuit at the fundamental frequency will be expressed as :

$$\begin{aligned} \bar{P}_{ac} &= \frac{CR}{L} \frac{V_1^2}{2} , \\ \bar{P}_{bc} &= -\left(\frac{\omega C}{2} - \frac{2}{\omega L} \right) \frac{V_1^2}{2} . \end{aligned} \quad (64)$$

The total active power \bar{A}_1 and reactive power B_1 will be

$$\begin{aligned} \bar{A}_1 &= \frac{V_1^2}{2} \left\{ \left(-\alpha + \frac{CR}{L} \right) + \frac{3}{4} \gamma V_1^2 + \frac{3}{2} \gamma V_2^2 + \beta V_2 \sin 2\theta \right\} , \\ \bar{B}_1 &= \frac{V_1^2}{2} \left\{ \frac{2}{\omega L} \left(1 - \frac{\omega^2}{4\omega_0^2} \right) - \beta V_2 \cos 2\theta \right\} , \end{aligned} \quad (65)$$

where $\omega_0^2 = 1/LC$.

In this relation, V_2 remains undetermined. It may be calculated approximately

from the impedance drop due to the circulating current in the tank circuit which is caused by the external periodic force $v_i = V_i \sin \omega t$.

Hence, neglecting the resistance in the circuit, the following relation will be obtained :

$$V_i = V_2(1 - \omega^2 LC),$$

And from this relation, using the approximation,

$$2\omega_0 \doteq \omega \quad (66)$$

the relation between the amplitude V_2 and V_i may be obtained ;

$$V_2 \doteq -V_i/3. \quad (67)$$

Using (66) and (67) in (65), we get the following expression :

$$\begin{aligned} \overline{A}_1(V_1\theta) &= \frac{V_1^2}{2} \left\{ \left(-\alpha + \frac{CR}{L} \right) + \frac{3}{4} \gamma V_1^2 + \frac{3}{18} \gamma V_i^2 - \frac{1}{3} \beta V_i \sin 2\theta \right\}, \\ \overline{B}_1(V_1\theta) &= \frac{V_1^2}{2} \left\{ \frac{1}{\omega_0 L} \left(1 - \frac{\omega^2}{4\omega_0^2} \right) + \frac{1}{3} \beta V_i \cos 2\theta \right\}. \end{aligned} \quad (68)$$

And the synchronized state of the oscillation to be correlated with the equilibrium of these powers is expressed as follows :

$$\begin{aligned} \overline{A}_1(V_{10}, \theta_0) &= \frac{V_{10}^2}{2} \left\{ \left(-\alpha + \frac{CR}{L} \right) + \frac{3}{4} \gamma V_{10}^2 + \frac{3}{18} \gamma V_i^2 - \frac{1}{3} \beta V_i \sin 2\theta_0 \right\} = 0, \\ \overline{B}_1(V_{10}, \theta_0) &= \frac{V_{10}^2}{2} \left\{ \frac{1}{\omega_0 L} \left(1 - \frac{\omega^2}{4\omega_0^2} \right) + \frac{1}{3} \beta V_i \cos 2\theta_0 \right\} = 0. \end{aligned} \quad (69)$$

Thus in the synchronized state, we obtain the following relation :

$$\left\{ \omega_0 \left(-\alpha L + CR \right) + \frac{3}{4} \omega_0 \gamma L V_{10}^2 + \frac{3}{18} \omega_0 \gamma L V_i^2 \right\}^2 = \left\{ \frac{1}{3} \omega_0 \beta L V_i \right\}^2 - \left\{ 1 - \left(\frac{\omega}{2\omega_0} \right)^2 \right\}^2. \quad (70)$$

Let us now turn to the stability problem.

From (68) we obtain

$$\begin{aligned} \left(\frac{\partial \overline{A}_1}{\partial V_1} \right)_{0,0} &= \frac{3}{4} \gamma V_{10}^2, \\ \left(\frac{1}{V_1} \frac{\partial \overline{A}_1}{\partial \theta} \right)_{0,0} &= \frac{1}{\omega_0 L} \left(1 - \frac{\omega^2}{4\omega_0^2} \right) V_{10}, \\ \left(\frac{\partial \overline{B}_1}{\partial V_1} \right)_{0,0} &= 0, \\ \left(\frac{1}{V_1} \frac{\partial \overline{B}_1}{\partial \theta} \right)_{0,0} &= \left(\alpha - \frac{CR}{L} \right) V_{10} - \frac{3}{4} \gamma V_{10}^3 - \frac{3}{18} \gamma V_{10} V_i^2. \end{aligned} \quad (71)$$

By virtue of these quantities, the stable conditions of the subharmonic synchronized state will be obtained from (30) and (31);

$$\left(-\alpha + \frac{CR}{L}\right) + \frac{3}{2} \gamma V_{10}^2 + \frac{3}{18} \gamma V_i^2 > 0, \quad (72)$$

and
$$-\frac{3}{4} \gamma V_{10}^4 \left\{ \left(-\alpha + \frac{CR}{L}\right) + \frac{3}{4} \gamma V_{10}^2 + \frac{3}{18} \gamma V_i^2 \right\} < 0. \quad (73)$$

Since $\gamma > 0$ in (73), we shall obtain only the following condition from these two:

$$\left(-\alpha + \frac{CR}{L}\right) + \frac{3}{4} \gamma V_{10}^2 + \frac{3}{18} \gamma V_i^2 > 0. \quad (74)$$

All the characteristics of the subharmonic synchronization can be easily deduced from (70) together with the condition (74).

VI. Coupled Tuning Circuits

In this chapter we shall deal with the case where two oscillatory circuits are coupled to each other through a common inductance and connected to a source which has a non-linear character.

The essential feature of the circuits are shown in Fig. 6. Current I flows from the source into the external circuits and, I_1 and I_2 circulate through the primary and secondary tuning circuit respectively.

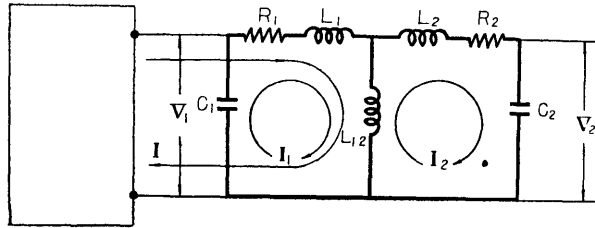


Fig. 6. Coupled tuning circuits

These circulating currents play the principal role in our treatment as we have seen in chapter I. Their magnitude and phase must be regarded as a function of t :

$$I_1 = I_1(t) e^{-j\theta_1(t)}, \quad I_2 = I_2(t) e^{-j\theta_2(t)}. \quad (75)$$

And these circulating currents are subjected to the effect of the fictitious electromotive force

$$-2L_{11} \frac{d}{dt} I_1, \quad -2L_{22} \frac{d}{dt} I_2. \quad (76)$$

which should be introduced in the primary and secondary tuning circuit respectively, where

$$L_{11} = L_1 + L_{12}, \quad L_{22} = L_2 + L_{12}. \quad (77)$$

Then the following relations will be obtained readily :

$$\begin{aligned} -2L_{11} \frac{d}{dt} \mathbf{I}_1 &= \left(R_1 + j\omega L_{11} + \frac{1}{j\omega C_1} \right) \mathbf{I}_1 + (R_1 + j\omega L_{11}) \mathbf{I} - j\omega L_{12} \mathbf{I}_2, \\ -2L_{22} \frac{d}{dt} \mathbf{I}_2 &= \left(R_2 + j\omega L_{22} + \frac{1}{j\omega C_2} \right) \mathbf{I}_2 - j\omega L_{12} (\mathbf{I} + \mathbf{I}_1), \end{aligned} \quad (78)$$

$$\text{and} \quad -j\omega C_1 \mathbf{V}_1 = \mathbf{I}_1, \quad j\omega C_2 \mathbf{V}_2 = \mathbf{I}_2. \quad (79)$$

By virtue of the relation (77) and (78), we can obtain the following relations :

$$\begin{aligned} -2C_1 \frac{d}{dt} \mathbf{V}_1 &= \left\{ \frac{R_1 C_1}{L_{11}} + j \left(\omega C_1 - \frac{1}{\omega L_{11}} \right) \right\} \mathbf{V}_1 - \left(1 + \frac{R_1}{j\omega L_{11}} \right) \mathbf{I} + j \frac{\omega L_{12} C_2}{L_{11}} \mathbf{V}_2, \\ -2C_2 \frac{d}{dt} \mathbf{V}_2 &= \left\{ \frac{R_2 C_2}{L_{22}} + j \left(\omega C_2 - \frac{1}{\omega L_{22}} \right) \right\} \mathbf{V}_2 - \frac{L_{12}}{L_{22}} \mathbf{I} + j \frac{\omega L_{12} C_1}{L_{22}} \mathbf{V}_1. \end{aligned} \quad (80)$$

These \mathbf{V}_1 and \mathbf{V}_2 are also the functions of t as

$$\mathbf{V}_1 = V_1(t) e^{j\phi_1(t)}, \quad \mathbf{V}_2 = V_2(t) e^{j\phi_2(t)}. \quad (81)$$

Let us introduce \mathbf{V}_1^* and \mathbf{V}_2^* defined by

$$\begin{aligned} \mathbf{V}_1^* &= V_1(t) e^{-j\phi_1(t)}, \\ \mathbf{V}_2^* &= V_2(t) e^{-j\phi_2(t)}. \end{aligned} \quad (82)$$

Then it will be easily verified from (80) that \mathbf{V}_1^* and \mathbf{V}_2^* satisfy the following relations :

$$\begin{aligned} -2C_1 \frac{d}{dt} \mathbf{V}_1^* &= \left\{ \frac{R_1 C_1}{L_{11}} - j \left(\omega C_1 - \frac{1}{\omega L_{11}} \right) \right\} \mathbf{V}_1^* - \mathbf{I} - j \frac{\omega L_{12} C_2}{L_{11}} \mathbf{V}_2^*, \\ -2C_2 \frac{d}{dt} \mathbf{V}_2^* &= \left\{ \frac{R_2 C_2}{L_{22}} - j \left(\omega C_2 - \frac{1}{\omega L_{22}} \right) \right\} \mathbf{V}_2^* - \frac{L_{12}}{L_{22}} \mathbf{I} - j \frac{\omega L_{12} C_1}{L_{22}} \mathbf{V}_1^*, \end{aligned} \quad (83)$$

in which the following approximation is used :

$$1 + R_1/j\omega L_{11} \approx 1.$$

Multiplying \mathbf{V}_1 on both sides of the first equation of (83) and \mathbf{V}_2 on both sides of the second, we can obtain the relations which may be called the "vector power relations" :

$$\begin{aligned}
-2C_1 V_1 \frac{d}{dt} \dot{V}_1 &= \left\{ \frac{R_1 C_1}{L_{11}} - j \left(\omega C_1 - \frac{1}{\omega L_{11}} \right) \right\} V_1 \dot{V}_1 - V_1 \dot{I} - j \frac{\omega L_{12} C_2}{L_{11}} V_1 \dot{V}_2, \\
-2C_2 V_2 \frac{d}{dt} \dot{V}_2 &= \left\{ \frac{R_2 C_2}{L_{22}} - j \left(\omega C_2 - \frac{1}{\omega L_{22}} \right) \right\} V_2 \dot{V}_2 - \frac{L_{12}}{L_{22}} V_2 \dot{I} - j \frac{\omega L_{12} C_1}{L_{22}} V_2 \dot{V}_1.
\end{aligned} \quad (84)$$

We can obtain the following relations from (84):

$$\begin{aligned}
-2C_1 V_1 \frac{dV_1}{dt} &= -V_1 I \cos \phi_1 + \frac{R_1 C_1}{L_{11}} V_1^2 - \frac{\omega L_{12} C_2}{L_{11}} V_1 V_2 \sin(\phi_2 - \phi_1), \\
2C_1 V_1^2 \frac{d\phi_1}{dt} &= -V_1 I \sin \phi_1 - \left(\omega C_1 - \frac{1}{\omega L_{11}} \right) V_1^2 - \frac{\omega L_{12} C_2}{L_{11}} V_1 V_2 \cos(\phi_2 - \phi_1), \\
-2C_2 V_2 \frac{dV_2}{dt} &= -\frac{L_{12}}{L_{22}} V_2 I \cos \phi_2 + \frac{R_2 C_2}{L_{22}} V_2^2 - \frac{\omega L_{12} C_1}{L_{22}} V_1 V_2 \sin(\phi_1 - \phi_2), \\
2C_2 V_2^2 \frac{d\phi_2}{dt} &= -\frac{L_{12}}{L_{22}} V_2 I \sin \phi_2 - \left(\omega C_2 - \frac{1}{\omega L_{22}} \right) V_2^2 - \frac{\omega L_{12} C_1}{L_{22}} V_1 V_2 \cos(\phi_1 - \phi_2).
\end{aligned} \quad (85)$$

In these treatments, we have regarded the source as an ideal constant current source which supplies the active power $-V_1 I \cos \phi_1$ and reactive power $-V_1 I \sin \phi_1$ to the primary circuit. But if the actual character of the source is

$$i = g(v_1) \quad (86)$$

we should calculate the active and reactive power from (24) as

$$\begin{aligned}
\bar{P}_a &= \frac{1}{2\pi} \int_0^{2\pi} g(\sqrt{2} V_1 \sin \tau) \sqrt{2} V_1 \sin \tau \, d\tau, \\
\bar{P}_b &= \frac{-1}{2\pi} \int_0^{2\pi} g(\sqrt{2} V_1 \sin \tau) \sqrt{2} V_1 \cos \tau \, d\tau,
\end{aligned} \quad (87)$$

and accordingly $-V_2 I \cos \phi_2$ and $-V_2 I \sin \phi_2$ in (85) must be replaced by

$$\frac{1}{2\pi} \int_0^{2\pi} g[\sqrt{2} V_1 \sin(\tau + \phi_1 - \phi_2)] V_2 \sin \tau \, d\tau, \quad (88)$$

and
$$\frac{-1}{2\pi} \int_0^{2\pi} g[\sqrt{2} V_1 \sin(\tau + \phi_1 - \phi_2)] V_2 \cos \tau \, d\tau,$$

respectively.

Steady state of the system is to be correlated with the singular point of the differential equations of the first order (85).

Let us rewrite the equations (85) in more general forms for the sake of brevity:

$$\begin{aligned}
-2C_1 V_1 \frac{dV_1}{dt} &= \bar{A}_1(V_1, \phi_1, V_2, \phi_2), \\
2C_1 V_1^2 \frac{d\phi_1}{dt} &= \bar{B}_1(V_1, \phi_1, V_2, \phi_2), \\
-2C_2 V_2 \frac{dV_2}{dt} &= \bar{A}_2(V_1, \phi_1, V_2, \phi_2), \\
2C_2 V_2^2 \frac{d\phi_2}{dt} &= \bar{B}_2(V_1, \phi_1, V_2, \phi_2).
\end{aligned} \quad (89)$$

In the steady state of the system, magnitude and phase of the voltage are all constant and may be determined from

$$\begin{aligned}\bar{A}_1(V_{10}, \phi_{10}, V_{20}, \phi_{20}) &= 0, \\ \bar{B}_1(V_{10}, \phi_{10}, V_{20}, \phi_{20}) &= 0, \\ \bar{A}_2(V_{10}, \phi_{10}, V_{20}, \phi_{20}) &= 0, \\ \bar{B}_2(V_{10}, \phi_{10}, V_{20}, \phi_{20}) &= 0.\end{aligned}\quad (90)$$

Now, \bar{A}_1 and \bar{B}_1 express the total active and reactive power in the primary circuit respectively and \bar{A}_2 , \bar{B}_2 express the similar meaning to the secondary circuit.

And (90) denotes the equilibrium of these powers.

In order to test the stability of the steady state, let us obtain the variational equation from (89), giving small variations to each variables, $V_1 + \delta V_1$, $\phi_1 + \delta \phi_1$, $V_2 + \delta V_2$, $\phi_2 + \delta \phi_2$.

The variational equations will be

$$\begin{aligned}-2C_1 V_{10} \frac{d}{dt} \delta V_1 &= \left(\frac{\partial \bar{A}_1}{\partial V_1} \right)_0 \delta V_1 + \left(\frac{\partial \bar{A}_1}{\partial \phi_1} \right)_0 \delta \phi_1 + \left(\frac{\partial \bar{A}_1}{\partial V_2} \right)_0 \delta V_2 + \left(\frac{\partial \bar{A}_1}{\partial \phi_2} \right)_0 \delta \phi_2, \\ 2C_1 V_{10}^2 \frac{d}{dt} \delta \phi_1 &= \left(\frac{\partial \bar{B}_1}{\partial V_1} \right)_0 \delta V_1 + \left(\frac{\partial \bar{B}_1}{\partial \phi_1} \right)_0 \delta \phi_1 + \left(\frac{\partial \bar{B}_1}{\partial V_2} \right)_0 \delta V_2 + \left(\frac{\partial \bar{B}_1}{\partial \phi_2} \right)_0 \delta \phi_2, \\ -2C_2 V_{20} \frac{d}{dt} \delta V_2 &= \left(\frac{\partial \bar{A}_2}{\partial V_1} \right)_0 \delta V_1 + \left(\frac{\partial \bar{A}_2}{\partial \phi_1} \right)_0 \delta \phi_1 + \left(\frac{\partial \bar{A}_2}{\partial V_2} \right)_0 \delta V_2 + \left(\frac{\partial \bar{A}_2}{\partial \phi_2} \right)_0 \delta \phi_2, \\ 2C_2 V_{20}^2 \frac{d}{dt} \delta \phi_2 &= \left(\frac{\partial \bar{B}_2}{\partial V_1} \right)_0 \delta V_1 + \left(\frac{\partial \bar{B}_2}{\partial \phi_1} \right)_0 \delta \phi_1 + \left(\frac{\partial \bar{B}_2}{\partial V_2} \right)_0 \delta V_2 + \left(\frac{\partial \bar{B}_2}{\partial \phi_2} \right)_0 \delta \phi_2.\end{aligned}\quad (91)$$

$$\left(\frac{\partial \bar{A}_1}{\partial V_1} \right)_0 \text{ denotes } \left(\frac{\partial \bar{A}_1}{\partial V_1} \right)_{\substack{V_1=V_{10}, V_2=V_{20}, \\ \phi_1=\phi_{10}, \phi_2=\phi_{20}}}, \text{ etc.}$$

Characteristic equation of (91) may be given as follows:

$$D_0 S^4 + D_1 S^3 + D_2 S^2 + D_3 S + D_4 = 0, \quad (92)$$

where

$$D_0 = 1. \quad (93)$$

$$D_1 = 2C_1 V_{10} \left\{ \left(\frac{\partial \bar{A}_1}{\partial V_1} \right)_0 - \frac{1}{V_{10}} \left(\frac{\partial \bar{B}_1}{\partial \phi_1} \right)_0 \right\} + 2C_2 V_{20} \left\{ \left(\frac{\partial \bar{A}_2}{\partial V_2} \right)_0 - \frac{1}{V_{20}} \left(\frac{\partial \bar{B}_2}{\partial \phi_2} \right)_0 \right\}, \quad (94)$$

$$\begin{aligned}D_2 &= \begin{vmatrix} \frac{1}{2C_1 V_{10}} \left(\frac{\partial \bar{A}_1}{\partial V_1} \right)_0 & \frac{1}{2C_1 V_{10}} \left(\frac{\partial \bar{A}_1}{\partial V_2} \right)_0 \\ \frac{1}{2C_2 V_{20}} \left(\frac{\partial \bar{A}_2}{\partial V_1} \right)_0 & \frac{1}{2C_2 V_{20}} \left(\frac{\partial \bar{A}_2}{\partial V_2} \right)_0 \end{vmatrix} + \begin{vmatrix} \frac{1}{2C_1 V_{10}^2} \left(\frac{\partial \bar{B}_1}{\partial \phi_1} \right)_0 & \frac{1}{2C_1 V_{10}^2} \left(\frac{\partial \bar{B}_1}{\partial \phi_2} \right)_0 \\ \frac{1}{2C_2 V_{20}^2} \left(\frac{\partial \bar{B}_2}{\partial \phi_1} \right)_0 & \frac{1}{2C_2 V_{20}^2} \left(\frac{\partial \bar{B}_2}{\partial \phi_2} \right)_0 \end{vmatrix} \\ &- \left\{ \frac{1}{2C_1 V_{10}} \left(\frac{\partial \bar{A}_1}{\partial V_1} \right)_0 + \frac{1}{2C_2 V_{20}} \left(\frac{\partial \bar{A}_2}{\partial V_2} \right)_0 \right\} \left\{ \frac{1}{2C_1 V_{10}^2} \left(\frac{\partial \bar{B}_1}{\partial \phi_1} \right)_0 + \frac{1}{2C_2 V_{20}^2} \left(\frac{\partial \bar{B}_2}{\partial \phi_2} \right)_0 \right\},\end{aligned}\quad (95)$$

$$\begin{aligned}
D_3 = & \left\{ \frac{1}{2C_1 V_{10}} \left(\frac{\partial \bar{A}_1}{\partial V_1} \right)_0 + \frac{1}{2C_2 V_{20}} \left(\frac{\partial \bar{A}_2}{\partial V_2} \right)_0 \right\} \begin{vmatrix} \frac{1}{2C_1 V_{10}^2} \left(\frac{\partial \bar{B}_1}{\partial \phi_1} \right)_0 & \frac{1}{2C_1 V_{10}} \left(\frac{\partial \bar{B}_1}{\partial \phi_2} \right)_0 \\ \frac{1}{2C_2 V_{20}^2} \left(\frac{\partial \bar{B}_2}{\partial \phi_1} \right)_0 & \frac{1}{2C_2 V_{20}} \left(\frac{\partial \bar{B}_2}{\partial \phi_2} \right)_0 \end{vmatrix} \\
& - \left\{ \frac{1}{2C_1 V_{10}^2} \left(\frac{\partial \bar{B}_1}{\partial \phi_1} \right)_0 + \frac{1}{2C_2 V_{20}^2} \left(\frac{\partial \bar{B}_2}{\partial \phi_2} \right)_0 \right\} \begin{vmatrix} \frac{1}{2C_1 V_{10}} \left(\frac{\partial \bar{A}_1}{\partial V_1} \right)_0 & \frac{1}{2C_1 V_{10}} \left(\frac{\partial \bar{A}_1}{\partial V_2} \right)_0 \\ \frac{1}{2C_2 V_{20}} \left(\frac{\partial \bar{A}_2}{\partial V_1} \right)_0 & \frac{1}{2C_2 V_{20}} \left(\frac{\partial \bar{A}_2}{\partial V_2} \right)_0 \end{vmatrix}, \quad (96)
\end{aligned}$$

$$\begin{aligned}
D_4 = & \begin{vmatrix} \frac{1}{2C_1 V_{10}} \left(\frac{\partial \bar{A}_1}{\partial V_1} \right)_0 & \frac{1}{2C_1 V_{10}} \left(\frac{\partial \bar{A}_1}{\partial V_2} \right)_0 \\ \frac{1}{2C_2 V_{20}} \left(\frac{\partial \bar{A}_2}{\partial V_1} \right)_0 & \frac{1}{2C_2 V_{20}} \left(\frac{\partial \bar{A}_2}{\partial V_2} \right)_0 \end{vmatrix} \times \begin{vmatrix} \frac{1}{2C_1 V_{10}^2} \left(\frac{\partial \bar{B}_1}{\partial \phi_1} \right)_0 & \frac{1}{2C_1 V_{10}} \left(\frac{\partial \bar{B}_1}{\partial \phi_2} \right)_0 \\ \frac{1}{2C_2 V_{20}^2} \left(\frac{\partial \bar{B}_2}{\partial \phi_1} \right)_0 & \frac{1}{2C_2 V_{20}} \left(\frac{\partial \bar{B}_2}{\partial \phi_2} \right)_0 \end{vmatrix} \\
& - \begin{vmatrix} \frac{1}{2C_1 V_{10}^2} \left(\frac{\partial \bar{A}_1}{\partial \phi_1} \right)_0 & \frac{1}{2C_1 V_{10}} \left(\frac{\partial \bar{A}_1}{\partial \phi_2} \right)_0 \\ \frac{1}{2C_2 V_{20}^2} \left(\frac{\partial \bar{A}_2}{\partial \phi_1} \right)_0 & \frac{1}{2C_2 V_{20}} \left(\frac{\partial \bar{A}_2}{\partial \phi_2} \right)_0 \end{vmatrix} \times \begin{vmatrix} \frac{1}{2C_1 V_{10}} \left(\frac{\partial \bar{B}_1}{\partial V_1} \right)_0 & \frac{1}{2C_1 V_{10}} \left(\frac{\partial \bar{B}_1}{\partial V_2} \right)_0 \\ \frac{1}{2C_2 V_{20}} \left(\frac{\partial \bar{B}_2}{\partial V_1} \right)_0 & \frac{1}{2C_2 V_{20}} \left(\frac{\partial \bar{B}_2}{\partial V_2} \right)_0 \end{vmatrix}. \quad (97)
\end{aligned}$$

By virtue of Hurwitz criterion, stability conditions may be given as follows:

$$D_1 > 0, \quad D_2 > 0, \quad D_3 > 0, \quad D_4 > 0,$$

and

$$D_3(D_1 D_2 - D_0 D_3) - D_1^2 D_4 > 0. \quad (98)$$

Acknowledgment

I wish to express my thanks to Guest Professor Nukiyama for his very helpful suggestion and to Mr. T. Tsunogae for his assistances on many matters.

Also I wish to express my obligation to Professor Kiyooka for his kindness in reading the original manuscript.