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Sub Title	
Author	渡部, 一郎(Watanabe, Ichiro)
Publisher	慶應義塾大学藤原記念工学部
Publication year	1953
Jtitle	Proceedings of the Fujihara Memorial Faculty of Engineering Keio University Vol.6, No.23 (1953.) ,p.86(8)- 91(13)
JaLC DOI	
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Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00060023-0008

On the Lateral Buckling of a Trapezoidal Cantilever of Uniform Thickness and a Trapezoidal Cantilever of I Section with Concentrated Load at the Free End(Part 1)

(Received September 17, 1953)

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Abstract

In this paper the author discusses the method of analysis for the determination of the critical lateral buckling load of a trapezoidal cantilever of uniform thickness and a trapezoidal cantilever of I section both with concentrated load at the free end, with special reference to the effect of the degree of convergence or divergence of the trapezoid on the critical buckling load.

The author, further, performed the calculus for the numerical example for each case to find the tendency of the variation of the critical load with degree of convergence or divergence coincides qualitatively for the two cases.

I. Introduction

In this paper the author treats the problem of lateral buckling of a trapezoidal cantilever of uniform thickness (Part I) and a trapezoidal cantilever of I section (Part II) both with concentrated load at the free end, with special reference to the effect of the degree of convergence or divergence of the trapezoid on the critical buckling load.

In Part I, i. e. as to the case of trapezoidal cantilever of uniform thickness, L. Prandtl previously had found theoretically the critical buckling load for the case of rectangular cantilever of uniform thickness.[1] Further, K. Federhofer had made theoretical analysis concerning to the cantilever of uniform thickness but varying breadth, the variation of which being expressed by the relation of $h=h_0(1-z/l)^n$, where h_0 represents the breadth of the cantilever at the fixed end, l denotes the length of the cantilever and z means the ordinate measured lengthwise from the fixed end[2]. If we put $n=0$ in the above expression, we obtain the solution for the triangular cantilever of uniform thickness with a concentrated load at the

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1) L. Prandtl; Kipperscheinung, Nürnberg 1899

2) K. Federhofer: Rep. Intern. Congr. App. Mech., Stockholm 1930

appex. The author now has made theoretical survey and obtained the buckling loads of trapezoidal cantilever of uniform thickness but varying convergent or divergent degrees loaded at the free end. The results were plotted for a numerical example together with the results derived from the above-mentioned analysis developed by L. Prandtl and K. Federhofer, to find the buckling loads increasing with increasing divergent degree (or decreasing convergent degrees) when the breadth of the cantilever at the fixed end is maintained unaltered.

In part II, i. e. as to the case of trapezoidal cantilever of I section, the author has treated the problem in a like manner, and also examined the effect of the degree of divergence or convergence upon the critical buckling load for a numerical example.

Part I

Lateral buckling of a trapezoidal cantilever of uniform thickness with concentrated load at the free end

I—1. The method of analysis

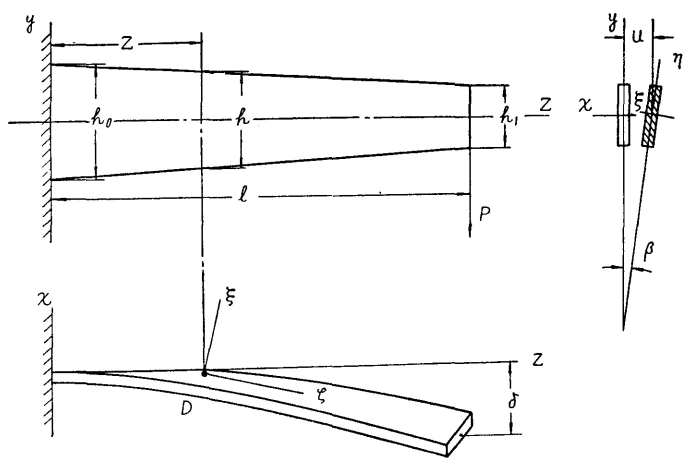

Fig. 1 Trapezoidal cantilever of uniform thickness

Fig. 1 shows a trapezoidal cantilever of uniform thickness with concentrated load P at the free end, the other end being fixed. P is considered to lie in the midplane of the web and further let the point of application of P to be the centre of the end section of the cantilever.

Further, we assume that the displacement is small and the stresses induced are within the elastic limit. Let the origin of the fixed coordinate x, y, z be on the fixed end of the cantilever as shown in the figure, and further we choose another rectangular coordinate ξ, η, ζ, the origin of which is selected as D, D being the centre of any cross section of the cantilever as shown. Denoting the displacements as u(x—component), v(y—component) and β, and taking into account of the right hand portion of the section through the point D, the moment around the axes passing through the point D and parallel to the x—, y— and z—axis may be described respectively as follows.

$$M_x=P(l-z), \qquad M_y=0, \qquad M_z=-P(\delta-u)$$

where l denotes the total length of the cantilever, while δ represents the displacement of the free end shown as in Fig. 1.

As the direction cosine between the rectangular axes of x, y, z, and ξ, η, ζ may be written as in table 1, the moment around the ξ, η, ζ axes may be written as follows.

Table 1. Direction cosines

	x	y	z
ξ	1,	β,	$-du/dz$
η	$-\beta$,	1,	$-dv/dz$
ζ	du/dz,	dv/dz,	1

$$M_\xi=P(l-z)$$

$$M_\eta=-\beta P(l-z)$$

$$M_\zeta=P(l-z)\frac{du}{dz}-P(\delta-u)$$

Putting these expressions into the general moment equation

$$B_1\frac{d^2u}{dz^2}=M_\eta, \qquad B_2\frac{d^2v}{dz^2}=-M_\xi, \qquad C\frac{d\beta}{dz}=M_\zeta,$$

we obtain the following equations.

$$\left.\begin{aligned} B_1\frac{d^2u}{dz^2}&=-\beta P(l-z)\\ B_2\frac{d^2v}{dz^2}&=-P(l-z)\\ C\frac{d\beta}{dz}&=P(l-z)\frac{du}{dz}-P(\delta-u)\end{aligned}\right\} \qquad (1)$$

Denoting the thickness and the height of the cantilever by b and h respectively, we have

$$B_1=\frac{hb^3}{12}E, \qquad B_2=\frac{bh^3}{12}E, \qquad C=\frac{hb^3}{3}(1-0.630\frac{b}{h})G.$$

If we put $h=h_0(1-c\frac{z}{l})$, $c=(h_0-h_1)/h_0$, then it becomes that c represents the degree of convergence or divergence of the trapezoidal cantilever. From the first equation of (1), we get

$$\frac{d^2u}{dz^2}=-\frac{\beta P}{B_1}(l-z) \qquad (2)$$

Differentiating the third equation of (1) by z and eliminating d^2u/dz^2 by means of eq. (2), we obtain

$$\left\{(l-cz)-0.630\frac{bl}{h_0}\right\}\frac{d^2\beta}{dz^2}-c\frac{d\beta}{dz}+\frac{36P^2l^2(l-z)^2}{h_0^2b^6EG(l-cz)}\beta=0 \tag{2.1}$$

Putting $36P^2l^2/h_0^2b^6EG=a$, $0.630\ bl/h_0=K$ for simplicity, the eq. (2.1) yields to

$$(l-cz)\left\{(l-cz)-K\right\}\frac{d^2\beta}{dz^2}-c(l-cz)\frac{d\beta}{dz}+a(l-z)^2\beta=0 \tag{2.2}$$

Putting $w=l-z$, the eq. (2.2) becomes finally

$$(cw+l-cl)\left\{(cw+l-cl)-K\right\}\frac{d^2\beta}{dw^2}+c(cw+l-cl)\frac{d\beta}{dw}+aw^2\beta=0 \tag{3}$$

We may solve the eq. (3) by infinite series. Thus, putting

$$\beta=Aw^m+Bw^{m+1}+Cw^{m+2}+Dw^{m+3}+Ew^{m+4}+Fw^{m+5}+Gw^{m+6}+Hw^{m+7}+Iw^{m+8}+Jw^{m+9}$$
$$+Kw^{m+10}+Lw^{m+11}+Mw^{m+12}+Nw^{m+13}+Ow^{m+14}+\cdots\cdots\cdots\cdots \tag{4}$$

and denoting

$$X=\{l^2(1-c)^2-Kl(1-c)\},\qquad Y=\{2cl(1-c)-Kc\}$$

for the sake of simplicity, we get the equation which defines the exponent m of the infinite series as follows:

$$m(m-1)=0 \tag{5}$$

Thus, we obtain $m=0$ and $m=1$. We are able to obtain, further, the following relations between the coefficients A, B, C, $\cdots\cdots\cdots\cdots$.

$$B=-\frac{Acl(1-c)}{(m+1)X}$$

$$C=\frac{Acl(1-c)\{mY+cl(1-c)\}}{(m+2)(m+1)X^2}-\frac{Am^2c^2}{(m+2)(m+1)X}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Using these expressions, the eq. (4) is expressed in a following manner.

$$\beta=A-\frac{Acl(1-c)}{X}w+\frac{Ac^2l^2(1-c)^2}{2!X^2}w^2+\cdots\cdots\cdots\cdots\cdots$$
$$+\alpha w-\frac{\alpha cl(1-c)}{2X}w^2+\cdots\cdots\cdots\cdots\cdots\cdots \tag{4.1}$$

One of the boundary conditions of the present problem is that $M_\xi=0$ at $z=l$, *i.e.* $(d\beta/dz)=-(d\beta/dw)=0$ when $w=0$. Therefore, it is evident that there exists the following relation between A and α of eq. (4.1).

$$\alpha=\frac{Acl(1-c)}{X}$$

Denoting

$$K=0.630\frac{bl}{h_0}=kl,\qquad k=0.630\frac{b}{h_0},$$

$$X=\{l^2(1-c)^2-Kl(1-c)\}=l^2\{(1-c)^2-k(1-c)\}=l^2x,$$

$$Y=2cl(1-c)-Kc=l\{2c(1-c)-kc\}=ly,$$

$$x=(1-c)^2-k(1-c),\qquad y=2c(1-c)-kc$$

for simplicity, the relations between the above-mentioned coefficients A, B, C, ················ are deduced to the following. Thus,

$$B=-\frac{Ac(1-c)}{lx},$$

$$C=\frac{Ac^2(1-c)^2}{2l^2x^2},$$

$$D=-\frac{Ac^2(1-c)^2\{y+c(1-c)\}}{6l^3x^3}+\frac{Ac^3(1-c)}{6l^3x^2},$$

$$E=-\frac{4c^2(1-c)^2\{y+c(1-c)\}\{2y+c(1-c)\}}{4!l^4x^4}-\frac{Ac^3(1-c)\{2y+5c(1-c)\}}{4!l^4x^3}-\frac{aA}{12l^2x},$$

···

The similar expressions are obtained between the coefficients $\alpha, \beta, \gamma,$ ···········. In practice, however, it is rather convenient to proceed as follows. Thus, using the numerical values of x and y evaluated for given values of c, we proceed to decide the values of the coefficients numerically by means of the following relations.

$$B=\frac{-Ac(1-c)}{(m+1)lx},$$

$$C=-\frac{B\{my+c(1-c)\}}{(m+2)lx}-\frac{Am^2c^2}{(m+2)(m+1)l^2x},$$

$$D=-\frac{C\{(m+1)y+c(1-c)\}}{l(m+3)x}-\frac{B(m+1)^2c^2}{l^2(m+3)(m+2)x},$$

$$E=-\frac{D\{(m+2)y+c(1-c)\}}{l(m+4)x}-\frac{C(m+2)^2c^2}{l^2(m+4)(m+3)x},$$

$$F=-\frac{E\{(m+3)y+c(1-c)\}}{l(m+5)x}-\frac{D(m+3)^2c^2}{l^2(m+5)(m+4)x}-\frac{aB}{l^2(m+5)(m+4)x},$$

$$G=-\frac{F\{(m+4)y+c(1-c)\}}{l(m+6)x}-\frac{E(m+4)^2c^2}{l^2(m+6)(m+5)x}-\frac{aC}{l^2(m+6)(m+5)x},$$

$$H=-\frac{G\{(m+5)y+c(1-c)\}}{l(m+7)x}-\frac{F(m+5)^2c^2}{l^2(m+7)(m+6)x}-\frac{aD}{l^2(m+7)(m+6)x},$$

$$I=-\frac{H\{(m+6)y+c(1-c)\}}{l(m+8)x}-\frac{G(m+6)^2c^2}{l^2(m+8)(m+7)x}-\frac{aE}{l^2(m+8)(m+7)x},$$

··

Thus, using the above expressions, all the coefficients are determined for given values of c, x and y corresponding to $m=0$ and 1 respectively. The coefficients thus determined are then put in the eq. (4.1). Applying another boundary condition, *i. e.* $\beta=0$ at the fixed end ($z=0$ or $w=l$), into this equation, and denoting

$$e=\frac{36}{h_0^2b^6EG}, \qquad a=\frac{36P^2l^2}{h_0^2b^6EG}=eP^2l^2$$

for the sake of simplicity, we obtain the polynomial expression as regard to $al^2=cP^2l^4$. The minimum root of the equation which may be obtained by numerical calculus represents the critical lateral buckling load P_{cr} in our case.

I—2. Numerical example

As cited previously, $c=(h_0-h_1)/h_0$ is the coefficient which represents the degree of convergence or divergence of the trapezoidal cantilever. The positive values of c means the convergent trapezoidal cantilever, while the negative values of c represents the divergent trapezoidal cantilever. As mentioned before, the cases for $c=0$ and $c=1$ are already solved by L. Prandtl and K. Federhofer respectively. In the present example, the author performed calculations for $c=-0.5$ to determine the critical buckling load for the assumed values of $b/h_0=1/30$ and $k=0.021$. The evaluations were performed taking up to the 4th order terms for eP^2l^4. Thus, for $c=-0.5$, $x=2.2185$, $y=1.4895$, the equation yields to

$$0.00000000083273(eP^2l^4)^4-0.0000039909(eP^2l^4)^3+0.00017439(eP^2l^4)^2-0.0325718(eP^2l^4)+0.9987=0.$$

Solving this equation, we have $eP^2l^4=37.5$ or

$$P_{cr}=1.020\frac{h_0b^3\sqrt{EG}}{l^2}$$

For $c=0$, $P_{cr}=0.6615\dfrac{h_0b^3\sqrt{EG}}{l^2}$

and for $c=1$, $P_{cr}=0.3963\dfrac{h_0b^3\sqrt{EG}}{l^2}$

The results are plotted in Fig. 2, taking $P_{cr}/\left(\dfrac{h_0b^3\sqrt{EG}}{l^2}\right)$ as ordinate and c as abscissa. The critical buckling load decreases for convergent trapezoidal cantilever and increases for divergent one for the same values of h_0, b and l. As seen in the figure, the slope becomes somewhat steeper for c negative.

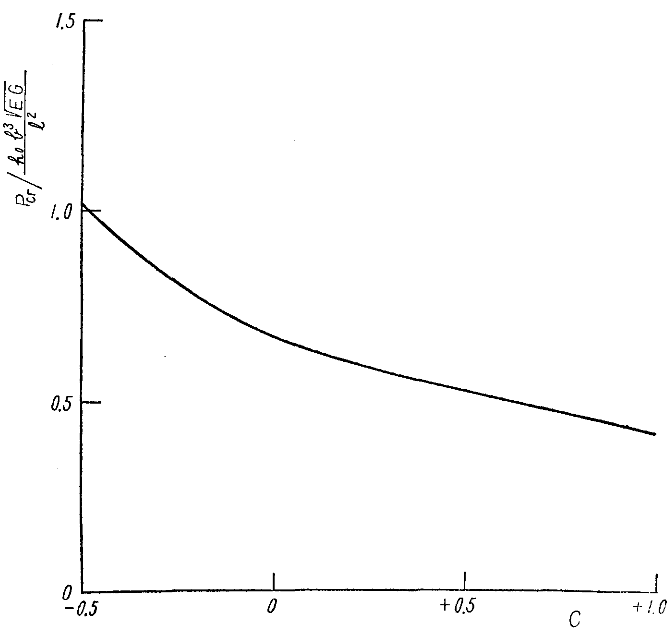

Fig. 2 Critical buckling load of a trapezoidal cantilever of uniform thickness