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Author	栖原, 豊太郎(Suhara, Toyotaro)
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Curved Beam of Variable Cross-Section

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Toyotaro SUHARA*

Abstract

A statically indeterminate problem of a curved beam in the form of circular arc with variable section along the length is solved, neglecting the effect of the axial thrust, and the expressions of the reactions at the ends, moments and deflections are obtained for a beam whose moment of inertia I is given by

$$I=I_0(1-k^2\sin^2\phi)^{j/2}$$

A thin curved beam ACB of variable cross-section, whose central line forms an arc of circle in the natural state, is fixed at the ends A and B with a distributed load q per unit horizontal length as shown in Fig.1. It is assumed that the central line of the beam lies in one vertical plane before and during the strain, and that all the forces and moments acting on the beam are also in that plane.

I Nomenclature

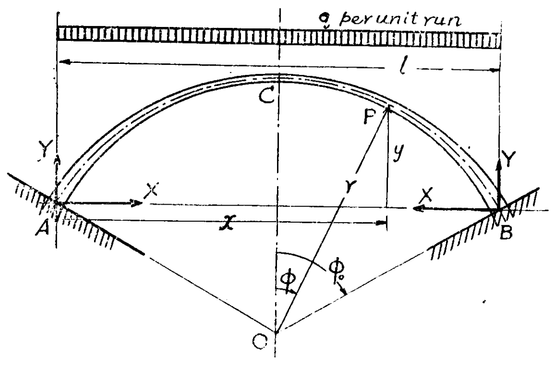

Fig. 1.

r the original radius of the circular arc ACB

x, y the co-ordinates of any point P in the central line of the beam in the natural state referred to A as the origin

σ the variable representing the length of arc AP, and $\sigma=r(\phi_0+\phi)$

ϕ, ϕ_0 the central angles subtending the arc CP and CB respectively in the natural state

δx, δy the displacement components in the strained state

$\delta\phi$ the rotation at any section P in the strained state

X, Y the horizontal and vertical reactions at the ends

M_0, M, M_c the moments at the ends, at any section P and at the middle section C, respectively

$EI=EI_cf(\phi)$ variable rigidity as a function of ϕ, I_c being the moment of inertia of section at C and E the Young's modulus

The following abbreviations are used throughout the paper:

$$s=\sin\phi,\quad c=\cos\phi,\quad t=\tan\phi,\quad \Delta\phi=\sqrt{(1-k^2\sin^2\phi)}$$

*栖原豊太郎, Dr. Eng., Professor at Keio University

$s_0=\sin\phi_0,\quad c_0=\cos\phi_0,\quad t_0=\tan\phi_0,\quad \Delta\phi_0=\sqrt{(1-k^2\sin^2\phi_0)}$

$$\psi_{mn}=\int_{-\phi_0}^{\phi}\frac{\cos^m\phi\sin^n\phi d\phi}{\sqrt{(1-k^2\sin^2\phi)^j}}=\int_{-\phi_0}^{\phi}\frac{c^m s^n d\phi}{\sqrt{(1-k^2s^2)^j}}$$

$$\Psi_{mn}=\int_{-\phi_0}^{\phi_0}\frac{\cos^m\phi\sin^n\phi d\phi}{\sqrt{(1-k^2\sin^2\phi)^j}}=\int_{-\phi_0}^{\phi_0}\frac{c^m s^n d\phi}{\sqrt{(1-k^2s^2)^j}}$$

in which j is a positive or negative integer, m and n positive or negative integers including zero, and $0<k^2<1$

$F(\phi)$, $E(\phi)$ the elliptic integrals of first and second kinds, with the modulus k

II Expression for variable moment of inertia of section of the beam

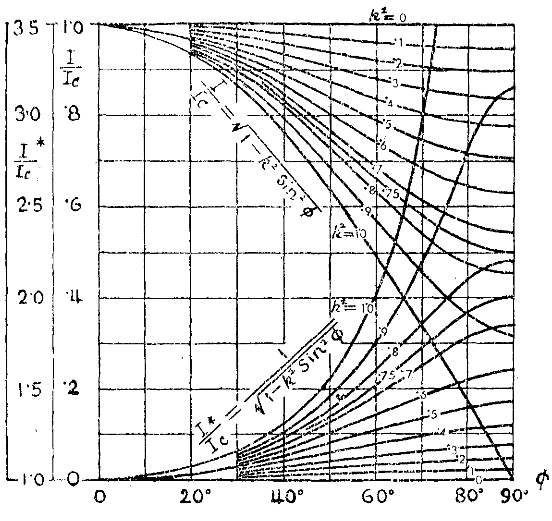

Fig. 2.

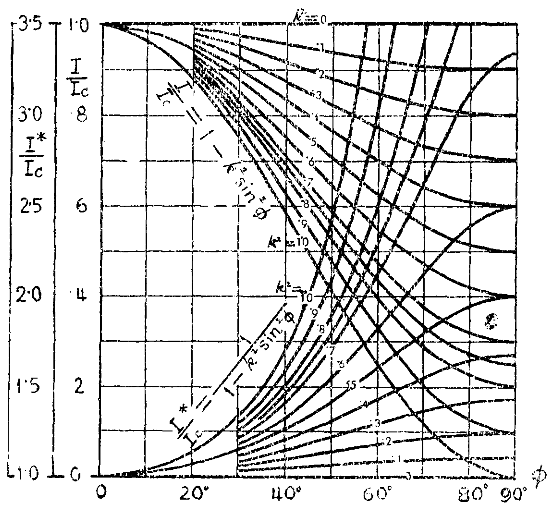

Fig. 3.

Let the moment of inertia of section of the beam as a function of ϕ be expressed by

$$I=I_c(1-k^2\sin^2\phi)^{j/2}$$
$$=I_c(1-k^2s^2)^{j/2} \qquad (1)$$

Fig. 2 shows the graph of the expression (1) for $j=1,\ -1$ and Fig. 3. for $j=2,\ -2$ with different values of k. Practically any shape of a symmetrical beam may be expressed by this formula.

III General expressions for the bending moments and reactions at the ends

The bending moment M at a section P is

$$M=Xr(c-c_0)-Yr(s_0-s)+M_0+\frac{1}{2}qr^2(s_0-s)^2 \qquad (2)$$

and the vertical reaction Y at the end is

$$Y=qrs_0 \qquad (3)$$

Eliminating Y between (2) and (3) we get

$$M=Xr(c-c_0)+M_0-\frac{1}{2}qr^2(s_0^2-s^2) \qquad (4)$$

Two unknowns X, the horizontal reaction, and M_0, the moment at the end, in the above equation may be determined from the partial derivatives of the expression of strain energy with respect to X and M_0 respectively,

thus $\frac{\partial}{\partial X}\int_{-\phi_0}^{\phi_0}\frac{M^2}{EI}d\sigma=0$ and $\frac{\partial}{\partial M_0}\int_{-\phi_0}^{\phi_0}\frac{M^2}{EI}d\sigma=0$ (5), (6)

From (4), (5) and (6) we get

$$(Xrc_0-M_0+\frac{1}{2}qr^2s_0^2)\Psi_{10}-Xr\Psi_{20}-\frac{1}{2}qr^2\Psi_{12}=0$$

$$(Xrc_0-M_0+\frac{1}{2}qr^2s_0^2)\Psi_{00}-Xr\Psi_{10}-\frac{1}{2}qr^2\Psi_{02}=0$$

and X and M_0 are determined as follows:

$$\frac{X}{qr}=\frac{\Psi_{00}\Psi_{12}-\Psi_{10}\Psi_{02}}{2(\Psi_{10}^2-\Psi_{00}\Psi_{20})} \tag{7}$$

$$\frac{M_0}{qr^2}=\frac{1}{2}\left(s_0^2-\frac{\Psi_{02}}{\Psi_{00}}\right)+\frac{X}{qr}\left(c_0-\frac{\Psi_{10}}{\Psi_{00}}\right)=\frac{(\Psi_{00}\Psi_{12}-\Psi_{10}\Psi_{02})c_0+\Psi_{02}\Psi_{20}-\Psi_{10}\Psi_{12}}{2(\Psi_{10}^2-\Psi_{00}\Psi_{20})}+\frac{s_0^2}{2} \tag{8}$$

The bending moment at any cross-section may be obtained from (4), (7) and (8), thus

$$\frac{M}{qr^2}=\frac{(\Psi_{00}\Psi_{12}-\Psi_{10}\Psi_{02})c+\Psi_{02}\Psi_{20}-\Psi_{10}\Psi_{12}}{2(\Psi_{10}^2-\Psi_{00}\Psi_{20})}+\frac{s^2}{2} \tag{9}$$

The moment at the middle section C of the beam is

$$\frac{M_c}{qr^2}=\frac{X}{qr}(1-c_0)+\frac{M_0}{qr^2}-\frac{s_0^2}{2}=\frac{\Psi_{00}\Psi_{12}-\Psi_{10}\Psi_{02}+\Psi_{02}\Psi_{20}-\Psi_{10}\Psi_{12}}{2(\Psi_{10}^2-\Psi_{00}\Psi_{20})} \tag{10}$$

IV General expressions for the rotation and displacement

The rotation $\delta\phi$ at any section P may be found from

$$EI=\frac{d\delta\phi}{d\sigma}=M,\ (d\sigma=rd\phi) \tag{11}$$

in which I and M are given by (1) and (9) respectively.

$$\frac{EI_0\delta\phi}{qr^3}=\frac{X}{qr}(\psi_{10}-\psi_{00}c_0)+\frac{M_0}{qr^2}\psi_{00}-\frac{1}{2}(\psi_{00}s_0^2-\psi_{02}) \tag{12}$$

or eliminating X and M_0

$$\frac{EI_c\delta\phi}{qr^3}=\frac{(\Psi_{02}\Psi_{20}-\Psi_{10}\Psi_{12})\psi_{00}+(\Psi_{00}\Psi_{12}-\Psi_{10}\Psi_{02})\psi_{10}+(\Psi_{10}^2-\Psi_{00}\Psi_{20})\psi_{02}}{2(\Psi_{10}^2-\Psi_{00}\Psi_{20})} \tag{13}$$

This expression satisfies the conditions at the fixed ends A and B, and at the middle section C,

$$(\delta\phi)_{\phi=\pm\phi_0}=(\delta\phi)_{\phi=0}=0$$

The displacement components δx and δy are

$$\delta x=\int_{-\phi_0}^{\phi} d\delta x=-r\int_{-\phi_0}^{\phi} \mathrm{s}\delta\phi\, d\phi$$

$$\frac{EI_c\delta x}{qr^4}=\frac{EI_c\delta\phi\cdot \mathrm{c}}{qr^3}-\frac{X}{qr}(\psi_{20}-\psi_{10}\mathrm{c}_0)-\frac{M_0}{qr^2}\psi_{10}-\frac{1}{2}(\psi_{12}-\psi_{10}\mathrm{s}_0^2)$$

$$=\left\{\begin{matrix}(\Psi_{10}\Psi_{12}-\Psi_{02}\Psi_{20})(\psi_{10}-\psi_{00}\mathrm{c})\\ -(\Psi_{10}^2-\Psi_{00}\Psi_{20})(\psi_{12}-\psi_{02}\mathrm{c})\\ +(\Psi_{10}\Psi_{02}-\Psi_{00}\Psi_{12})(\psi_{20}-\psi_{10}\mathrm{c})\end{matrix}\right\}\div 2(\Psi_{10}^2-\Psi_{00}\Psi_{20}) \tag{14}$$

This becomes zero at $\phi=\pm\phi_0$ and $\phi=0$.

$$\delta y=\int_{-\phi_0}^{\phi} d\delta y=r\int_{-\phi_0}^{\phi} \mathrm{c}\delta\phi d\phi$$

$$\frac{EI_c\delta y}{qr^4}=\frac{EI_c\delta\phi\cdot \mathrm{s}}{qr^3}-\frac{X}{qr}(\psi_{11}-\psi_{10}\mathrm{c}_0)-\frac{M_0}{qr^2}\psi_{01}+\frac{1}{2}(\psi_{21}-\psi_{01}\mathrm{c}_0^2)$$

$$=\left\{\begin{matrix}(\Psi_{10}\Psi_{12}-\Psi_{02}\Psi_{20})(\psi_{01}-\psi_{00}\mathrm{s})\\ -(\Psi_{10}^2-\Psi_{00}\Psi_{20})(\psi_{03}-\psi_{02}\mathrm{s})\\ +(\Psi_{10}\Psi_{02}-\Psi_{00}\Psi_{12})(\psi_{11}-\psi_{10}\mathrm{s})\end{matrix}\right\}\div 2(\Psi_{10}^2-\Psi_{00}\Psi_{20}) \tag{15}$$

This becomes zero at $\phi=\pm\phi_0$.

The vertical depression of the point C is

$$\frac{EI_c}{qr^4}(\delta y)_{\phi=0}=\frac{(\Psi_{10}\Psi_{12}-\Psi_{02}\Psi_{20})\psi_{01(0)}+(\Psi_{10}\Psi_{02}-\Psi_{00}\Psi_{12})\psi_{11(0)}}{2(\Psi_{10}^2-\Psi_{00}\Psi_{20})}-\frac{\psi_{03(0)}}{2} \tag{16}$$

where $\psi_{01(0)}$, $\psi_{11(0)}$ and $\psi_{03(0)}$ denote the values of the integrals ψ_{01}, ψ_{11} and ψ_{03} respectively with the upper limit $\phi=0$.

The expressions of the reactions, bending moments and deflections for beams of special forms are deduced in the following, with some numerical examples.

V Curved beam of uniform section

In this case $I=I_c$ and the general expressions (7) and (3) for reactions X and Y, and (8), (9), (10) for moments M_0, M, M_c reduce to

$$\frac{X}{qr}=\frac{\mathrm{s}_0(3\phi_0-2\phi_0\mathrm{s}_0^2-3\mathrm{c}_0\mathrm{s}_0)}{6(\phi_0^2+\phi_0\mathrm{c}_0\mathrm{s}_0-2\mathrm{s}_0^2)},\qquad \frac{Y}{qr}=\mathrm{s}_0 \tag{17),(18}$$

$$\frac{M_0}{qr^2}=\frac{3\phi_0(2\mathrm{s}_0^2-1)+2\phi_0\mathrm{c}_0\mathrm{s}_0(3+\mathrm{s}_0^2)-3\mathrm{s}_0^2-5\mathrm{s}_0^4}{12(\phi_0^2+\phi_0\mathrm{c}_0\mathrm{s}_0-2\mathrm{s}_0^2)} \tag{19}$$

$$\frac{M}{qr^2}=\frac{X}{qr}(\mathrm{c}-\mathrm{c}_0)+\frac{M_0}{qr^2}-\frac{1}{2}(\mathrm{s}_0^2-\mathrm{s}^2)$$

$$=\frac{2(3\phi_0-2\phi_0\mathrm{s}_0^2-3\mathrm{c}_0\mathrm{s}_0)\mathrm{s}_0\mathrm{c}+6(\phi_0^2+\phi_0\mathrm{c}_0\mathrm{s}_0-2\mathrm{s}_0^2)\mathrm{s}^2-(3\phi_0^2-3\mathrm{s}_0^2-\mathrm{s}_0^4)}{12(\phi_0^2+\phi_0\mathrm{c}_0\mathrm{s}_0-2\mathrm{s}_0^2)} \tag{20}$$

$$\frac{M_c}{qr^2}=\frac{-3\phi_0^2+2\phi_0\mathrm{s}_0(3-2\mathrm{s}_0^2)+\mathrm{s}_0^2(3-6\mathrm{c}_0+\mathrm{s}_0^2)}{12(\phi_0^2+\phi_0\mathrm{c}_0\mathrm{s}_0-2\mathrm{s}_0^2)} \tag{21}$$

and (12), (13) for rotation $\delta\phi$ and (14), (15) for displacement components δx, δy reduce to

$$\frac{EI_c\delta\phi}{qr^3}=\phi\left(-\frac{Xc_0}{qr}+\frac{M_0}{qr^2}+\frac{1}{4}-\frac{s_0^2}{2}\right)+\frac{Xs}{qr}-\frac{1}{4}cs$$

$$=\frac{\left\{\begin{array}{l}\phi(3\phi_0c_0s_0-3s_0^2+s_0^4)+2s_0s(3\phi_0-2\phi_0s_0^2-3c_0s_0)\\ -cs(3\phi_0^2+3\phi_0c_0s_0-6s_0^2)\end{array}\right\}}{12(\phi_0^2+\phi_0s_0c_0-2s_0^2)} \tag{22}$$

$$\frac{EI_c\delta x}{qr^4}=\frac{EI_c\delta\phi\cdot c}{qr^3}-\frac{X}{2qr}(\phi_0+\phi-s_0c_0-2c_0s+cs)$$

$$-\frac{M_0}{qr^2}(s_0+s)-\frac{1}{6}(s_0+s)^2(s-2s_0) \tag{23}$$

$$\frac{EI_c\delta y}{qr^4}=\frac{EI_c\delta\phi\cdot s}{qr^3}+(c-c_0)^2\left[\frac{X}{2qr}+\frac{M_0}{(c-c_0)qr^2}-\frac{c+2c_0}{6}\right] \tag{24}$$

The vertical depression at the point C is from (16)

$$\frac{EI_c}{qr^4}(\delta y)_{\phi=0}=(1-c_0)^2\left[\frac{X}{2qr}+\frac{M_0}{(1-c_0)qr^2}-\frac{1+2c_0}{6}\right]$$

$$=\frac{(1-c_0)\left[\phi_0^2(1-2c_0-2c_0^2)+\phi_0s_0(1+5c_0)-s_0^2(4-c_0)\right]}{12(\phi_0^2+\phi_0c_0s_0-2s_0^2)} \tag{25}$$

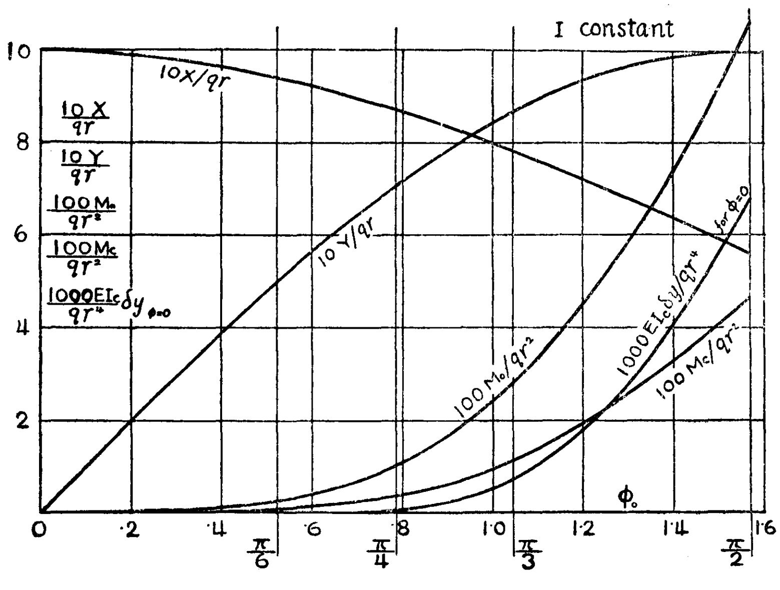

Fig. 4.

Fig. 4 shows the graphs of eqs. (17), (18), (19), (21) and (25) plotted against the central angle ϕ_0 of the beam.

VI Curved beam whose moment of inertia of section is expressed by $I=I_e\sqrt{1-k^2\sin^2\phi}$ (26)

In this case, the expressions (7) and (3) for reactions X and Y, and (8), (9), (10) for moments M_0, M, M_c reduce to, with $j=1$ in (1)

$$\frac{X}{qr}=\frac{\sin^{-1}(ks_0)[F(\phi_0-2E(\phi_0)]+ks_0\Delta\phi_0F(\phi_0)}{4k\{F(\phi_0)E(\phi_0)-k'^2F^2(\phi_0)-[\sin^{-1}(ks_0)]^2\}} \tag{27}$$

$$\frac{Y}{qr}=s_0 \tag{28}$$

$$\frac{M_0}{qr^2}=\frac{X}{qr}\Big[c_0-\frac{\sin^{-1}(ks_0)}{kF(\phi_0)}\Big]+\frac{1}{2k}\Big[\frac{E(\phi_0)}{F(\phi_0)}-(\Delta\phi_0)^2\Big] \tag{29}$$

$$\frac{M}{qr^2}=\frac{X}{qr}\,(c-c_0)+\frac{M_0}{qr^2}-\frac{1}{2}(s_0^2-s^2) \tag{30}$$

$$\frac{M_c}{qr^2}=\frac{X}{qr}(1-c_0)+\frac{M_0}{qr^2}-\frac{1}{2}s_0^2 \tag{31}$$

The rotation $\delta\phi$ is from (12)

$$\frac{EI_e}{qr^3}\delta\phi=\frac{X}{kqr}\Big[\sin^{-1}(ks)-kc_0F(\phi)\Big]-\frac{M_0}{qr^2}F(\phi)+\frac{\Delta\phi_0F(\phi)-E(\phi)}{2k^2} \tag{32}$$

and this becomes zero at $\phi=0$ and $\phi=\pm\phi_0$.

The displacement components δx and δy are from (14) and (15)

$$\frac{EI_e\delta x}{qr^4}=\left\{\begin{aligned}&\Big[\frac{X(c+c_0)}{kqr}-\frac{M_0}{kqr^2}+\frac{1-2(\Delta\phi_0)^2}{4k^3}\Big]\Big[\sin^{-1}(ks_0)+\sin^{-1}(ks)\Big]\\&+\Big[\frac{X}{qr}\Big(\frac{k'^2}{k^2}-c_0c\Big)+\frac{M_0c}{qr^2}+\frac{(\Delta\phi_0)^2c}{2k^2}\Big]\Big[F(\phi_0)+F(\phi)\Big]\\&-\Big[\frac{X}{k^2qr}+\frac{c}{2k^2}\Big]\Big[E(\phi_0)+E(\phi)\Big]+\frac{s_0\Delta\phi_0+s\Delta\phi}{4k^2}\end{aligned}\right.$$

$$=\left\{\begin{aligned}&-\frac{X(1-c)}{kqr}\sin^{-1}(ks_0)+\Big[\frac{X}{kqr}(c+c_0)-\frac{M_0}{kqr^2}+\frac{2k^2s_0^2-1}{4k^3}\Big]\sin^{-1}(ks)\\&+\Big[\frac{Xc_0}{qr}-\frac{(\Delta\phi_0)^2}{2k^2}\Big](1-c)F(\phi_0)\\&+\Big[\frac{X}{qr}\Big(\frac{k'^2}{k^2}-c_0c\Big)+\frac{M_0c}{qr^2}+\frac{c(\Delta\phi)^2}{2k^2}\Big]F(\phi)\\&+\frac{1-c}{2k^2}E(\phi_0)-\Big(\frac{X}{k^2qr}+\frac{c}{2k^2}\Big)E(\phi)+\frac{s\Delta\phi}{4k^2}\end{aligned}\right. \tag{33}$$

and $\delta x=0$ at $\phi=0$ & $\phi=\pm\phi_0$.

$$\frac{EI_e\delta y}{qr^4}=\left\{\begin{aligned}&\Big[\frac{Xc_0}{kqr}-\frac{M_0}{kqr^2}+\frac{k'^2-2(\Delta\phi_0)^2}{4k^3}\Big]\Big[\mathrm{sh}^{-1}\frac{kc_0}{k'}-\mathrm{sh}^{-1}\frac{kc}{k'}\Big]\\&+\frac{Xs}{kqr}\Big[\sin^{-1}(ks_0)+\sin^{-1}(ks)\Big]\\&-\Big[\frac{Xc_0}{qr}-\frac{M_0}{qr^2}-\frac{(\Delta\phi_0)^2}{2k^2}\Big]s\,[F(\phi_0)+F(\phi)]\\&-\frac{s}{2k^2}[E(\phi_0)+E(\phi)]+\frac{X(\Delta\phi-\Delta\phi_0)}{k^2qr}-\frac{c\Delta\phi-c_0\Delta\phi_0}{4k^2}\end{aligned}\right. \tag{34}$$

The depression at the center C is from (16)

$$\frac{EI_c}{qr^4}(\delta y)_{\phi=0}=\left[\frac{Xc_0}{kqr}-\frac{M_0}{kqr^2}+\frac{k'^2-2(\Delta\phi_0)^2}{4k^3}\right]\left(\mathrm{sh}^{-1}\frac{kc_0}{k'}-\mathrm{sh}^{-1}\frac{k}{k'}\right)$$
$$+\frac{X(1-\Delta\phi_0)}{k^2qr}-\frac{1-c_0\Delta\phi_0}{4k^2} \tag{35}$$

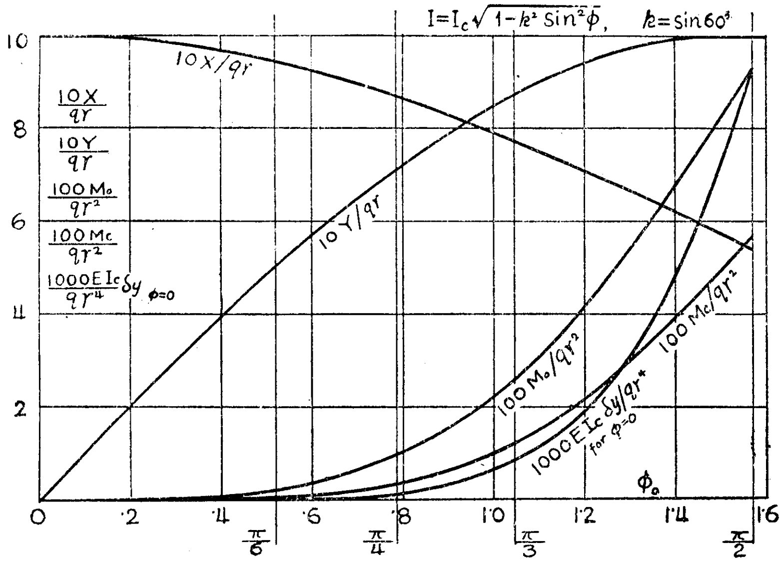

Fig. 5.

Fig. 5 shows the graphs of eqs. (27), (28), (29), (31) and (35) plotted against the central angle ϕ_0 with $k=\sin(\pi/3)$

VII Curved beam whose moment of inertia of section is expressed by $I=I_c/\sqrt{1-k^2\sin^2\phi}$ (36)

In this case, the expressions (7) and (3) for reactions X and Y, and (8), (9), (10) for moments M_0, M, M_c reduce to, with $j=-1$ in (1)

$$\frac{X}{qr}=\frac{\left\{\begin{array}{l}E(\phi_0)\left[(7-8k^2)\sin^{-1}(ks_0)+ks_0\Delta\phi_0(1-8k^2+6k^2s_0^2)\right]\\ -4[\sin^{-1}(ks_0)+ks_0\Delta\phi_0][k'^2F(\phi_0)-k^2s_0c_0\Delta\phi_0]\end{array}\right\}}{16kE(\phi_0)[k'^2F(\phi_0)-(1+k^2)E(\phi_0)-k^2s_0c_0\Delta\phi_0]+12k[\sin^{-1}(ks_0)+ks_0\Delta\phi_0]^2} \tag{37}$$

$$\frac{Y}{qr}=s_0 \tag{38}$$

$$\frac{M_0}{qr^2}=\frac{X}{qr}\left[c_0-\frac{ks_0\Delta\phi_0+\sin^{-1}(ks_0)}{2kE(\phi_0)}\right]-\frac{k'^2F(\phi_0)+(2k^2-1-3k^2s_0^2)E(\phi_0)-k^2s_0c_0\Delta\phi_0}{6k^2E(\phi_0)} \tag{39}$$

$$\frac{M}{qr^2}=\frac{X}{qr}(c-c_0)+\frac{M_0}{qr^2}-\frac{1}{2}(s_0^2-s^2) \tag{40}$$

$$\frac{M_c}{qr^2}=\frac{X}{qr}(1-c_0)+\frac{M_0}{qr^2}-\frac{1}{2}s_0^2 \tag{41}$$

The rotation $\delta\phi$ is, from (12), (13)

$$\frac{EI_c}{qr^3}\delta\phi=\begin{aligned}[t]&\frac{X}{2kqr}\Big[\sin^{-1}(ks_0)+\sin^{-1}(ks)\Big]+\frac{k'^2}{k^2}\Big[F(\phi_0)+F(\phi)\Big]\\&-\Big(\frac{Xc_0}{qr}-\frac{M_0}{qr^2}-\frac{1}{3}+\frac{1}{6k^2}+\frac{1}{2}s_0^2\Big)\Big[E(\phi_0)+E(\phi)\Big]\\&+\frac{X}{2qr}(s_0\Delta\phi_0+s\Delta\phi)-\frac{1}{6}(s_0c_0\Delta\phi_0+sc\Delta\phi)\end{aligned}$$

$$=\left.\begin{aligned}&\frac{X}{qr}\Big[\frac{1}{2}s\Delta\phi+\frac{1}{2k}\sin^{-1}(ks)-c_0E(\phi)\Big]+\frac{M_0}{qr^2}E(\phi)\\&+\Big(\frac{2k^2-1}{6k^2}-\frac{s_0^2}{2}\Big)E(\phi)+\frac{k'^2}{6k^2}F(\phi)-\frac{1}{6}sc\Delta\phi\end{aligned}\right\}\tag{42}$$

and the displacement components δx and δy become

$$\frac{EI_c}{qr^4}\delta x=\left.\begin{aligned}&+\frac{X}{qr}\left\{\begin{aligned}&\Big[\frac{X}{qr}(c+c_0)-\frac{M_0}{qr^2}-\frac{1}{8k^2}+\frac{s_0^2}{2}\Big]\frac{\sin^{-1}(ks_0)+\sin^{-1}(ks)}{2k}\\&+\frac{k'^2}{3k^2}[F(\phi_0)+F(\phi)]-\Big(c_0c+\frac{1+k^2}{3k^2}\Big)[(E\phi_0)+E(\phi)]\\&+\frac{1}{2}(c+c_0)(s_0\Delta\phi_0+s\Delta\phi)-\frac{1}{3}(s_0c_0\Delta\phi_0+sc\Delta\phi)\end{aligned}\right.\\&+\frac{M_0}{qr^2}\Big\{c[E(\phi_0)+E(\phi)]-\frac{1}{2}(s_0\Delta\phi_0+s\Delta\phi)\Big\}\\&+\frac{k'^2c}{6k^2}[F(\phi_0)+F(\phi)]+\Big(\frac{1}{3}-\frac{1}{6k^2}-\frac{s_0^2}{2}\Big)c[E(\phi_0)+E(\phi)]\\&-\frac{c}{6}(s_0c_0\Delta\phi_0+sc\Delta\phi)-\frac{1}{8}(s_0^3\Delta\phi_0+s^3\Delta\phi)\\&+\Big(\frac{1}{4}s_0^2+\frac{1}{16k^2}\Big)(s_0\Delta\phi_0+s\Delta\phi)\end{aligned}\right\}\tag{43}$$

The above (42) and (43) become zero at $\phi=0$ and $\phi=\pm\phi_0$.

$$\frac{EI_c}{qr^4}\delta y=\begin{aligned}[t]&\Big(\frac{Xc_0}{qr}-\frac{M_0}{qr^2}-\frac{k'^2}{8k^2}-\frac{c_0^2}{2}\Big)\frac{k'^2}{2k}\Big(\mathrm{sh}^{-1}\frac{kc_0}{k'}-\mathrm{sh}^{-1}\frac{kc}{k'}\Big)\\&+\frac{X}{qr}\Big[\frac{(\Delta\phi)^3-(\Delta\phi_0)^3}{3k^2}-\frac{c_0}{2}(c\Delta\phi-c_0\Delta\phi_0)+\frac{s^2\Delta\phi}{2}+\frac{s}{2k}\sin^{-1}(ks)-c_0sE(\phi)\Big]\\&+\frac{M_0}{qr^2}\Big[sE(\phi)+\frac{1}{2}(c\Delta\phi-c_0\Delta\phi_0)\Big]\\&+\Big(\frac{2k^2-1}{6k^2}-\frac{s_0^2}{2}\Big)sE(\phi)+\frac{k'^2}{6k^2}sF(\phi)-\frac{1}{6}s^2c\Delta\phi\\&+\frac{1}{8}c\Delta\phi\Big(2c_0^2+s^2-\frac{1+k^2}{2k^2}\Big)-\frac{1}{8}c_0\Delta\phi_0\Big(c_0^2-\frac{k'^2}{2k^2}\Big)\end{aligned}\tag{44}$$

This becomes zero at the ends, $\phi=\pm\phi_0$. The vertical depression at the center C, $\phi=0$, becomes

$$\frac{EI_c}{qr^4}(\delta y)_{\phi=0}=\left|\begin{array}{l}\left(\frac{Xc_0}{qr}-\frac{M_0}{qr^2}-\frac{k'^2}{8k^2}-\frac{c_0^2}{2}\right)\frac{k'^2}{2k}\left(\text{sh}^{-1}\frac{kc_0}{k'}-\text{sh}^{-1}\frac{k}{k'}\right)\\ +\frac{X}{qr}\left[\frac{1-(\Delta\phi_0)^3}{3k^2}-\frac{c_0}{2}(1-c_0\Delta\phi_0)\right]+\frac{M_0}{2qr^2}(1-c_0\Delta\phi_0)\\ +\frac{1}{4}c_0^2-\frac{1+k^2}{16k^2}-\frac{1}{8}c_0\Delta\phi_0\left(c_0^2-\frac{k'^2}{2k^2}\right)\end{array}\right| \tag{45}$$

VIII Curved beam whose moment of inertia of section is expressed by $I=I_c(1-k^2\sin^2\phi)$ (46)

In this case, the exprssions (7) and (3) for reactions X and Y, and (8), (9), (10) for moments M_0, M, M_c reduce to, with $j=2$ in (1)

$$\frac{X}{qr}=\frac{1}{2k}\cdot\frac{k'\phi_0\text{th}^{-1}(ks_0)-ks_0\tan^{-1}(k't_0)}{k'[\text{th}^{-1}(ks_0)]^2-\tan^{-1}(k't_0)[\phi_0-k'\tan^{-1}(k't_0)]} \tag{47}$$

$$\frac{Y}{qr}=s_0 \tag{48}$$

$$\frac{M_0}{qr^2}=-\frac{(\Delta\phi_0)^2}{2k^2}+\frac{kk'(s_0+\phi_0c_0)\text{th}^{-1}(ks_0)+(k'^2\phi_0-k^2s_0c_0)\tan^{-1}(k't_0)-k'\phi_0^2}{2k^2\{k'[\text{th}^{-1}(ks_0)]^2-\tan^{-1}(k't_0)[\phi_0-k'\tan^{-1}(k't_0)]\}} \tag{49}$$

$$\frac{M}{qr^2}=-\frac{(\Delta\phi)^2}{2k^2}+\frac{kk'(s_0+\phi_0c)\text{th}^{-1}(ks_0)+(k'^2\phi_0-k^2s_0c)\tan^{-1}(k't_0)-k'\phi_0^2}{2k^2\{k'[\text{th}^{-1}(ks_0)]^2-\tan^{-1}(k't_0)[\phi_0-k'\tan^{-1}(k't_0)]\}} \tag{50}$$

$$\frac{M_c}{qr^2}=-\frac{1}{2k^2}+\frac{kk'(s_0+\phi_0)\text{th}^{-1}(ks_0)+(k'^2\phi_0-k^2s_0)\tan^{-1}(k't_0)-k'\phi_0^2}{2k^2\{k'[\text{th}^{-1}(ks_0)]^2-\tan^{-1}(k't_0)[\phi_0-k'\tan^{-1}(k't_0)]\}} \tag{51}$$

The rotation is

$$\frac{EI_c}{qr^3}\delta\phi=\frac{\left\{\begin{array}{l}k'\text{th}^{-1}(ks_0)[\phi_0\text{th}^{-1}(ks)-\phi\,\text{th}^{-1}(ks_0)]\\ +[k'\tan^{-1}(k't_0)-\phi_0][\phi_0\tan^{-1}(k't)-\phi\tan^{-1}(k't_0)]\\ +ks_0[\text{th}^{-1}(ks_0)\tan^{-1}(k't)-\text{th}^{-1}(ks)\tan^{-1}(k't_0)]\end{array}\right.}{2k^2\{k'[\text{th}^{-1}(ks_0)]^2-\tan^{-1}(k't_0)[\phi_0-k'\tan^{-1}(k't_0)]\}} \tag{52}$$

The displacement components δx, δy are

$$\frac{EI_c}{qr^4}\delta x=\frac{\left\{\begin{array}{l}k'[\phi_0+kc\cdot\text{th}^{-1}(ks_0)][\phi_0\text{th}^{-1}(ks)-\phi\,\text{th}^{-1}(ks_0)]\\ +kc[-\phi_0+k'\tan^{-1}(k't_0)][\phi_0\tan^{-1}(k't)-\phi\tan^{-1}(k't_0)]\\ +k^2(\phi_0+s_0c)[\text{th}^{-1}(ks_0)\tan^{-1}(k't)-\text{th}^{-1}(ks)\tan^{-1}(k't_0)]\\ -k\tan^{-1}(k't_0)(\phi_0 s-\phi s_0)\\ -kk'\text{th}^{-1}(ks_0)[s_0\text{th}^{-1}(ks)-s\,\text{th}^{-1}(ks_0)]\\ -kk'\tan^{-1}(k't_0)[s_0\tan^{-1}(k't)-s\tan^{-1}(k't_0)]\end{array}\right.}{2k^3\{k'[\text{th}^{-1}(ks_0)]^2-\tan^{-1}(k't_0)[\phi_0-k'\tan^{-1}(k't_0)]\}} \tag{53}$$

$$\frac{EI_c}{qr^4}\delta y=-\frac{c-c_0}{2k^2}+$$

$$+\frac{\begin{Bmatrix}[\phi_0^2-ks_0\mathrm{th}^{-1}(ks_0)-k'\phi_0\tan^{-1}(k't_0)][\tan^{-1}(kc_0/k')-\tan^{-1}(kc/k')]\\+[-k'\phi_0\mathrm{th}^{-1}(ks_0)+ks_0\tan^{-1}(k't_0)](\lg\Delta\phi_0-\lg\Delta\phi)\\+kk'\mathrm{th}^{-1}(ks_0)s[\phi_0\mathrm{th}^{-1}(ks)-\phi\mathrm{th}^{-1}(ks_0)]\\+k^2s_0s[\mathrm{th}^{-1}(ks_0)\tan^{-1}(k't)-\mathrm{th}^{-1}(ks)\tan^{-1}(k't_0)]\\+k(k'-\phi_0)s[\phi_0\tan^{-1}(k't)-\phi\tan^{-1}(k't_0)]\end{Bmatrix}}{2k^3\{k'[\mathrm{th}^{-1}(ks_0)]^2-\tan^{-1}(k't_0)[\phi_0-k'\tan^{-1}(k't_0)]\}} \tag{54}$$

The depression at the point C is

$$\frac{EI_c}{qr^4}(\delta y)_{\phi=0}=-\frac{1-c_0}{2k^2}+$$

$$+\frac{\begin{Bmatrix}[\phi_0^2-ks_0\mathrm{th}^{-1}(ks_0)-k'\phi_0\tan^{-1}(k't_0)][\tan^{-1}(kc_0/k')-\tan^{-1}(k/k')]\\+[-k'\phi_0\mathrm{th}^{-1}(ks_0)+ks_0\tan^{-1}(k't_0)]\lg\Delta\phi_0\end{Bmatrix}}{2k^3\{k'[\mathrm{th}^{-1}(ks_0)]^2-\tan^{-1}(k't_0)[\phi_0-k'\tan^{-1}(k't_0)]\}} \tag{55}$$

IX Curved beam whose moment of inertia of section is expressed by $I=I_c/(1-k^2\sin^2\phi)$ (56)

In this case, the expressions (7) and (3) for reactions X and Y, and (8), (9), (10) for moments M_0, M, M_c reduce to, with $j=-2$ in (1)

$$\frac{X}{qr}=\frac{\begin{Bmatrix}\phi_0s_0\left[-2+\frac{7}{10}k^2+\frac{2}{5}k^4+(\frac{4}{3}-\frac{1}{2}k^4)s_0^2\right.\\\left.+s_0^2c_0\left[2-\frac{3}{2}k^2-k^2(\frac{7}{3}-\frac{1}{2}k^2)s_0^2+\frac{11}{15}k^4s_0^4\right]\right]\end{Bmatrix}}{\begin{Bmatrix}-\frac{1}{2}(2-k^2)(4-k^2)\phi_0^2+\phi_0s_0c_0[-4+3k^2+k^2(2-k^2)s_0^2]\\+2s_0^2\left[4+k^2+\frac{1}{4}k^4-k^2s_0^2(\frac{11}{3}+\frac{3}{4}k^2)+\frac{13}{9}k^4s_0^4-\frac{1}{2}k^4s_0^6\right]\end{Bmatrix}} \tag{57}$$

$$\frac{Y}{qr}=s_0 \tag{58}$$

$$\frac{M_0}{qr^2}=\frac{X}{qr}\left[c_0-\frac{2s_0-\frac{2}{3}k^2s_0^3}{(2-k^2)\phi_0-k^2s_0c_0}\right]+\frac{1}{2}s_0^2-\frac{(1-\frac{3}{4}k^2)(\phi_0-s_0c_0)+\frac{1}{2}k^2s_0^3c_0}{2(2-k^2)\phi_0-2k^2s_0c_0} \tag{59}$$

$$\frac{M}{qr^2}=\frac{X}{qr}\left[c-\frac{2s_0-\frac{2}{3}k^2s_0^3}{(2-k^2)\phi_0-k^2s_0c_0}\right]+\frac{1}{2}s^2-\frac{(1-\frac{3}{4}k^2)(\phi_0-s_0c_0)+\frac{1}{2}k^2s_0^3c_0}{2(2-k^2)\phi_0-2k^2s_0c_0} \tag{60}$$

$$\frac{M_c}{qr^2}=\frac{X}{qr}\left[1-\frac{2s_0-\frac{2}{3}k^2s_0^3}{(2-k^2)\phi_0-k^2s_0c_0}\right]-\frac{(1-\frac{3}{4}k^2)(\phi_0-s_0c_0)+\frac{1}{2}k^2s_0^3c_0}{2(2-k^2)\phi_0-2k^2s_0c_0} \tag{61}$$

The rotation is

$$\frac{EI_c}{qr^3}\delta\phi=\left\{\begin{array}{l}\frac{X}{qr}(s-\frac{1}{3}k^2s^3)+\frac{1}{4}(1-\frac{3}{4}k^2)(\phi-sc)+\frac{1}{8}k^2s^2c\\ -\left[\frac{X}{qr}(s_0-\frac{1}{3}k^2s_0^3)+\frac{1}{8}k^2s_0^3c_0+\frac{1}{4}(1-\frac{3}{4}k^2)(\phi_0-s_0c_0)\right]\frac{(1-\frac{1}{2}k^2)\phi+\frac{1}{2}k^2sc}{(1-\frac{1}{2}k^2)\phi_0+\frac{1}{2}k^2s_0c_0}\end{array}\right\} \tag{62}$$

The displacement components δx and δy are

$$\frac{EI_c}{qr^4}\delta x=\left\{\begin{array}{l}\frac{EI_cc}{qr^3}\delta\phi-\frac{M_0}{qr^2}\left[s+s_0-\frac{1}{3}k^2(s^3+s_0^3)\right]\\ -\frac{X}{qr}\left[\frac{1}{2}(\phi+\phi_0)(1-\frac{1}{4}k^2)-\frac{1}{2}s_0c_0-sc_0+\frac{1}{2}sc+\frac{1}{8}k^2(sc\cos2\phi-s_0c_0\cos2\phi_0)+\frac{1}{3}k^2c_0(s^3+s_0^3)\right]\\ +\frac{1}{6}(s+s_0)^2(2s_0-s)-\frac{1}{6}k^2s_0^2(s^3+s_0^3)+\frac{1}{10}k^2(s^5+s_0^5)\end{array}\right\} \tag{63}$$

$$\frac{EI_c}{qr^4}\delta y=\left\{\begin{array}{l}\frac{EI_cs}{qr^3}\delta\phi+\frac{M_0}{qr^2}(c-c_0)\left[1-\frac{1}{3}k^2(3-c^2-cc_0-c_0^2)\right]\\ +\frac{X}{2qr}(c-c_0)^2\left[1-\frac{1}{6}k^2(6-3c^2-2cc_0-c_0^2)\right]\\ -\frac{1}{6}(c-c_0)^2(c-2c_0)\\ +\frac{1}{30}k^2(c-c_0)^2\left[(c+2c_0)(5-3c^2)-2c_0^2(2c+c_0)\right]\end{array}\right\} \tag{64}$$

The depression at the point C is

$$\frac{EI_c}{qr^4}(\delta y)_{\phi=0}=\left\{\begin{array}{l}\frac{M_0}{qr^2}(1-c_0)\left[1-\frac{1}{3}k^2(1-c_0)(2+c_0)\right]\\ +\frac{X}{2qr}(1-c_0)^2\left[1-\frac{1}{6}k^2(1-c_0)(3+c_0)\right]\\ -\frac{1}{6}(1-c_0)^2(1+2c_0)+\frac{k^2}{15}(1-c_0)^3(1+3c_0+c_0^2)\end{array}\right\} \tag{65}$$

(To be continued)