Title	Thermal stress in a spheroid with steady axisymmetric distribution of temperature
Sub Title	
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Publisher	慶應義塾大学藤原記念工学部
Publication year	1953
Jtitle	Proceedings of the Fujihara Memorial Faculty of Engineering Keio University Vol.6, No.20 (1953.) ,p.10(10)- 26(26)
JaLC DOI	
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Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00060020-0010

Thermal Stress in a Spheroid with Steady Axisymmetric Distribution of Temperature*

(Received September 2, 1953)

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Abstract

The general exact solution is given for the thermal stress in a spheroid with steady axisymmetric distribution of temperature, in closed form so far as the temperature is expressed in finite terms.

As an application of the preceding analysis, numerical calculations of the three dimensional thermal stress in a turbine disc of the form of an oblate spheroid are carried out for several cases.

I. Nomenclature

The following nomenclature is used in the paper ;

(u, v, w) = Cartesian components of displacement

e = dilatation

F = thermo displacement potential

φ, λ = displacement potential

(α, β, γ) = spheroidal co-ordinates

$\left.\begin{matrix}(u_\alpha, u_\beta, u_\gamma) \\ (\sigma_\alpha, \cdots, \tau_{\alpha\beta})\end{matrix}\right\}$ = curvilinear components of displacement and stress, respectively

$\left.\begin{matrix}q=\cosh\alpha, \bar{q}=\sinh\alpha \\ p=\cos\beta, \bar{p}=\sin\beta\end{matrix}\right\}$ = auxiliary position paramenters for prolate spheroidal co-ordinates

$\left.\begin{matrix}h=h_1=h_2=1/\sqrt{q^2-p^2} \\ h_3=\dfrac{1}{\bar{q}\bar{p}}\end{matrix}\right\}$ = local scale coefficients for prolate spheroidal co-ordinates

$q_0, \bar{q}_0$ = values of q and $\bar{q}$ at $\alpha = \alpha_0$

$s=\dfrac{q_0}{\bar{q}_0}$ = shape ratio for prolate spheroid

Δ = Laplacian Operator

G, ν = shear modulus and Poisson's ratio

ε = linear thermal expansion coefficient

T = temperature distribution

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* Read at the 590th Congress of the Japan Society of Mechanical Engineers, June, 24, 1953

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II. Introduction

Thermal stress in a turbine disc is an important problem for designers and many investigations have been made. These papers stand on the assumption that the thickness of the disc is so small compared with its diameter that stress for one circular section of the disc will hold for any such section. In many cases, however, the foregoing assumption can not be applied and three dimensional aspects of the thermal stress must be studied.

In this paper, the author assumed the disc to be an oblate spheroid composed of homogeneous isotropic material and obtained a general exact solution for the thermal stress due to a steady axisymmetric distribution of temperature, in closed form as far as the temperature distribution is given in finite terms.

For convenience, the analysis are carried out at first for a prolate spheroid and then the results so obtained are transformed into an oblate spheroid.

III. Stress Distribution in a Prolate Spheroid with Steady Axisymmetric Distribution of Temperature

This problem is equal to determining a displacement field which satisfies the following thermo displacement equation

$$\left.\begin{aligned}&[\Delta u, \Delta v, \Delta w] + \frac{1}{1-2\nu}\,\mathrm{grad}\, e - \frac{2(1+\nu)}{1-2\nu}\,\mathrm{grad}\, \varepsilon T = 0\,,\\ &e = \mathrm{div}\,[u, v, w]\,,\end{aligned}\right\} \tag{1}$$

and is free from tractions on the surface of the spheroid.

The particular solution of eq. (1)[1] will be obtained by taking

$$[u, v, w] = \frac{1+\nu}{1-\nu}\,\varepsilon\,\mathrm{grad}\, F, \tag{2}$$

where F is thermo displacement potential and

$$\Delta F = T. \tag{3}$$

For the removal of the stress residuals on the surface due to the particular solution, the Boussinesq's Approach[2] will be used, which in the case of rotationaly symmetry about the Z axis represents the sum of the following two displacement fields referring to the cylindrical co-ordinates (ρ, γ, z);

$$\left.\begin{aligned}&[u_\rho, u_\gamma, u_z] = \mathrm{grad}\,\phi,\\ &[u_\rho, u_\gamma, u_z] = z\,\mathrm{grad}\,\lambda - [0, 0, (3-4\nu)\lambda],\end{aligned}\right\} \tag{4}$$

where $\quad \Delta\phi(\rho, z) = \Delta\lambda(\rho, z) = 0\,.$

These solutions will be referred to as basic solutions 1 and 2 respectively.

The prolate spheroidal co-ordinates system is defined by the equations of transformation

1) Timoshenko and Goodier; Theory of Elasticity. (McGraw-Hill 1951)

2) The Boussinesq's Approach was originated by Boussinesq. J. Boussinesq; Applications des Potentials (Gauthier-Villars, Paris, France, 1885)

$$\left.\begin{aligned} x &= c\sinh\alpha\sin\beta\cos\gamma, \\ y &= c\sinh\alpha\sin\beta\sin\gamma, \quad c = 1, \\ z &= c\cosh\alpha\cos\beta. \end{aligned}\right\} \tag{5}$$

The surfaces $\alpha =$ const. $\beta =$ const. and $\gamma =$ const. form a triply orthogonal family of prolate spheroids, hyperboloids of two sheets and meridional half planes.

For convenience, the following auxiliary variables are introduced

$$\left.\begin{aligned} q &= \cosh\alpha, & \bar{q} &= \sinh\alpha = \sqrt{q^2-1}, \\ p &= \cos\beta, & \bar{p} &= \sin\beta = \sqrt{1-p^2}, \end{aligned}\right\} \tag{6}$$

where

$$\left.\begin{aligned} 1 &\leqslant q < \infty, & 0 &\leqslant \bar{q} < \infty, \\ -1 &\leqslant p \leqslant 1, & 0 &\leqslant \bar{p} \leqslant 1. \end{aligned}\right\} \tag{7}$$

The displacement and stress fields of the thermo displacement potential in the spheroidal co-ordinates are expressed as follows.[3)]

$$\left.\begin{aligned} u_\alpha &= \frac{1+\nu}{1-\nu}\varepsilon h\bar{q}F_q, \\ u_\beta &= -\frac{1+\nu}{1-\nu}\varepsilon h\bar{p}F_p. \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} \sigma_\alpha/2G &= \frac{1+\nu}{1-\nu}\varepsilon\,[\,h^2\bar{q}^2F_{qq} + h^4\bar{p}^2(qF_q - pF_p) - T\,], \\ \sigma_\beta/2G &= \frac{1+\nu}{1-\nu}\varepsilon\,[\,h^2\bar{p}^2F_{pp} + h^4\bar{q}^2(qF_q - pF_p) - T\,], \\ \sigma_\gamma/2G &= \frac{1+\nu}{1-\nu}\varepsilon\,[\,h^2(qF_q - pF_p) - T\,], \\ \tau_{\alpha\beta}/2G &= \frac{1+\nu}{1-\nu}\varepsilon\,[-h^2F_{pq} + h^4(qF_p - pF_q)]\,\bar{p}\bar{q}. \end{aligned}\right\} \tag{9}$$

Also, the displacement and stress fields of the Boussinesq's solution in the co-ordinates are expressed as follows.[4)]

$$\left.\begin{aligned} u_\alpha &= h\bar{q}\phi_q, \\ u_\beta &= -h\bar{p}\phi_p. \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} \sigma_\alpha/2G &= h^2\bar{q}^2\phi_{qq} + h^4\bar{p}^2(q\phi_q - p\phi_p), \\ \sigma_\beta/2G &= h^2\bar{p}^2\phi_{pp} + h^4\bar{q}^2(q\phi_q - p\phi_p), \\ \sigma_\gamma/2G &= h^2(q\phi_q - p\phi_p), \\ \tau_{\alpha\beta}/2G &= [-h^2\phi_{pq} + h^4(q\phi_p - p\phi_q)]\,\bar{p}\bar{q}. \end{aligned}\right\} \tag{11}$$

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3) Subscripts attached to functions which originally bear no subscript denote partial differentiation.

4) The equations from (10) to (13) are the rotationaly symmetry case of the formulation designated by Sadowsky and Sternberg. In this paper, their notations are adhered to. M. A. Sadowsky and E. Sternberg; Stress Concentration Around an Ellipsoidal Cavity in an Infinite Body Under Arbitrary Plane Stress Perpendicular to the Axis of the Cavity. J. of Appl. Mech. **69**, A-191, (1947)

$$\left.\begin{aligned} u_\alpha &= h[q\lambda_q - (3-4\nu)\lambda]\,p\bar{q}, \\ u_\beta &= h[-p\lambda_p + (3-4\nu)\lambda]\,\bar{p}q. \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} \sigma_\alpha/2G &= h^2(q\lambda_{qq} - 2\lambda_q)\,p\bar{q}^2 - 2\nu h^2(q\bar{p}^2\lambda_p - p\bar{q}^2\lambda_q) \\ &\quad + h^4(q\lambda_q - p\lambda_p)\,pq\bar{p}^2, \\ \sigma_\beta/2G &= h^2(p\lambda_{pp} - 2\lambda_p)\,q\bar{p}^2 + 2\nu h^2(q\bar{p}^2\lambda_p - p\bar{q}^2\lambda_q) \\ &\quad + h^4(q\lambda_q - p\lambda_p)\,pq\bar{q}^2, \\ \sigma_\gamma/2G &= (1-2\nu)\,h^2pq\,(q\lambda_q - p\lambda_p) - 2\nu h^2(q\lambda_p - p\lambda_q), \\ \tau_{\alpha\beta}/2G &= h^2[(1-2\nu)(p\lambda_p + q\lambda_q) - pq\lambda_{pq}]\,\bar{p}\bar{q} \\ &\quad + h^4 p\bar{p}q\bar{q}(q\lambda_p - p\lambda_q). \end{aligned}\right\} \tag{13}$$

Any steady axisymmetric distribution of temperature in the prolate spheroid is

$$T = \sum_0^\infty A_n P_n(p)\, P_n(q)\,, \tag{14}$$

where P_n is an ordinary Legendre function of the first kind. For the later convenience, eq. (14) is rearranged in the following form;

$$T = 2\sum_0^\infty B_n[(2n+3)\,P_n(p)\,P_n(q) + (2n-5)\,P_{n-2}(p)\,P_{n-2}(q)]\,, \tag{15}$$

where

$$\begin{aligned} B_n &= \frac{1}{2(2n+3)}[A_n - 2(2n-1)\,B_{n+2}] \\ &= \frac{2n-1}{2}\sum_{m=0}^\infty (-1)^m \frac{A_{n+2m}}{(2n+4m-1)(2n+4m+3)}\,. \end{aligned} \tag{16}$$

Making use of the eq. (3), the thermo displacement potential corresponding to the temperature distribution (15) is found to be

$$F = (q^2 + p^2)\sum_{n=0}^\infty \mathrm{B}_n[P_n(p)\,P_n(q) - P_{n-2}(p)\,P_{n-2}(q)]\,. \tag{17}$$

For simplicity, the thermal stress due to one term of eq. (15), namely

$$T_n \doteq 2B_n[(2n+3)\,P_n(p)\,P_n(q) + (2n-5)\,P_{n-2}(p)\,P_{n-2}(q)]$$

will be considered, but the generality remains undisturbed.

The complete solution, R_n, which represents the displacement vector and stress tensor fields will be written symbolically

$$R_n = R_{o,n} + \sum_{m=0}^{[\frac{n}{2}]} C_{n-2m+2}\; R_{1,n-2m+2} + \sum_{m=0}^{[\frac{n}{2}]} D_{n-2m+1}\; R_{2,n-2m+1}\;, \tag{18}$$

where R_{on} is the solution obtained from the thermo displacement potential F and R_{1n}, R_{2n} are the solutions obtained from the first and second solutions of Boussinesq's Approach respectively. The C, D are coefficients of superposition and $\left[\frac{n}{2}\right]$

designates the maximum integer in the bracket.

In the expression of $\sigma_\alpha/2G$ and $\tau_{\alpha\beta}/2G$ in eqs. (9), (11) and (13) there are terms multiplied by h^4 which are apparently undesirable for the determination of constants. These terms can be put into the forms multiplied by h^2 through the proper form of harmonic functions such as

$$\varphi = P_n(p)\,P_n(q) - P_{n-2}(p)\,P_{n-2}(q)\,. \tag{19}$$

Making use of the following recurrence relations

$$\left.\begin{aligned}(2n+1)\,P_n &= \frac{dP_{n+1}}{dp} - \frac{dP_{n-1}}{dp}\,,\\ (2n+1)\,p\,\frac{dP_n}{dp} &= n\,\frac{dP_{n+1}}{dp} + (n+1)\,\frac{dP_{n-1}}{dp}\,,\end{aligned}\right\} \tag{20}$$

and noting eq. (19), the terms multiplied by h^4 now become

$$\left.\begin{aligned}h^4\,(q\varphi_q - p\varphi_p) &= -h^2\,\frac{2n-1}{n(n-1)}\,\frac{dP_{n-1}}{dp}\,\frac{dP_{n-1}}{dq}\,,\\ h^4\,(q\varphi_p - p\varphi_q) &= h^2\left[\frac{1}{n}\,\frac{dP_n}{dp}\,\frac{dP_n}{dq} + \frac{1}{n-1}\,\frac{dP_{n-2}}{dp}\,\frac{dP_{n-2}}{dq}\right].\end{aligned}\right\} \tag{21}$$

After the repeated use of the following recurrence relations and eq. (20)

$$\left.\begin{aligned}&nP_n - (2n-1)\,p\,P_{n-1} + (n-1)\,P_{n-2} = 0\,,\\ &(p^2-1)\,\frac{dP_n}{dp} = n(pP_n - P_{n-1})\,.\end{aligned}\right\} \tag{22}$$

and by making use of the eq. (8) – (13) and (21), the displacement and stress fields can be computed. For brevity, the displacement fields are omitted and only the stress fields are recorded.

$$R_{0,n}\,;\qquad \left.\begin{aligned}F &= (q^2+p^2)\,B_n\,[P_n(p)\,P_n(q) - P_{n-2}(p)\,P_{n-2}(q)]\,,\\ T_n &= 2B_n\,[(2n+3)\,P_n(p)\,P_n(q) + (2n-5)\,P_{n-2}(p)\,P_{n-2}(q)]\,.\end{aligned}\right\} \tag{23}$$

$$\left.\begin{aligned}\sigma_\alpha/2G &= \frac{1+\nu}{1-\nu}\,\varepsilon B_n h^2 \sum_{m=0}^{3} K^n_{n+2-2m}(q)\,P_{n+2-2m}(p),\\ \sigma_\beta/2G &= -\frac{1+\nu}{1-\nu}\,\varepsilon B_n h^2 \sum_{m=0}^{3} K^n_{n+2-2m}(p)\,P_{n+2-2m}(q),\\ \sigma_\gamma/2G &= -\frac{1+\nu}{1-\nu}\,\varepsilon B_n\Big[4(n+1)P_n(p)\,P_n(q) + 4(n-2)P_{n-2}(p)P_{n-2}(q)\\ &\qquad + \frac{2n-1}{n(n-1)}\,(q^2+p^2)\,\frac{dP_{n-1}}{dp}\,\frac{dP_{n-1}}{dq}\Big]\,,\\ \tau_{\alpha\beta}/2G &= \frac{1+\nu}{1-\nu}\,\varepsilon B_n h^2 \sum_{m=0}^{3} U^n_{n+2-2m}(q)\,\frac{dP_{n+2-2m}}{dp}\,,\end{aligned}\right\} \tag{24}$$

where

$$K^n_{n+2}(x) = \frac{(n+1)(n+2)}{(2n+1)^2(2n+3)}\Big[\{n(n+3)+4\}\,\frac{dP_{n+1}}{dx} - \{n(n+5)+5\}\,\frac{dP_{n-1}}{dx}\Big]\,,$$

$$
\left.
\begin{aligned}
K^n_n(x) &= \frac{(n+1)(n+2)}{(2n+1)(2n+3)(2n+5)}\{n(n-1)-4\}\frac{dP_{n+3}}{dx} \\
&\quad + \Big[\frac{n(n+3)-2}{(2n+1)(2n-1)} + \frac{2\{n(n-1)-1\}\{2n(n+1)-1\}}{(2n-1)(2n+1)(2n+3)} \\
&\qquad - \frac{(n+1)(n+2)\{n(n+3)+6\}}{(2n+1)(2n+3)(2n+5)}\Big]\frac{dP_{n+1}}{dx} \\
&\quad + \Big[-\frac{2\{n(n+3)+1\}\{2n(n+1)-1\}}{(2n-1)(2n+1)(2n+3)} - \frac{2(n-1)n\{2n(n-2)+7\}}{(2n-3)^2(2n-1)(2n+1)} \\
&\qquad + \frac{4n(n-2)}{(2n-3)(2n+1)} - \frac{(n+1)(n+2)}{(2n+1)(2n+3)}\Big]\frac{dP_{n-1}}{dx} \\
&\quad + \frac{(n-1)n\{4n(2n+1)-13\}}{(2n-3)^2(2n-1)(2n+1)}\frac{dP_{n-3}}{dx}, \\
K^n_{n-2}(x) &= -\frac{(n-1)n\{4n(2n-5)-1\}}{(2n-3)(2n-1)(2n+1)^2}\frac{dP_{n+1}}{dx} + \Big[\frac{4(n-1)n(n+1)(n+2)}{(2n-3)(2n-1)(2n+1)^2} \\
&\quad - \frac{2\{n^2-5(n-1)\}\{2n(n-3)+3\}}{(2n-5)(2n-3)(2n-1)} - \frac{2(n-1)\{2n(n+1)+1\}}{(2n-1)(2n+1)^2} \\
&\quad - \frac{2(n-1)(n+1)}{(2n-3)(2n-1)} + \frac{(n-3)(n-2)}{(2n-5)(2n-3)} - \frac{8(n-1)n}{(2n-3)(2n+1)^2}\Big]\frac{dP_{n-1}}{dx} \\
&\quad + \Big[\frac{2\{2n(n-3)+3\}\{n(n-1)-1\}}{(2n-5)(2n-3)(2n-1)} - \frac{(n-3)(n-2)\{n^2-5(n-2)\}}{(2n-7)(2n-5)(2n-3)} \\
&\qquad - \frac{n(n-5)+2}{(2n-3)(2n-1)}\Big]\frac{dP_{n-3}}{dx} \\
&\quad - \frac{(n-3)(n-2)\{n(n-1)-4\}}{(2n-7)(2n-5)(2n-3)}\frac{dP_{n-5}}{dx}, \\
K^n_{n-4}(x) &= -\frac{(n-3)(n-2)}{(2n-5)(2n-3)^2}\Big[\{n(n-7)+11\}\frac{dP_{n-1}}{dx} \\
&\qquad - \{n(n-5)+8\}\frac{dP_{n-3}}{dx}\Big]. \\
U^n_{n+2}(x) &= -\frac{(n+1)^2}{(2n+1)(2n+3)}\frac{dP_n}{dx}, \\
U^n_n(x) &= -\frac{(n+1)^2}{(2n+1)(2n+3)}\frac{dP_{n+2}}{dx} - 2\Big[\frac{n(n+1)-1}{(2n+1)(2n+3)} \\
&\quad + \frac{n^2(n-1)+1}{n(2n-1)(2n+1)}\Big]\frac{dP_n}{dx} + \frac{8n(n-1)-5}{(2n-3)(2n-1)(2n+1)}\frac{dP_{n-2}}{dx}, \\
U^n_{n-2}(x) &= \frac{8n(n-1)-5}{(2n-3)(2n-1)(2n+1)}\frac{dP_n}{dx} + 2\Big[\frac{n(n-1)^2-1}{(2n-3)(2n-1)(n-1)} \\
&\quad + \frac{n(n-3)+1}{(2n-5)(2n-3)}\Big]\frac{dP_{n-2}}{dx} + \frac{(n-2)^2}{(2n-5)(2n-3)}\frac{dP_{n-4}}{dx}, \\
U^n_{n-4}(x) &= \frac{(n-2)^2}{(2n-5)(2n-3)}\frac{dP_{n-2}}{dx}.
\end{aligned}
\right\} \quad (25)
$$

$R_{1,n}$;

$$\varphi = P_n(p)\,P_n(q) - P_{n-2}(p)\,P_{n-2}(q). \quad (26)$$

$$
\left.
\begin{aligned}
\sigma_\alpha/2G &= h^2\,[L^n_n(q)\,P_n(p) + L^n_{n-2}(q)\,P_{n-2}(p)], \\
\sigma_\beta/2G &= -h^2\,[L^n_n(p)\,P_n(q) + L^n_{n-2}(p)\,P_{n-2}(q)],
\end{aligned}
\right\} \quad (27)
$$

$$\left.\begin{aligned}\sigma_\gamma/2G &= -\frac{2n-1}{(n-1)n}\frac{dP_{n-1}}{dp}\frac{dP_{n-1}}{dq}\\ \tau_{\alpha\beta}/2G &= \bar{p}\,\bar{q}\,h^2\Big[V^n_n(q)\frac{dP_n}{dp}+V^n_{n-2}(q)\frac{dP_{n-2}}{dp}\Big],\end{aligned}\right\}$$

where

$$\left.\begin{aligned}L^n_n(x) &= \frac{1}{2n+1}\Big[(n-1)n\frac{dP_{n+1}}{dx}-\{n(n+1)+1\}\frac{dP_{n-1}}{dx}\Big],\\ L^n_{n-2}(x) &= -\frac{1}{2n-3}\Big[\{(n-1)(n-2)+1\}\frac{dP_{n-1}}{dx}-(n-1)n\frac{dP_{n-3}}{dx}\Big],\\ V^n_n(x) &= -\frac{n-1}{n}\frac{dP_n}{dx},\qquad V^n_{n-2}(x)=\frac{n}{n-1}\frac{dP_{n-2}}{dx}.\end{aligned}\right\}\quad(28)$$

$R_{2,\,n-1}$;

$$\lambda = P_{n-1}(p)\,P_{n-1}(q) - P_{n-3}(p)\,P_{n-3}(q). \quad (29)$$

$$\left.\begin{aligned}\sigma_\alpha/2G &= h^2\sum_{m=0}^{2}[M^{n-1}_{n-2m}(q)-2\nu N^{n-1}_{n-2m}(q)]\,P_{n-2m}(p),\\ \sigma_\beta/2G &= -h^2\sum_{m=0}^{2}[M^{n-1}_{n-2m}(p)-2\nu N^{n-1}_{n-2m}(p)]\,P_{n-2m}(q),\\ \sigma_\gamma/2G &= -(1-2\nu)\frac{2n-3}{(n-2)(n-1)}\,pq\frac{dP_{n-2}}{dp}\frac{dP_{n-2}}{dq}\\ &\quad -2\nu\Big[\frac{1}{n-1}\frac{dP_{n-1}}{dp}\frac{dP_{n-1}}{dq}+\frac{1}{n-2}\frac{dP_{n-3}}{dp}\frac{dP_{n-3}}{dq}\Big],\\ \tau_{\alpha\beta}/2G &= \bar{p}\,\bar{q}\,h^2\sum_{m=0}^{2}[W^{n-1}_{n-2m}(q)+(1-2\nu)X^{n-1}_{n-2m}(q)]\frac{dP_{n-2m}}{dp},\end{aligned}\right\}\quad(30)$$

where

$$\left.\begin{aligned}M^{n-1}_n(x) &= \frac{(n-1)n^2(n-4)}{(2n-1)^2(2n+1)}\frac{dP_{n+1}}{dx}+\frac{n\{n(3n-4)+4\}}{(2n-3)(2n-1)(2n+1)}\frac{dP_{n-1}}{dx}\\ &\quad -\frac{(n-1)n}{(2n-3)(2n-1)^2}\{n(n+1)+1\}\frac{dP_{n-3}}{dx},\\ M^{n-1}_{n-2}(x) &= \frac{(n-1)^2n(n-4)}{(2n-1)^2(2n+1)}\frac{dP_{n+1}}{dx}+\Big[\frac{(n-1)\{n(n-1)+6\}}{(2n-3)(2n-1)(2n+1)}\\ &\quad -\frac{(n-3)(n-2)^2(n-6)}{(2n-5)^2(2n-3)}-\frac{n-2}{(2n-1)(2n-5)}\Big]\frac{dP_{n-1}}{dx}\\ &\quad -\Big[\frac{(n-1)^2n(n+3)}{(2n-3)(2n-1)^2}+\frac{(n-2)\{n(n-5)+12\}}{(2n-7)(2n-5)(2n-3)}\\ &\quad +\frac{n-1}{(2n-5)(2n-1)}\Big]\frac{dP_{n-3}}{dx}+\frac{(n-3)(n-2)^2(n+1)}{(2n-7)(2n-5)^2}\frac{dP_{n-5}}{dx},\\ M^{n-1}_{n-4}(x) &= -\frac{(n-3)(n-2)\{n(n-7)+13\}}{(2n-5)^2(2n-3)}\frac{dP_{n-1}}{dx}\\ &\quad -\frac{(n-3)\{n(3n-14)+19\}}{(2n-7)(2n-5)(2n-3)}\frac{dP_{n-3}}{dx}\\ &\quad +\frac{(n-3)^2(n-2)(n+1)}{(2n-7)(2n-5)^2}\frac{dP_{n-5}}{dx},\\ N^{n-1}_n(x) &= -\frac{(n-1)n}{(2n-1)^2}[2nP_n-P_{n-2}],\end{aligned}\right\}\quad(31)$$

$$
\left.\begin{aligned}
N^{n-1}_{n-2}(x) &= \frac{(n-1)n}{(2n-1)^2} P_n + 2\left[\frac{(n-1)^2 n}{(2n-1)^2} + \frac{(n-3)(n-2)^2}{(2n-5)^2}\right] P_{n-2} \\
&\qquad - \frac{(n-3)(n-2)}{(2n-5)^2} P_{n-4}\,, \\
N^{n-1}_{n-4}(x) &= -\frac{(n-3)(n-2)}{(2n-5)^2}\left[P_{n-2}+2(n-3)P_{n-4}\right], \\
W^{n-1}_{n}(x) &= -\frac{(n-2)}{(2n-1)^2}\left[(n-1)\frac{dP_n}{dx} + n\frac{dP_{n-2}}{dx}\right], \\
W^{n-1}_{n-2}(x) &= -\frac{(n-2)n}{(2n-1)^2}\frac{dP_n}{dx} - \left[\frac{(n-2)n^2}{(2n-1)^2(n-1)}\right. \\
&\qquad \left. - \frac{(n-1)(n-3)^2}{(2n-5)^2(n-2)}\right]\frac{dP_{n-2}}{dx} + \frac{(n-1)(n-3)}{(2n-5)^2}\frac{dP_{n-4}}{dx}\,, \\
W^{n-1}_{n-4}(x) &= \frac{n-1}{2n-5}\left[(n-3)\frac{dP_{n-2}}{dx} + (n-2)\frac{dP_{n-4}}{dx}\right], \\
X^{n-1}_{n}(x) &= \frac{1}{(2n-1)^2}\left[2(n-1)\frac{dP_n}{dx} + \frac{dP_{n-2}}{dx}\right], \\
X^{n-1}_{n-2}(x) &= \frac{1}{(2n-1)^2}\frac{dP_n}{dx} - 2\left[\frac{n}{(2n-1)^2} + \frac{n-3}{(2n-5)^2}\right]\frac{dP_{n-2}}{dx} \\
&\qquad - \frac{1}{(2n-5)^2}\frac{dP_{n-4}}{dx}\,, \\
X^{n-1}_{n-4}(x) &= -\frac{1}{(2n-5)^2}\left[\frac{dP_{n-2}}{dx} - 2(n-2)\frac{dP_{n-4}}{dx}\right].
\end{aligned}\right\}
$$

IV. The Satisfaction of the Boundary Conditons

The arbitrary constants C, D are to be determined from the following conditions on the surface of the spheroid

$$\sigma_\alpha = \tau_{\alpha\beta} = 0 \tag{32}$$

which led to the $2\left[\frac{n}{2}+1\right]$[5] $+1 = n+3$ simultaneous linear equatons for even value of n and $2\left[\frac{n}{2}+1\right]+2 = n+3$ for odd as in eqs. (34) and (36), whereas the number of the arbitrary constants are only $2\left[\frac{n}{2}+1\right]$ for the temperature distribution expressed by the following equation,

$$
\begin{aligned}
T = 2 \sum_{m=0}^{[\frac{n}{2}]} B_{n-2m}\big[&(2n-4m+3)\, P_{n-2m}(p)\, P_{n-2m}(q) \\
&+(2n-4m-5)\, P_{n-2m-2}(p) P_{n-2m-2}(q)\big].
\end{aligned} \tag{33}
$$

For even value of n, the equations arising from the boundary conditions are

$$
\left.\begin{aligned}
&\left[C_{n+2}L^{n+2}_{n+2}(q_0) + D_{n+1}\overline{M}^{n+1}_{n+2}(q_0) + \frac{1+\nu}{1-\nu}\,\varepsilon B_n K^{n}_{n+2}(q_0)\right]P_{n+2}(p)=0\,, \quad [1] \\
&\left[C_{n+2}L^{n+2}_{n}(q_0)+C_n L^{n}_{n}(q_0)+D_{n+1}\overline{M}^{n+1}_{n}(q_0)+ \cdots\right]P_n(p)=0\,, \quad [2]
\end{aligned}\right|
$$

---

5) $\left[\frac{n}{2}+1\right]$ designates the maximum integer in the bracket.

$$\left.\begin{aligned}
&\cdots\cdots\cdots\\
&\left[C_2 \overset{2}{\underset{0}{L}}(q_0)+D_1\overset{1}{\underset{0}{\overline{M}}}(q_0)+\cdots\right]P_0(p)=0\ ,\ \left[\tfrac{n}{2}+2\right]\\
&\left[C_{n+2}\overset{n+2}{\underset{n+2}{V}}(q_0)+D_{n+1}\overset{n+1}{\underset{n+2}{\overline{W}}}(q_0)+\frac{1+\nu}{1-\nu}\varepsilon B_n\overset{n}{\underset{n+2}{U}}(q_0)\right]p\,q_0\frac{dP_{n+2}}{dp}=0\ ,\ \left[\tfrac{n}{2}+3\right]\\
&\cdots\cdots\cdots\\
&\left[C_2 \overset{2}{\underset{2}{V}}(q_0)+D_1\overset{1}{\underset{2}{\overline{W}}}(q_0)+\cdots\right]p\,q_0\frac{dP_2}{dp}=0\ ,\ \left[n+3\right]
\end{aligned}\right\}\quad(34)$$

where $\overline{M}=M+2\nu N,\quad \overline{W}=W+(1-2\nu)X.$ (35)

For odd value of n, the equations arising from the boundary conditions are

$$\left.\begin{aligned}
&\left[C_{n+2}\overset{n+2}{\underset{n+2}{L}}(q_0)+D_{n+1}\overset{n+1}{\underset{n+2}{\overline{M}}}(q_0)+\frac{1+\nu}{1-\nu}\varepsilon B_n\overset{n}{\underset{n+2}{K}}(q_0)\right]P_{n+2}(p)=0\ ,\quad [1]\\
&\cdots\cdots\cdots\\
&\left[C_3 \overset{3}{\underset{1}{L}}(q_0)+D_2\overset{2}{\underset{1}{\overline{M}}}(q_0)+\cdots\right]P_1(p)=0\ ,\quad \left[\tfrac{n}{2}+2\right]\\
&\left[C_{n+2}\overset{n+2}{\underset{n+2}{V}}(q_0)+D_{n+1}\overset{n+1}{\underset{n+2}{\overline{W}}}(q_0)+\frac{1+\nu}{1-\nu}\varepsilon B_n\overset{n}{\underset{n+2}{U}}(q_0)\right]p\,q_0\frac{dP_{n+2}}{dp}=0\ ,\ \left[\tfrac{n}{2}+3\right]\\
&\cdots\cdots\cdots\\
&\left[C_3 \overset{3}{\underset{1}{V}}(q_0)+D_2\overset{2}{\underset{1}{\overline{W}}}(q_0)+\cdots\right]p\,q_0\frac{dP_1}{dp}=0\quad \left[n+3\right]
\end{aligned}\right\}\quad(36)$$

The systems of equations so obtained are compatible, because the rank of the augumented matrix is but $2[\frac{n}{2}+1]$ for even or odd value of n and this will be examined subsequently. Then for the temperature distribution given by eq. (33), the constants C, D can be computed successively from the following equations.

$$\left.\begin{aligned}
&\overset{n+2-2m}{\underset{n+2-2m}{L}}C_{n+2-2m}+\overset{n+1-2m}{\underset{n+2-2m}{\overline{M}}}D_{n+1-2m}\\
&\qquad=-\frac{1+\nu}{1-\nu}\varepsilon\sum_{r=0}^{3}B_{n+2r-2m}\overset{n+2r-2m}{\underset{n+2-2m}{K}}-C_{n+4-2m}\overset{n+4-2m}{\underset{n+2-2m}{L}}\\
&\qquad\quad-\sum_{r=0}^{1}D_{n+2r+3-2m}\overset{n+2r+3-2m}{\underset{n+2-2m}{\overline{M}}}\ ,\\
&\overset{n+2-2m}{\underset{n+2-2m}{V}}C_{n+2-2m}+\overset{n+1-2m}{\underset{n+2-2m}{\overline{W}}}D_{n+1-2m}\\
&\qquad=-\frac{1+\nu}{1-\nu}\varepsilon\sum_{r=0}^{3}B_{n+2r-2m}\overset{n+2r-2m}{\underset{n+2-2m}{U}}-C_{n+4-2m}\overset{n+4-2m}{\underset{n+2-2m}{V}}\\
&\qquad\quad-\sum_{r=0}^{1}D_{n+2r+3-2m}\overset{n+2r+3-2m}{\underset{n+2-2m}{\overline{W}}}\ .\qquad (m=0,1,\cdots,[\tfrac{n}{2}])
\end{aligned}\right\}\quad(37)$$

$\sigma_\alpha/2G$ and $\tau_{\alpha\beta}/2G$ of the solution R_1, eq. (27), can be written in the other forms as follows.

$$\left.\begin{aligned}\sigma_\alpha/2Gh^2&=\bar{q}^2\left\{P_n(p)\frac{d^2P_n}{dq^2}-P_{n-2}(p)\frac{d^2P_{n-2}}{dq^2}\right\}-\frac{2n-1}{n(n-1)}\bar{p}^2\frac{dP_{n-1}}{dp}\frac{dP_{n-1}}{dq},\\ \tau_{\alpha\beta}/2Gh^2&=-\bar{p}\,\bar{q}\left\{\frac{n-1}{n}\frac{dP_n}{dp}\frac{dP_n}{dq}-\frac{n}{n-1}\frac{dP_{n-2}}{dp}\frac{dP_{n-2}}{dq}\right\}.\end{aligned}\right\}\tag{38}$$

Now making the following substitution in eq. (38),

$$p\to q,\qquad \bar{p}^2\to-\bar{q}^2,\qquad \bar{p}\,\bar{q}\to-\bar{q}^2,\tag{39}$$

and by the aid of the recurrence formulas (20), it can be shown that $\sigma_\alpha/2Gh^2$ and $\tau_{\alpha\beta}/2Gh^2$ so substituted are equal.[6)]

$$[\sigma_\alpha/2Gh^2]_{p\to q}=[\tau_{\alpha\beta}/2Gh^2]_{p\to q}\tag{40}$$

Comparing eq. (38) with (40), it is easily seen that the following equation holds.

$$\frac{d^2P_n}{dq^2}P_n(q)-\frac{d^2P_{n-2}}{dq^2}P_{n-2}(q)=-\frac{2n-1}{n(n-1)}\left(\frac{dP_{n-1}}{dq}\right)^2+\frac{n-1}{n}\left(\frac{dP_n}{dq}\right)^2-\frac{n}{n-1}\left(\frac{dP_{n-2}}{dq}\right)^2.\tag{41}$$

It is also verified that $\sigma_\alpha/2Gh^2$ and $\tau_{\alpha\beta}/2Gh^2$ of the solution R_0 and R_2 can be equated if the substitution (39) are made and eq. (40) holds. This means that $\sigma_\alpha/2Gh^2$ and $\tau_{\alpha\beta}/2Gh^2$ are equal after the substitution of $P_n(p)\to P_n(q)$, $\bar{p}\,\bar{q}\frac{dP_n}{dp}\to-\bar{q}^2\frac{dP_n}{dq}$ in eqs. (24), (27)' and (30). Adding up the left hand side of the eq. (34) or (36) after the substitution of $P_n(p)\to P_n(q)$ and $\bar{p}\,\bar{q}\frac{dP_n}{dp}\to\bar{q}^2\frac{dP_n}{dq}$, the sum is, therefore, zero for the arbitrary values of B, C and D as the results of the foregoing discussion. This completes the proof for even value of n that the rank of the augumented matrix is $2\left[\frac{n}{2}+1\right]$.

One more linearly dependent relation, however, is necessary for odd value of n. Consider the resultant force in Z direction on the surface of the spheroid due to the stress functions.

$$Z_{resu't}=-2\pi\int_{-1}^{1}[\sigma_\alpha\,\bar{q}^2p-\tau_{\alpha\beta}\,q\,\bar{q}\,\bar{p}]dp.\tag{42}$$

To compute the resultant force, h^2 should be expanded in terms of $P_n(p)$ and this is achieved by the aid of Neuman's formula.[7)]

$$qh^2=\frac{q}{q^2-p^2}=\sum_{n=0}^{\infty}(4n+1)Q_{2n}(q)P_{2n}(p).\tag{43}$$

The resultant force in Z direction on the surface of the spheroid due to a stress function R_1, $\varphi=P_{2n+1}(p)P_{2n+1}(q)-P_{2n-1}(p)P_{2n-1}(q)$, can be computed by the aid of eqs. (38), (43) and the orthogonality of the Legendre functions. It is interesting to note that the integral (42) results in the relation analogous to the eq. (41).

6) If one or both of the Legendre function of the first kind are changed to that of the second kind in eq. (38), eqs. (40) and (41) hold true as far as n is greater than two. Also these relation hold for the solutions R_0 and R_2.

$$\left.\begin{aligned}
&-2\pi\int_{-1}^{1}\sigma_\alpha\bar{q}^2 p\,dp = -4\pi\bar{q}^4\Big[\frac{d^2P_{2n+1}}{dq^2}Q_{2n+1}-\frac{d^2P_{2n-1}}{dq^2}Q_{2n-1}+\frac{4n+1}{2n(2n+1)}\frac{dP_{2n}}{dq}\frac{dQ_{2n}}{dq}\Big],\\
&2\pi\int_{-1}^{1}\tau_{\alpha\beta}q\bar{q}\,\bar{p}\,dp = 4\pi\bar{q}^4\Big[\frac{2n}{2n+1}\frac{dP_{2n+1}}{dq}\frac{dQ_{2n+1}}{dq}-\frac{2n+1}{2n}\frac{dP_{2n-1}}{dq}\frac{dQ_{2n-1}}{dq}\Big],\\
&Z_{result} = -4\pi q^4\Big[\frac{d^2P_{2n+1}}{dq^2}Q_{2n+1}-\frac{d^2P_{2n-1}}{dq^2}Q_{2n-1}+\frac{4n+1}{2n(2n-1)}\frac{dP_{2n}}{dq}\frac{dQ_{2n}}{dq}\\
&\qquad -\frac{2n}{2n+1}\frac{dP_{2n+1}}{dq}\frac{dQ_{2n+1}}{dq}+\frac{2n-1}{2n}\frac{dP_{2n-1}}{dq}\frac{dQ_{2n-1}}{dq}\Big]=0^{8)}.
\end{aligned}\right\} \quad (44)$$

The comparison of eq. (44) with (38) shows that, when n is odd, the resultant force in Z direction due to σ_α is obtained by multiplying $-4\pi\frac{\bar{q}^4}{h^2}$ to σ_α and the transformation of $P_n(P)\rightarrow Q_n(q)$. The resultant force in Z direction due to $\tau_{\alpha\beta}$ is also obtained by multiplying $-4\pi\frac{\bar{q}^4}{h^2\bar{p}\bar{q}}$ to $\tau_{\alpha\beta}$ and the transformation of $P_n(p)\rightarrow Q_n(q)$. These hold true for the other stress functions, R_2 and R_0.

Adding up the left hand side of eq. (36) after the substitution of $P_n(p)\rightarrow Q_n(q)$, $\bar{p}\,\bar{q}\frac{dP_n}{dp}\rightarrow\bar{q}^2\frac{dQ_n}{dq}$, the sum is $-\frac{1}{4\pi\bar{q}^2}Z_{result}$ and is zero for the arbitrary value of B,C and D. This relation together with the foregoing relation using eq. (39) completes the proof for odd value of n that the rank of the augumented matrix is $2\left[\frac{n}{2}+1\right]$.

V. Thermal Stress in a Turbine Disc with Steady Axisymmetric Distribution of Temperature

The disc is assumed to be an oblate spheroid composed of homogeneous isotropic material.

The oblate spheroidal coordinates are defined by the following equations of transformation

$$\left.\begin{aligned}
x&=d\cosh\alpha\sin\beta\cos\gamma,\\
y&=d\cosh\alpha\sin\beta\sin\gamma,\\
z&=d\sinh\alpha\cos\beta.
\end{aligned}\right\} \quad (45)$$

For convenience, the following auxiliary variables are introduced

$$\left.\begin{aligned}
&q=\sinh\alpha, \quad \bar{q}=\cosh\alpha=\sqrt{q^2+1},\\
\text{where}\quad &o\le q<\infty, \quad 1\le\bar{q}<\infty.
\end{aligned}\right\} \quad (46)$$

7) H. Miyamoto; Stress Concentration Around a Spheroidal Cavity in an Infinite Elastic Body. Trans. of the Jap. Soc. of Mech. Eng. 1987, 60, (1953).

8) This equation can be verified for arbitrary value of n by the aid of the recurrence formula.

Eq. (45) can be obtained from the eq. (5) which defined the prolate spheroidal coordinates by the following transformations,

$$\left.\begin{array}{lll} \text{prolate spheroid} \rightarrow \text{oblate spheroid} & & \\ (\alpha)_{\text{proslate}} = & i\frac{\pi}{2} + (\alpha)_{\text{oblate}} & \\ \text{ch}\ \alpha \rightarrow i\ \text{sh}\ \alpha, & \text{or}\quad q \rightarrow iq, & \\ \text{sh}\ \alpha \rightarrow i\ \text{ch}\ \alpha, & \text{or}\quad \bar{q} \rightarrow i\bar{q}, & \\ c = 1 \rightarrow d = -i & & \\ \beta \rightarrow \beta, & \text{or}\quad p, \bar{p}, \rightarrow p, \bar{p}, & \\ \gamma \rightarrow \gamma. & & \end{array}\right\} \qquad (47)$$

By the aid of eq. (47) the results obtained for the prolate spheroid can be transformed to the oblate spheroid.

VI. Numerical Examples

(1) K, M, U are very complicated when they are expressed in the general form such as eq. (25) and (31), but they take simple forms when n are given. Some of these are shown

$$K_4^2=\frac{24}{25}P_3'-\frac{228}{175},\quad K_2^2=-\frac{8}{105}P_5'+\frac{2}{15}P_3'-\frac{158}{35},$$

$$K_0^2=\frac{18}{75}P_3'-\frac{26}{25},\quad K_6^4=\frac{820}{297}P_5'-\frac{410}{297}P_3',$$

$$K_4^4=\frac{80}{429}P_7'+\frac{7733}{9009}P_5'-\frac{55552}{17325}P_3'+\frac{524}{525},$$

$$K_2^4=-\frac{188}{945}P_5'-\frac{1036}{945}P_3'+\frac{164}{105},\quad K_0^4=\frac{2}{75}P_3'+\frac{8}{75},$$

$$M_2^1=-\frac{8}{45}P_3'+\frac{6}{15},\quad M_0^1=-\frac{4}{45}P_3'+\frac{8}{15},$$

$$M_4^3=\frac{16}{35}P_3'-\frac{36}{35},\quad M_2^3=\frac{16}{36}P_3'-\frac{47}{21},\quad M_0^3=-\frac{2}{45}P_3'-\frac{11}{15},$$

$$M_6^5=\frac{360}{1573}P_7'+\frac{528}{1287}P_5'-\frac{1290}{1089}P,$$

$$M_4^5=\frac{300}{1573}P_7'+\frac{792}{9009}P_5'-\frac{6493}{4235}P_3'+\frac{48}{35},$$

$$M_2^5=-\frac{4}{21}P_5'-\frac{43}{105}P_3'+\frac{36}{35},$$

$$U_4^2=-\frac{9}{35}P_2',\quad U_2^2=-\frac{9}{35}P_4'-\frac{13}{21}P_2',\quad U_6^4=-\frac{25}{99}P_4',$$

$$U_4^4=-\frac{25}{99}P_6'-\frac{530}{693}P_4'+\frac{91}{315}P_2',\quad U_2^4=\frac{91}{315}P_4'+\frac{4}{3}P_2'.$$

(2) When the temperature distribution on the surface of the oblate spheroids are given by $T/T_{max} = \bar{p}^2$, the distribution of $\sigma_\alpha/2G\varepsilon T_{max}$ on the surface and the stress and temperature distribution on the plane $Z=0$ are calculated for five shape ratios such as 0.05, 0.10, 0.15, 0.20, and 0.25. It is assumed that $\nu=0.25$ in

all the calculations. The results are shown in Fig. 1—5.

(3) Under the same temperature condition with (2), $\sigma_\alpha/2G\varepsilon T_{max}$ and $\sigma_z/2G\varepsilon T_{max}$ at the point B, $\sigma_\gamma/2G\varepsilon T_{xam}$ and $\sigma\rho/2G\varepsilon T_{max}$ at A, temperature at the center C are plotted as functions of shape ratio S in Fig. 6.

If S=1, the shape is sphere. It is reported [1] that E. Almansi [9] studied the thermal stress of a sphere but the paper is not available. So thermal stress in a sphere with stedy axisymmetric distribution of temperature is computed in Appendix which affords a chek for the limiting case S=1.

VII. Acknowledgement

The author wishes to express his deepest gratitude to Prof. T. Suhara who gave him valuable and detailed guidance throughout the course of the work.

Also his hearty thankfulness should be made known to Mr. and Mrs. Mita for giving warm encouragement to the author.

Appendix

Theramal Stress in a Sphere with Steady Axisymmetric Distribution of Temperature

After the same procedure as developed in the foregoing analysis, the following results are obtained. For the temperature distribution

$$T=\sum_{n=0}^{\infty} A_n\rho^n P(p),$$

where $\rho=r/a$, $p=\cos\beta$, and a is the radius of the sphere, the coresponding stress fields are

$$\sigma_\rho/2G=-\sum_{n=0}^{\infty} n(n-1)a_n(\rho^n-\rho^{n-2})P_n(p)\ ,$$

$$\sigma_\beta/2G=\sum_{n=0}^{\infty}\frac{a_n}{2n+1}\left[n(n-1)(\rho^n-\rho^{n-2})\frac{dP_{n+1}}{dp}\left\{(n^2+5n+3)\rho^n-(n^2+n+1)\rho^{n-2}\right\}\frac{dP_{n-1}}{dp}\right],$$

$$\tau_{\rho\beta}/2G=-\sum_{n=0}^{\infty}(n-1)a_n(\rho^n-\rho^{n-2})\,\bar{p}\frac{dP_n}{dp},$$

where $a_n=\dfrac{(1+\nu)}{2[(n^2+n+1)+(2n+1)\nu]}\varepsilon A_n\,.$

Apparently, $\sigma_\rho/2G$ and $\tau_{\rho\beta}/2G$ vanish for arbitrary values of n on the surface of the sphere and also all the stresses vanish for $n=0$ and $n=1$. The case $n=0$ means uniform temperature and the case $n=1$ means a linearly varying temperature distribution in Z direction, both of which apparently do not produce any stress.

It is interesting to note the fact that the distributions of σ/σ_{max} and τ/τ_{max} are not affected by Poisson's ratio ν so far as the temperature distribution is expressed by a single term, $A_n\rho^n P_n(p)$.

9) E. Almansi; Atti reale accad. sci. Torino, 32, 963, (1896-1897) Mem. reale accad. sci. Torino, series 2. 47, (1897)

Postscript

In this paper, thermal stress in a spheroid with steady axisymmetric distribution of temperature has been considered, but the thermal stress arising from distinct uniform temperature changes applied to a spheroidal inclusion and surrounding medium was solved by R. H. Edwards and to an ellipsoidal inclusion by K. Robinson.

(a) R. H. Edwards; Stress Concentration Around Spheroidal Inclusions and Cavities, J. Appl. Mech I 8, 19,(1951)

(b) K. Robinson; Elastic Energy of an Ellipsoidal Inclusion in an Infinite Solid, J. Appl, Physics, 22, 1045, (1951)

The author wishes to express his thanks to E. Sternberg Prof. of Illinois Institute of Technology for so kindly advising him to refer to these papers.

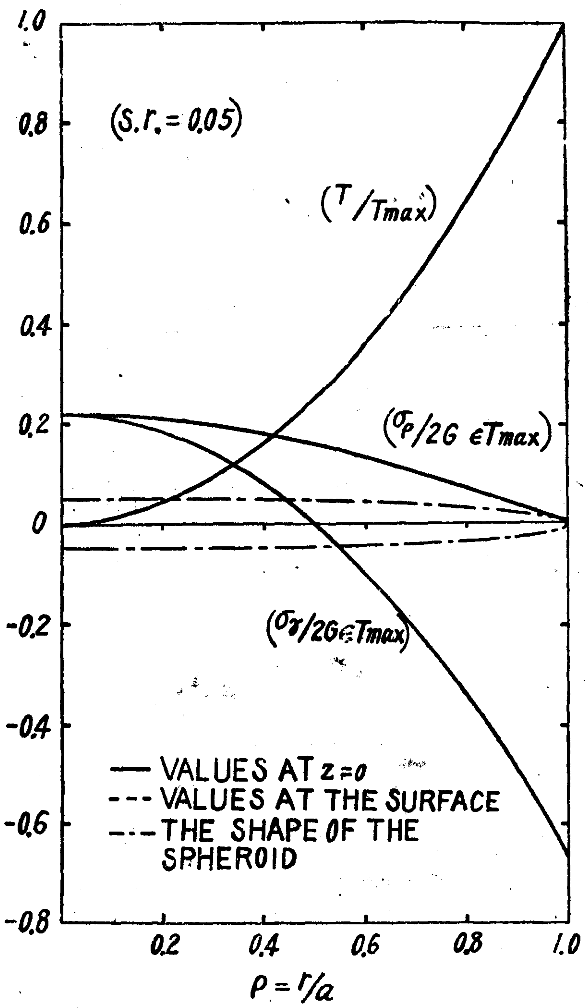


Fig. 1.

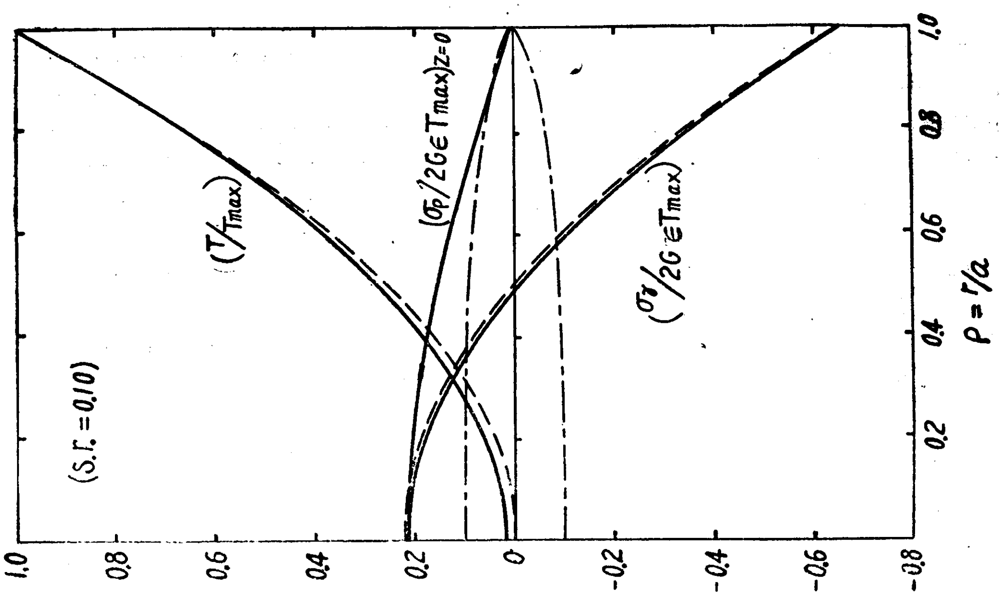

(S.r. = 0.10)
(τ/τ_{max})
$(\sigma_\rho/2G\epsilon\tau_{max})_{z=0}$
$(\sigma_\vartheta/2G\epsilon\tau_{max})$
$\rho = r/a$
1.0
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8

Fig. 2.

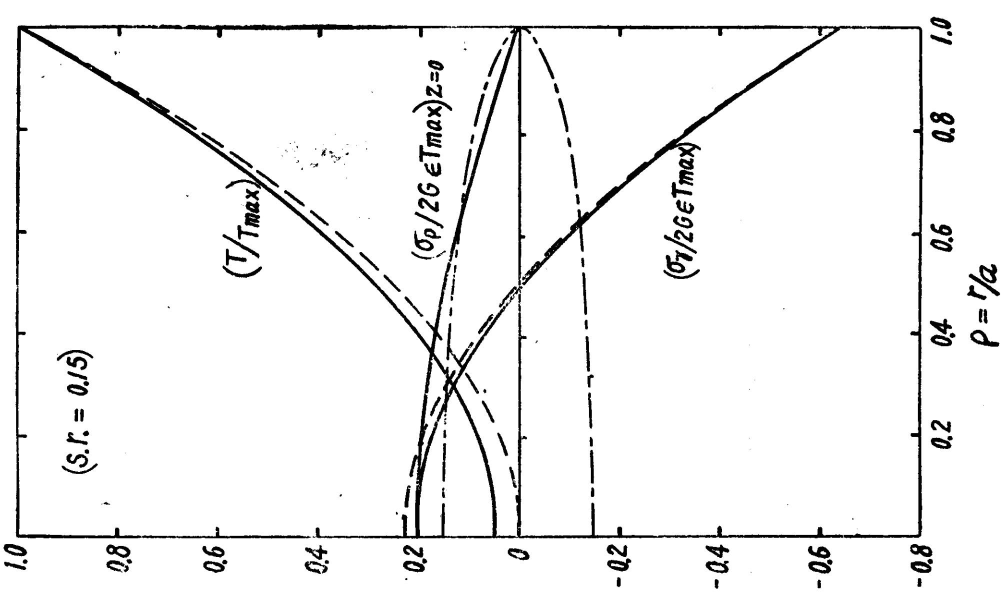

(S.r. = 0.15)
(τ/τ_{max})
$(\sigma_\rho/2G\epsilon\tau_{max})_{z=0}$
$(\sigma_\vartheta/2G\epsilon\tau_{max})$
$\rho = r/a$
1.0
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8

Fig. 3.

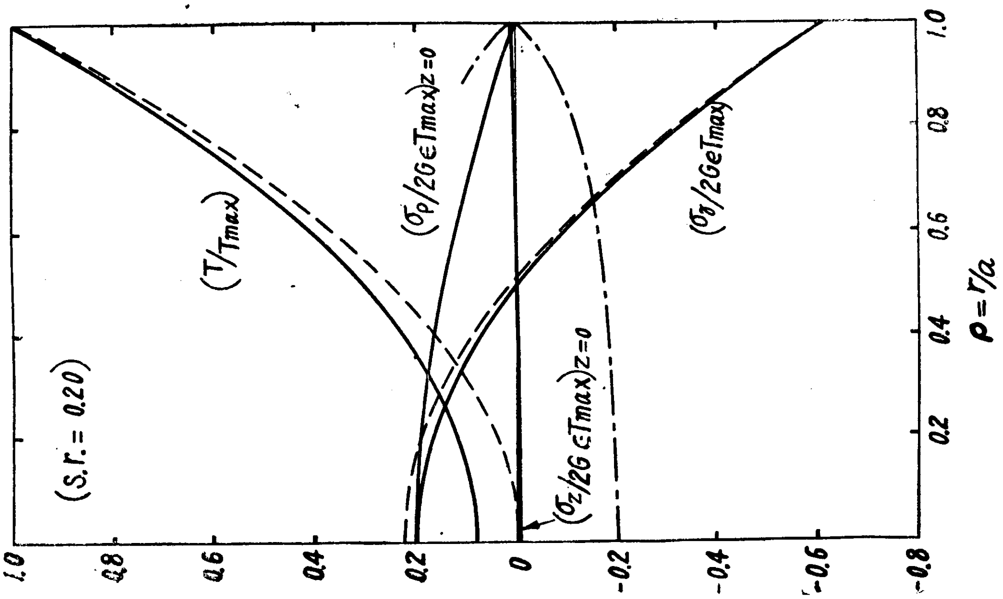

(S.r. = 0.20)
(T/Tmax)
(σρ/2GεTmax)z=0
(σz/2GεTmax)z=0
(σr/2GεTmax)
ρ = r/a
1.0
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8

Fig. 4.

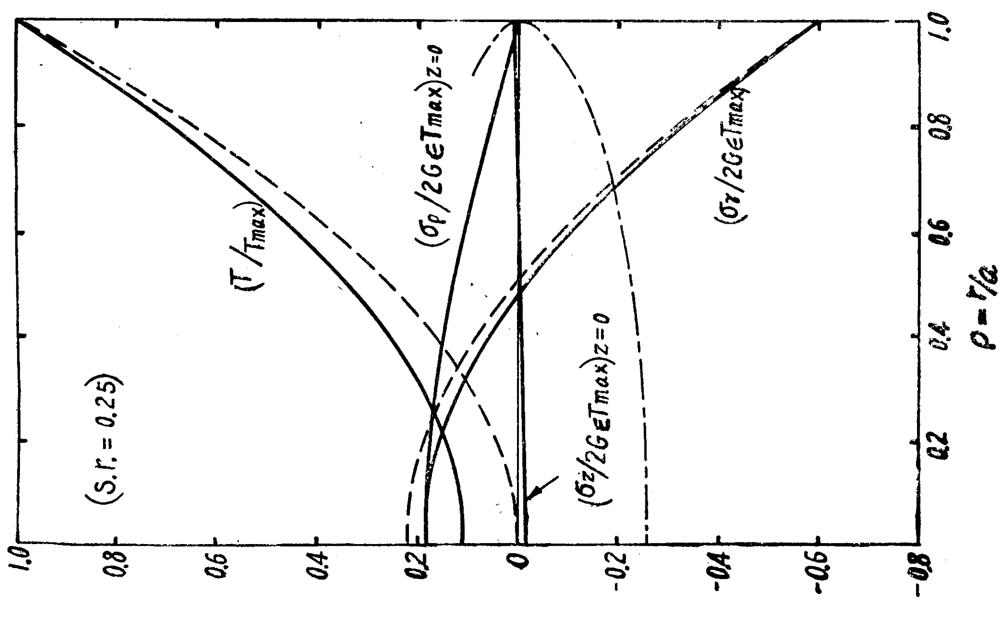
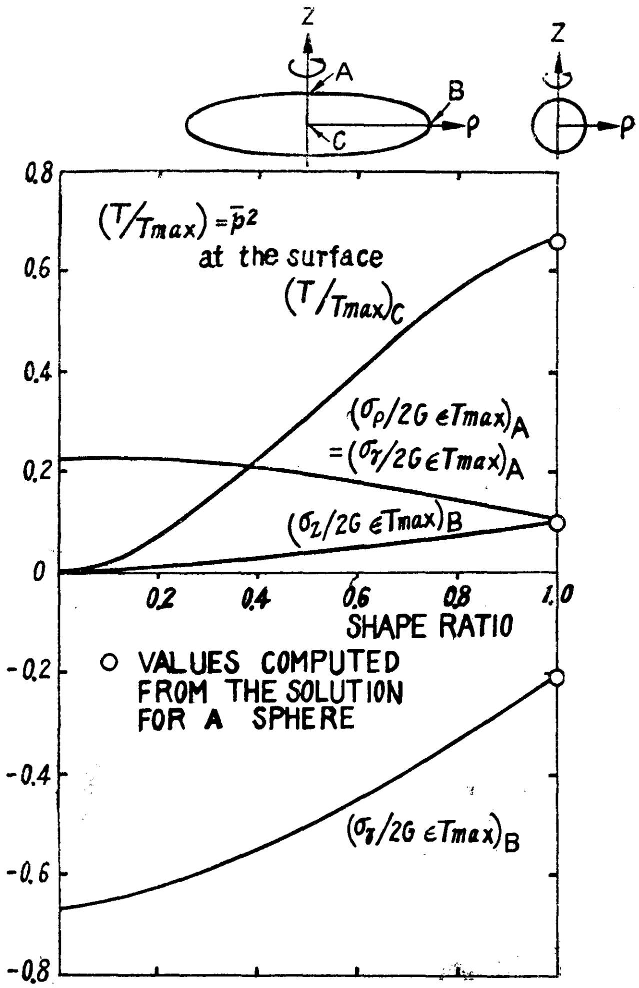

(S.r. = 0.25)
(T/Tmax)
(σρ/2GεTmax)z=0
(σz/2GεTmax)z=0
(σr/2GεTmax)
ρ = r/a
1.0
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8

Fig. 5.



Z
A
B
C
ρ
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
(T/Tmax) = ρ̄²
at the surface
(T/Tmax)C
(σρ/2G εTmax)A
= (σγ/2G εTmax)A
(σz/2G εTmax)B
(σγ/2G εTmax)B
0.2
0.4
0.6
0.8
1.0
SHAPE RATIO
O VALUES COMPUTED FROM THE SOLUTION FOR A SPHERE

Fig. 6.