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"On Mackay's Nonlinear Circuit"

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Abstract

R. S. Mackay[1] performed an experiment which is not only interesting as curiosities, but also has theoretical and practical importance. An ordinary incandescent lamp connected to A. C. line in series with an iron-cored choke and a condenser flashes intermittently of its own accord. The performance has been analysed under suitable assumptions and approximations. The results are qualitatively in good agreement with the experiment.

I. Introduction

Figure 1 shows the simple series circuit of a lamp, an iron-cored choke, a condenser and an A. C. generator where N is the number of turns of the iron-cored choke, ϕ the flux in the choke, C the capacity of the condenser, R the resistance of the lamp, i the current through the circuit and $v\sin(\omega t+\varphi)$ the voltage of the A. C. generator.

There are two kinds of nonlinear characteristics in the circuit: the one is that between "flux and current" in the iron-cored choke and the other is that between "voltage and current" in the lamp.

The iron-cored choke and the condenser make an ordinary "ferroresonance circuit" and the nonlinearity of the lamp performs a switching action so that the lamp flashes intermittently of itself.

There follows an analysis of the performance under suitable assumptions and approximations.

II. Analysis of the Performance

A. The differential equation of the circuit is

$$N\frac{d\phi}{dt}+\frac{1}{C}\int idt+Ri=v\sin(\omega t+\varphi) \qquad (1)$$

where the current is assumed to be related with the flux as

$$i=\alpha\phi+\beta\phi^3 \qquad (\alpha, \beta: \text{constants}) \qquad (2)$$

neglecting the magnetic hysteresis of the iron-cored choke.

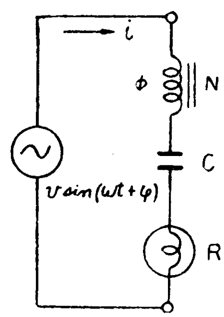

Fig. 1. Mackay's nonlinear cicuit

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1) R. S. Mackay: "Interesting Nonlinear Effects", J. App. Phys. 24 (1953) 311.

The relation between the resistance and the current of the lamp will be obtained by the following consideration.

The temperature θ (°K) of the filament of the lamp is given by

$$D\frac{p\theta}{pt}=\mathrm{R}i^2-\Psi(\theta,\theta_0) \tag{3}$$

where D is the heat capacity of the filament, $\Psi(\theta,\theta_0)$ is the outgoing heat (per unit time) from the filament by radiation, gas cooling and heat conduction, θ_0 (°K) being the surrounding temperature.

We assume that $\Psi(\theta,\theta_0)$ is a continuous function of θ defined for $\theta\geqq\theta_0$; $\Psi(\theta,\theta_0)=0$ when $\theta=\theta_0$ and $\frac{\Psi(\theta,\theta_0)}{\theta}$ increases monotonically with θ: an example being

$$\Psi(\theta,\theta_0)=\sigma(\theta^n-\theta_0^n) \qquad (\sigma: \text{constant}, n\geqq 1).$$

Moreover we assume that

$$\frac{R}{R_0}=\left(\frac{\theta}{\theta_0}\right)^\gamma \tag{4}$$

where R_0 is the resistance of the filament at temperture θ_0 and γ is a constant near 1.2 but we regard it as unity for simplicity. Then we have from (3) and (4)

$$\frac{dR}{pt}=\frac{R_0}{D\theta_0}i^2R-\frac{R_0}{D\theta_0}F(R,R_0) \tag{5}$$

here the function $\Psi(\theta,\theta_0)$ is replaced by $F(R,R_0)$ and the latter satisfies the similar conditions to the former.

The fundamental equations now established from (1) (2) and (5) are

$$\left\{\begin{aligned}&\ddot{\phi}+\frac{R}{\omega N}(\alpha+3\beta\phi^2)\dot{\phi}+\frac{1}{\omega^2CN}(1+\omega C\dot{R})(\alpha\phi+\beta\phi^3)\\ &\qquad\qquad=\frac{v}{\omega N}\cos(\tau+\varphi), &(6)\\ &\dot{R}=\frac{R_0}{\omega D\theta_0}(\alpha\phi+\beta\phi^3)^2R-\frac{R_0}{\omega D\theta_0}F(R,R_0) &(7)\end{aligned}\right.$$

where $\tau=\omega t$, $\dot{\phi}=\frac{d\phi}{d\tau}$, $\dot{R}=\frac{dR}{d\tau}$.

In the present case the rate of change of the filament resistance with respect to time is so small that

$$C\frac{dt}{dR}\ll 1. \tag{8}$$

We may therefore negrect $\omega C\dot{R}$ compared with unity in eq. (6) and the fundamental equations become

$$\left\{\begin{aligned}&\ddot{\phi}+r(1+\frac{3q}{p}\phi^2)\dot{\phi}+(p\phi+q\phi^3)=e\cos(\tau+\varphi), &(9)\\ &\dot{r}=\frac{Nr_0}{D\theta_0\alpha}(\alpha^2\phi^2+2\alpha\beta\phi^4+\beta^2\phi^6)r-\frac{r_0}{\omega D\theta_0}f(r,r_0) &(10)\end{aligned}\right.$$

where $r=\frac{R\alpha}{\omega N}$, $r_0=\frac{R_0\alpha}{\omega N}$, $p=\frac{\alpha}{\omega^2CN}$, $q=\frac{\beta}{\omega^2CN}$, $e=\frac{v}{\omega N}$.

The function $F(R,R_0)$ is replaced again by $f(r,r_0)$ and the latter satisfies the similar conditions to the former, namely, $f(r,r_0)$ is a continuous function of r defined for $r\geqq r_0$; $f(r,r_0)=0$ when $r=r_0$ and $\frac{f(r,r_0)}{r}$ increases monotonically with r.

B. In the first step, we seek for a periodic solution of the first approximation of eq. (9) putting[2)]

$$\phi(\tau)=x(\tau)\sin\tau \tag{11}$$

where $x(\tau)$ is a slowly varing function of τ.

From (11) we have

$$\dot{\phi}=\dot{x}\sin\tau+x\cos\tau,$$
$$\ddot{\phi}=2\dot{x}\cos\tau-x\sin\tau,$$
$$\phi^2\dot{\phi}=\frac{3}{4}x^2\dot{x}\sin\tau+\frac{1}{4}x^3\cos\tau,$$
$$\phi^3=\frac{3}{4}x^3\sin\tau$$

neglecting higher derivatives of x and higher harmonic terms such as $\sin 3\tau$, $\cos 3\tau$.

Substituting these in eq. (9) and equating the coefficients of $\sin\tau$, $\cos\tau$ separately on each side of eq. (9) we get

$$e\cos\varphi=2\dot{x}+rx+\frac{3q}{4p}rx^3,$$
$$e\sin\varphi=-r(1+\frac{9q}{4p}x^2)\dot{x}+(1-p)x-\frac{3q}{4}x^3.$$

Eliminating φ from these equations and neglecting higher powers of $\dot{x}$, we obain

$$r\dot{I}\{2(1+\frac{3q}{4p}I)+(1+\frac{9q}{4p}I)(\overline{p-1}+\frac{3q}{4})\}$$
$$+[\{r^2(1+\frac{3q}{4p}I)^2+(\overline{p-1}+\frac{3q}{4}I)^2\}I-E]=0 \tag{12}$$

whre $I=x^2$, $E=e^2$

We write the left side of eq. (12) as $h(I,\dot{I})$ for brevity.

The points of equilibrim of (12) are given by $h(I,0)=0$ that is by

$$V(I)=E \tag{13}$$

where

$$V(I)=\frac{9q^2(r^2+p^2)}{16p^2}I^3-\frac{3q\{(1-p)p-r^2\}}{2p}I^2+\{r^2+(1-p)^2\}I. \tag{14}$$

Now the condition of the occurrence of the ferroresonance is that $V(I)$ has a positive maximum point for some value of I and a positive minimum point

2) See, for example, N. Minorsky: "Introduction to Nonlinear Mechanics", J. W. Edwards, Ann Arbor, U. S. A. (1947)

for another value of I as being shown in Fig. 2.

After simple calculation we see that the condition is expressed as

$$(r+\frac{\sqrt{3}}{2})^2+(p-\frac{1}{2})^2<1 \tag{15}$$

which corresponds to the hatched region in the parametric plane of Fig. 3.

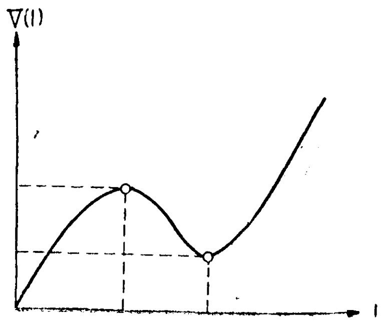

Fig. 2. Ferroresonance curve in the (I, V.) -plane

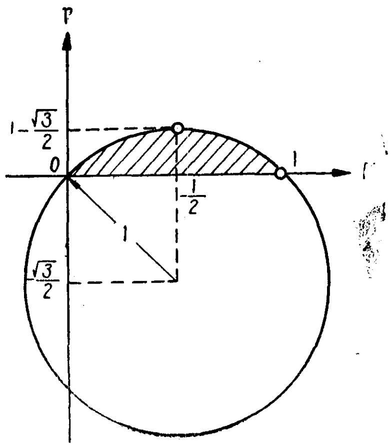

Fig. 3. Ferroresonance region in the parametric plane

By the way, we will consider the stability of these points of equilibrium in brief.

Putting $I=I_0+\delta I$, I_0 being a positive root of $h(I,0)=0$, anb substituting eq. (12) we know that I_0 is stable if

$$\frac{\delta \dot{I}}{\delta I}\bigg|_{\left(\substack{I=I_0\\ \dot{I}=0}\right)} = -\frac{\dfrac{\partial h(I,\dot{I})}{\partial I}}{\dfrac{\partial h(I,\dot{I})}{\partial \dot{I}}}\Bigg|_{\left(\substack{I=I_0\\ \dot{I}=0}\right)}$$

is negative and unstable if positive. After simple calculation we see that I_0 is stable if $\dfrac{dV(I)}{dI}\Big|_{(I=I_0)}$ is positive and unstable if negative.

Therefore after all we know that the condition of the occurrence of the ferroresonance is given by (15),

Curves of eq. (13) in the $(E.I)$ — plane corresponding to various values of the parameter r are sketched in Fig.4 where

$$*r=\sqrt{1-(p-\frac{1}{2})^2}-\frac{\sqrt{3}}{2} \tag{16}$$

is the maximum value of r which satisfies the condition (15) for a given value of p.

The unstable region is indicated in Fig.4 by hatching, the boundary of which being given by

$$E = \frac{3\,qI^2}{2} \quad \frac{\{4(1-p)-3\,q\,I\}}{4\,p+9\,q\,I} \tag{17}$$

Clearly the double hatched region is physically of no significance and E_1 and E_2 are decided by

$$E_1 = \frac{16\,(1-p)^3}{81\,q}$$

$$E_2 = \frac{16\,p\,(\sqrt{3}\sqrt{(1+2\,p)(3-2\,p)}-3)^3}{81\,q\,(3+2\,p-\sqrt{3}\sqrt{(1+2\,p)(3-2\,p)})^2}$$

respectively.

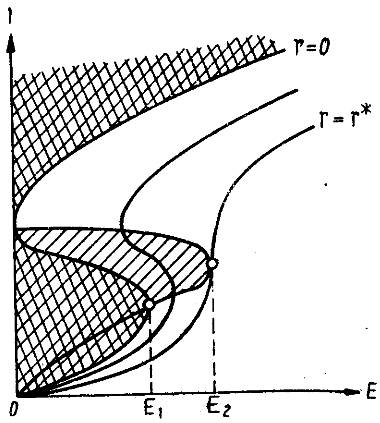

Fig. 4. Ferroresonance curves in the (E,I) - plane corresoponding to various values of the parameter r

On the other hand we have from eq. (13)

$$r^2 = \frac{p^2}{(4\,p+3\,q\,I)^2\,I}[\,16\,E-\{4\,(1-p)\,3\,q\,I\}^2\,I\,] \tag{18}$$

which is shown in Fig. 5 for various values of the parameter E.

The locus of extremums of this family of curves is given by

$$r^2 = \frac{\{4\,(1-p)-3\,q\,I\}\{9qI-4\,(1-p)\}\,p^2}{(4\,p+3\,q\,I)(4\,p+9\,q\,I)} \tag{19}$$

and the interior of the locus, indicated by hatching, corresponds to unstable region.

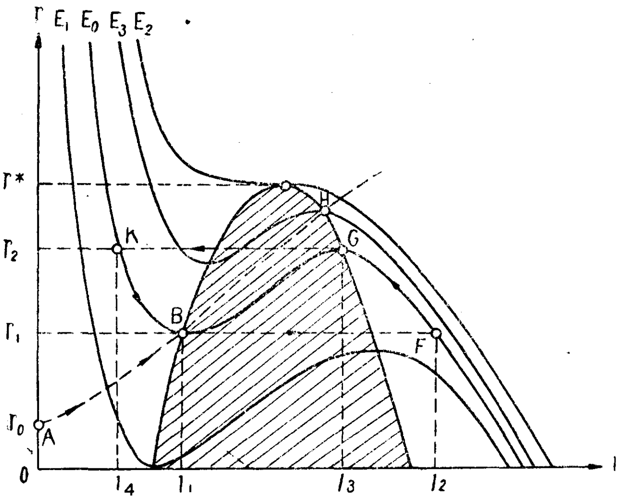

Fig. 5. Ferroresonance curves and the behaviour of the representative point in the (I,r) — plane

C. In the next place we will consider eq. (10). Substituting ϕ of (11) into (10) and using the mean value of

$$\alpha^2\phi^2 + 2\alpha\beta\phi^4 + \beta^2\phi^6$$

with respect to time over the period of ϕ, since the period of 'on and off' is fairly long compared with that of ϕ, we obtain

$$\dot{r} = \frac{N\, r_0}{16\, D\, \theta_0\, \alpha}\, g(I)\, r - \frac{r_0}{\omega\, D\, \theta_0}\, f(r, r_0) \qquad (20)$$

where $g(I) = 8\alpha^2 I + 12\alpha\beta I^2 + 5\beta^2 I^3$, $g(I)$ being a monotonically increasing function of I.

Now we will consider the behaviour of the representative point of the state given by (20) in the (I,r) -plane. (See Fig.5.)

Let the representative point be at A ($E=0, I=0, R=R_0$) initially. If E increases slowly then I also increases slowly and the rep. pt. moves along the locus given by

$$g(I) = \frac{16\alpha}{\omega N} \frac{f(r,r_0)}{r} \qquad (21)$$

since the state may be regarded as it is always in stable equilibrium, namely $\dot{r} = 0$ at any instant.

As easily seen from (21), r increases monotonically with I and the broken line in Fig.5 shows the locus of (21). We see from the figure that under suitable circumstances of the circuit constants the broken line intersects the boundary of the unstable region at, say B ($E=E_0, I=I_1, R=R_1$) and H ($E=E_3$). We will call these circumstances condition (C_1) for brevity.

As soon as the rep. pt. arrives at B it jumps abruptly to F ($E=E_0, I=I_2, r=r_1$) over the unstable region since the values E and R must remain constant as E_0 and R_1 respectively. Then the lamp fires on suddenly becauae of the increased current.

Now F is below the broken line so that $\dot{r}>0$ and the rep. pt. moves upwards along the curve $E=E_o$. It is clear from the figure that under suitable circumstances of the circuit constants the rep. pt. meets with the boundary of the unstable region at, say $G(E=E_o, I=I_3, r=r_2)$ before it meets with the broken line on which $\dot{r}=0$. We will call these circumstances condition (C_2) for brevity.

The rep. pt. no sooner arrives at G than jumps to K $(E=E_o, I=I_4, r=r_2)$ since the values E and r must remain constant as E_o and r_2 reipectively. Then the lamp goes out suddenly because of the decreased current.

Point K is evidently above the broken line so that $\dot{r}<0$ and hence the rep. pt. moves downwards along the curve $E=E_o$ and reaches at last to B. After that the rep. pt. repeats the cycling mentioned above.

After all, we see that the lamp which satisfies the two conditions (C_1) and (C_2) flashes intermittently of itself whenever $1>p>0,\ q>0$.

It is clear from the figure that for any value of E between E_o and E_3 the lamp flashes intermittently,but above E_3 it is always fully on and below E_o always fully off. The relation r_1, r_2 and E_o is given by (17) and (19), that is, by the parametric expressions

$$E=\frac{3qI^2}{2}\ \frac{\{4(1-p)-3qI\}}{4p+9qI}$$

and (22)

$$r^2=\frac{\{4(1-p)-3qI\}\ \{9qI-4(1-p)\}p^2}{(4p+3qI)\ (4p+9qI)}$$

in which there are two values of r corresponding to one value of E_o. We denote the larger one r_2 and the smaller one r_1 as shown in Fig.6. We will consider the behaviour of the rep. pt. on the (E,I)-plane. (See Fig.7.)The rep. pt. is situated at $0(E=0, I=0, r=r_o)$ initially. As E increases it moves along the broken line and reaches to B under the condition (C_1).

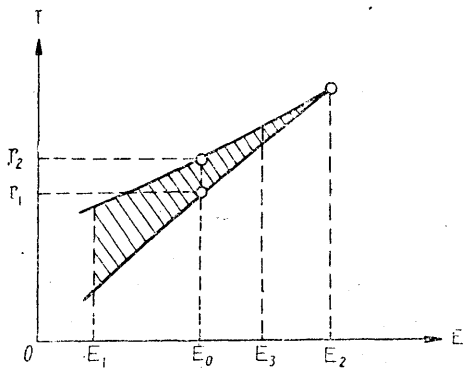

Fig. 6. The relation between r_1, r_2 and E_o

Then it jumps to F over the unstable region and the lamp fires on suddenly. Because of the increased current, r increases and the rep. pt. moves downwards. Under the condition (C_2)it reaches to G and jumps to K over the unstable region. Then the lamp goes out and because of the cooling, r decreases. Hence the rep. pt. moves upwards and reaches to B. After that the rep. pt. repeats the same cycling.

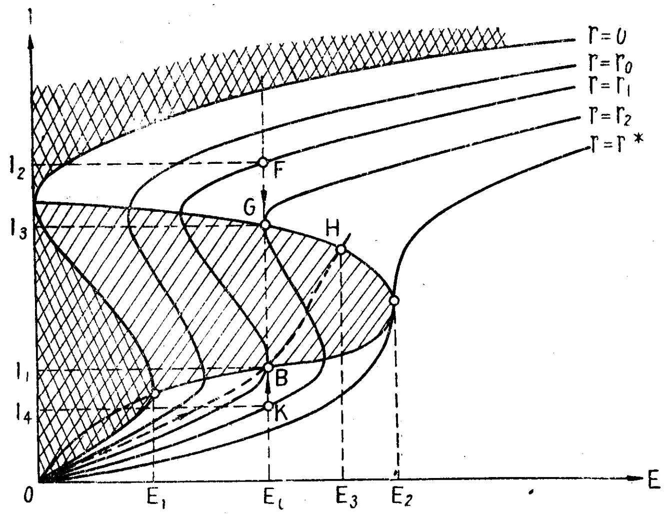

Fig. 7. Ferroresonance curves and the behaviour of the representative point in the (E,I)-plane

D. The period of 'on and off' can be obtained by integrating eq. (20). Let $I=\chi_1(r,E)$ and $I=\chi_2(r,E)$ be the two branches of the inverse function of eq. (18) corresponding to the cooling process and heating process respectively. (See Ffg.5.) Then the period T corresponding to the parameter E $(E_3 \geq E \geq E_0)$ is given by (in scale of t)

$$T=\int_{r_1(E)}^{r_2(E)} \frac{dr}{\frac{r_0}{D\theta_0}f(r,r_0)-\frac{\omega N r_0}{16D\theta_0\alpha}g_1(r,E)r}$$

$$+\int_{r_1(E)}^{r_2(E)} \frac{dr}{\frac{\omega N r_0}{16D\theta_0\alpha}g_2(r,E)r-\frac{r_0}{D\theta_0}f(r,r_0)} \tag{23}$$

where

$$g_1(r,E)=g(\chi_1(r,E))$$
$$g_2(r,E)=g(\chi_2(r,E))$$

and $r_1(E)$ and $r_2(E)$ are the smaller and the larger values of r in (22) corresponding to the parameter E respectively. The first and second integrals of (23) correspond to cooling and heating processes respectively and main contributions to the integrations are done by the first terms of their denominators. Therefore we may consider that the time interval of 'off' (corresponding to the cooling process) is dominated by the integral

$$\int_{r_1(E)}^{r_2(E)} \frac{d\,r}{\frac{r_0}{D\theta_0}f(r\;r_0)} \tag{24}$$

Since $f(r, r_0)$ increases with r; $r_1(E)$ and $r_2(E)$ increase with E, while $r_2(E)-r_1(E)$ decreases with E as seen from Fig.6, the time interval of 'off' decreases with increasing E.

III. Conclusion

As the line voltage increases from zero, E also increases and at $E=E_o$ the lamp fires on suddenly and begins to flash intermittently. The time interval of 'off' decreases with increasing E and at $E=E_3$ the lamp is fully on. Then the brightness of the lamp increases with E. If the condition (C_1) is not satisfied, the brightness increases gradually with the line voltage and the lamp does not flash intermittently. This case is usually observed when we change the lamp, which is flashing intermittently, into a smaller wattage one. On the other hand, if the condition (C_2) is not satisfied, the lamp fires on once and does not goes out. This case is usually observed when we change the lamp, which is flashing intermittently, into a larger wattage one. All these behaviours of the circuit are in good agreement with the experiment.

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