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On a Fourier-Bessel Expansion of Special Kind

(Received April 4, 1953)

Fumiki KITO*

Abstract

The Author discusses the validity of expansion of an arbitrary function $f(x)$ in a form of infinite series of Fourier-Bessel expansion of special kind. The proposition is closely related to validity theorem of Fourier-Bessel expansion treated by Mac Robert in 1922.

I. Introduction

In a previous paper by Author[1] an infinite series in a form of Fourier-Bessel expansion of special kind has been used. Here we wish to discuss the validity of this Fourier-Bessel expansion. Given a set of functions

$$Q(k_s) = J_n(k_s x)\,G_n'(k_s a) - G_n(k_s x)\,J_n'(k_s a) \tag{1}$$

where k_s $(s = 1, 2, 3, \cdots\cdots\cdots)$ are successive roots of the transcendental equation

$$S(k) \equiv J_n'(kb)\,G_n'(ka) - G_n'(kb)\,J_n'(ka) = 0 \tag{2}$$

it can be shown, by actual calculation, applying formula of Lommel integrals and (2), that

$$\int_a^b Q(k_s)\,Q(k_j)\,x\,dx = 0 \qquad (s \neq j) \tag{3}$$

$$\int_a^b [Q(k_s)]^2\,x\,dx = M(k_s) \tag{4}$$

where

$$M(k) = \frac{1}{2k^2}\left[(1 - \frac{n^2}{k^2b^2})\left\{\frac{J_n'(ka)}{J_n'(kb)}\right\}^2 - (1 - \frac{n^2}{k^2a^2})\right] \tag{5}$$

The above equations shows us that the set of functions $Q(k_s)$ $(s = 1, 2, 3, \cdots\cdots)$ are orthogonal to each other. n is a positive integer. a and b are positive real numbers such that $b > a$. J, G denotes a set of Bessel functions of order n. We may expect the expansion of an arbitrary function of real variable $f(x)$ in a form of Fourier-Bessel expansion

$$f(x) = \sum A_s Q(k_s) \tag{6}$$

under the usual assumptions as to the nature of function $f(x)$. But the question of validity of the expansion remains to be proved.

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1) F. Kito, On Vibration of a Cylinderical Shell Immersed in Water, This Journal Vol. 5, No 17

Now, a similar question relative to the set of functions

$$Q(x) = J_n(kx)\,G_n(ka) - G_n(kx)\,J_n(ka)$$

where k_s are successive roots of the equation

$$S(k) \equiv J_n(kb)\,G_n(ka) - G_n(kb)\,J_n(ka) = 0$$

has been treated by Mac Robert.[2)] In what follows, we shall show that the above-mentioned problem of the validity of expansion (6) can be proved by following close line to the proposition of Mac Robert.

II. Expression for the Value of Norm $M(k_s)$

In what follows, we infer by k_s values of the transcendental equation (2), while k means the current variable. We omit to write the order n, since it has the same fixed integral value throuhgout the present paper.

Now, we have, by actual differentiation,

$$S'(k) = b[J''(kb)\,G'(ka) - G''(kb)\,J'(ka)] \\ + a\,[J'(kb)\,G''(ka) - G'(kb)\,J''(ka)]$$

Taking into account the equation

$$J''(kb) + \frac{1}{kb}J'(kb) + (1 - \frac{n^2}{k^2b^2})\,J(kb) = 0$$

etc., and also the equation (2), we obtain

$$S'(k_s) = -b\,(1 - \frac{n^2}{k_s^2b^2})[J(k_sb)\,G'(k_sa) - J'(k_sa)\,G(k_sb)] \\ -a\,(1 - \frac{n^2}{k_s^2a^2})[G(k_sa)\,J'(k_sb) - J(k_sa)\,G'(k_sb)]$$

Putting into this expression, the relation

$$\frac{G'(k_sa)}{J'(k_sa)} = \frac{G'(k_sb)}{J'(k_sb)}$$

(which is equivalent to (2)), we find that

$$S'(k_s) = -2k_s\,M(k_s)\frac{J'(k_sb)}{J'(k_sa)} \tag{7}$$

Moreover, we put

$$U(k) = J'(kb)\,G(ka) - G'(kb)\,J(ka)$$

then, for $k = k_s$, we have

$$U(k_s) = J'(k_sb)\,[\,G(k_sa) - \frac{G'(k_sb)}{J'(k_sb)}J(k_sa)\,] \\ = J'(k_sb)\,[\,G(k_sa) - \frac{G'(k_sa)}{J'(k_sa)}J(k_sa)\,] \\ = \frac{J'(k_sb)}{J'(k_sa)}\cdot\frac{1}{k_sa}$$

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2) T. M. Mac Robert, The Asymptotic Expansion of the Confluent Hypergeometric Function, and a Fourier-Bessel Expansion, Proc. Roy. Soc. Edinb., 1922
I. N. Sneddon, Fourier Transforms, 1951, p. 56

So that the relation (7) may also be written

$$S'(k_s) = -2k_s^2 a\, M(k_s)\, U(k_s) \tag{8}$$

III. Contour-Integral Expression for the Sum of the Series, and Proof of the Validity of Expansion

Putting

$$R(k) = J(kr)\, G'(ka) - G(kr)\, J'(ka) \tag{9}$$

we have

$$f(r) = \sum A_s\, R(k_s)$$

where

$$A_s = \frac{1}{M(k_s)} \int_a^b x f(x)\, Q(k_s)\, dx$$

Using the expressions given in the previons section, we have

$$\begin{aligned} f(r) &= \sum \int_a^b x f(x) \frac{Q(k_s) R(k_s)}{M(k_s)}\, dx \\ &= -2a \int_a^b x f(x) \sum \frac{k_s^2\, Q(k_s)\, R(k_s)\, U(k_s)}{S'(k_s)}\, dx \end{aligned} \tag{10}$$

Now, let us consider the contour integral

$$I = \int_E^A F(\zeta)\, d\zeta$$

where

$$F(\zeta) = \frac{ab\zeta^2\, Q(\zeta)\, R(\zeta)\, U(\zeta)}{S(\zeta)} \tag{11}$$

and the integral is to be taken along the real axis from the point $E(\zeta=-M)$ to the point $A(\zeta=+M)$. Since there are poles at $\zeta=\pm k_s$, the contour line of integration is indented with small semicircular arcs about these poles. $F(\zeta)$ being the odd function, the integrals along the straight parts of this contour cancel out. So that we have, as the radius of circular indentation tend to zero; —

$$I = -2\pi i\, ab \sum \frac{k_s^2\, Q(k_s)\, R(k_s)\, U(k_s)}{S'(k_s)} \tag{12}$$

The functions $Q(\zeta)$, $R(\zeta)$, $S(\zeta)$ and $U(\zeta)$ are uniform functions of a complex variable ζ (at least as we confine ourselves to half plane above the real axis), having no singular point except at $\zeta = 0$ or $\zeta = \infty$. $Q(\zeta)$, $R(\zeta)$ and $S(\xi)$ are even functions while $U(\zeta)$ is an odd function of ζ. The first terms of their asymptotic expansions are respectively; —

$$\frac{1}{\zeta\sqrt{xa}} \sin\{\zeta(x-a)\}\ , \qquad \frac{1}{\zeta\sqrt{ra}} \sin\{\zeta(r-a)\}$$

$$\frac{1}{\zeta\sqrt{ab}} \sin\{\zeta(b-a)\}\ , \qquad \frac{1}{\zeta\sqrt{ab}} \cos\{\zeta(b-a)\}$$

The first term of the asymptotic expansion of $F(\zeta)$ is, therefore

$$\theta(\zeta) = -\frac{b}{\sqrt{xr}} \cot\{\zeta(b-a)\} \sin\{\zeta(x-a)\} \sin\{\zeta(r-a)\} \tag{13}$$

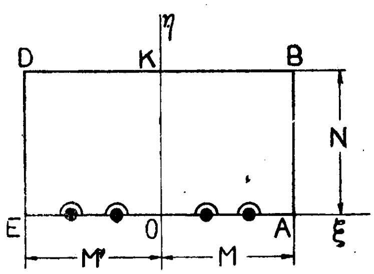

Fig. 1

These circumstances being quite similar to the case treated by Mac Robert, we can, by following his reasoning, which is very ingenious, prove that,

$$\int_E^A F(\zeta)\,d\zeta = \int_E^A \theta(\zeta)\,d\zeta + \varepsilon \tag{14}$$

where $\varepsilon \to 0$ as $M \to \infty$. (For this reasoning, a rectangular domain ABKDEOA as shown in Fig. 1 is considered, where points E, A lie between k_ν and $k_{\nu+1}$, M and N being made to tend to infinity.) The function $\theta(\zeta)$ being identical to that considered by Mac Robert, we can evaluate the integral of the left hand side of the equation (14) by his method (which consists in taking semicircular indentations at the poles of $\theta(\zeta)$, that is at the zeros of $\sin\{\zeta(b-a)\}$.—

$$2\pi i \frac{b}{\sqrt{xr}} \cdot \frac{1}{b-a} \sum \left[\sin\left\{ \frac{s\pi(x-a)}{(b-a)} \right\} \sin\left\{ \frac{s\pi(r-a)}{(b-a)} \right\} \right]$$

$$= \frac{ib}{\sqrt{xr}} \left[\frac{\sin\left\{ \pi(\mu+\frac{1}{2})\frac{x-r}{b-a} \right\}}{2\frac{b-a}{\pi} \sin\frac{\pi(x-r)}{2(b-a)}} - \frac{\sin\left\{ \pi(\mu+\frac{1}{2})\frac{x+r-2a}{b-a} \right\}}{2\frac{b-a}{\pi} \sin\frac{\pi(x+r-2a)}{2(b-a)}} \right]$$

where μ denotes the number of positive zeros of $\sin[\zeta(b-a)]$ between 0 and M.

Finaly, multiply by $xf(x)/(\pi ib)$ and integrate from a to b, and then let μ tend to infinity. Then we see, by (10), (12) and (14), that if $a<r<b$

$$\sum_{s=1}^{\infty} A_s R(k_s) = \frac{1}{2}\{f(r+0) + f(r-0)\}$$

Since the theory of Dirichlet Integral is used in this inference, the usual assumptions as to the nature of the real function must be laid upon $f(x)$, as in ordinary Fourier Series.