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On Vibration of a Cylindrical Shell Immersed in Water

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Abstract

When a cylindrical shell, which is immersed in water makes a vibratory motion, the surrounding water also vibrates, thus causing the so-called effect of virtual mass upon the vibration. The Author has made theoretical formula for estimation of the amount of virtual mass, for several cases in which rigid walls and vibrating shell-walls are arranged in various ways as sketched in Fig. 1.

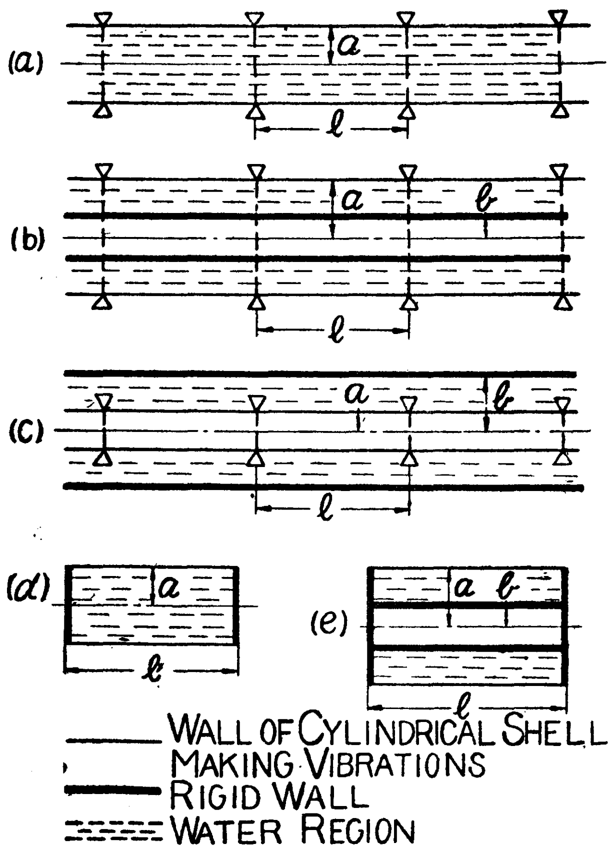

Fig. 1-1

I. Introduction

The Author has previously made studies on the effect of virtual mass of surrounding water upon the vibrations of shells, gratings of flat plates, etc.[2)3)] As a continuation of this theoretical study, the Author here gives theoretical formula for the amount of virtual mass of water, when a cylindrical shell is immersed in a water region and is making a vibratory motion. Various arrangements of cylindrical shell in vibration and rigid boundary walls are shown in Fig. 1.

In this figure the slender lines show wall of shell which is vibrating, while the thick lines show rigid

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2) On Vibration of Cylindrical Shell, which is Filled with Water, This Journal,

3) On Effect of Virtual Mass of a Grating of Flat Plates, This Journal,

boundaries. (a), (b), (c) and (f) correspond to the case of continuous cylindrical shells, supported at equal distances l. (d), (e) and (h) show cases of finite closed bonndary walls. (g) and (i) are cases in which rigid plane boundary walls perpendidular to the axis of the cylindrical shell extend to infinity, water region also extending to infinity.

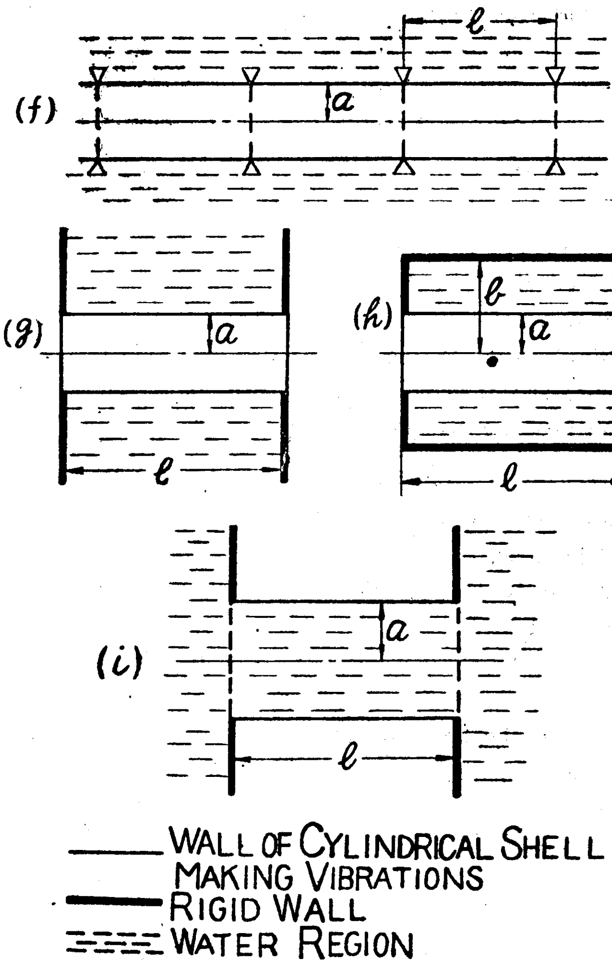

Fig. 1-2

II. General Considerations

As pointed out in previous Reports, we may approximately apply the potential equation $\nabla^2\phi=0$ instead of the wave equation

$$\nabla^2\phi=\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

so long as the wave length in water is large in comparison with the main dimensions of the cylinder.

The radial displacement w at any instant, of the cylindricall shell is expressed by

$$w=W\cos kx\sin n\theta\cos\omega t$$

where $k=\pi/l$, n a whole number giving no. of nodal lines, and ω the angular frequency of vibration.

The fundamental solution ϕ_1 for case (a) is given by

$$\phi_1=AI_n(kr)\cos kx\sin n\theta\sin\omega t \tag{1}$$

where

$$A=-\frac{\omega W}{kI_n'(ka)}$$

While for case (f), it is given by

$$\phi_1 = A\, K_n\,(kr) \cos kx \,\sin n\theta \,\sin\omega t \tag{2}$$

where

$$A = -\frac{\omega W}{kK_n'(ka)}$$

In general, the function ϕ_1 has the form,

$$\phi_1 = AF(r) \cos kx \sin n\theta \sin\omega t \tag{3}$$

For other cases we put $\phi=\phi_1+\phi_2$, and determine ϕ_2 in such a way that on the surface of vibrating shell we have $\partial\phi_2/\partial r=0$, while on the rigid boundary we have $\partial\phi/\partial n=\partial(\phi_1+\phi_2)/\partial n=0$. For cases (b) and (c), the expression for ϕ_2 consisting of a single term can be obtained.

For cases (c), (e) and (h), ϕ_2 are expressed as an infinite series of so-called Fourier-Bessel expansion, in the form : –

$$\phi_2 = \sum B_i \cosh m_i x\, f_i(r) \sin n\theta \sin\omega t \tag{4}$$

In cases (g) and (i), ϕ_2 can be expressed, at least formally, as a form of Fourier-Bessel integral.

Total amount of kinetic energy of water is given by calculating the value of the integral

$$T_w = \frac{1}{2}\int\int\phi \frac{\partial\phi}{\partial n} ds \tag{5}$$

taken over the boundary surface, where $\partial\phi/\partial n$ is the velocity component normal to the surface. The only boundary surface where $\partial\phi/\partial n$ is not zero is the surface of the vibrating surface of the cylindrical shell. On the other hand, we have, on the surface of the shell ;

$$\frac{\partial\phi}{\partial n} = \frac{\partial w}{\partial t} = -W\omega \cos kx \sin n\theta \sin\omega t \tag{6}$$

Putting the values (4) and (6) into (5), we have

$$I_w = \frac{1}{2} AW\omega \left[\frac{l}{2}F(a) + \sum\frac{2k}{(k^2+m_i^2)} \cosh\frac{m_i l}{2} \cdot \frac{B_i}{A} f_i(a)\right] \pi a \sin^2\omega t \tag{7}$$

For kinetic energy of shell wall, which is moving by radial displacement

$$w = W \cos kx \sin n\theta \cos\omega t$$

and whose density is ρ_m and thickness h, we have

$$T_m = \rho_m \omega\; W^2 ah \frac{\pi l}{4} \sin^2\omega t \tag{8}$$

The ratio of virtual mass to actual mass is given by

$$\varepsilon = \frac{T_w}{T_m} = \left(\frac{\rho_w}{\rho_m}\right)\left(\frac{a}{h}\right) K \tag{9}$$

where

$$K = \frac{A}{W\omega} \cdot \frac{1}{a}\left[F(a) + \sum\frac{B_i}{lA} f_i(a)\; \frac{4k}{(k^2+m_i^2)} \cosh\frac{m_i l}{2}\right] \tag{10}$$

and $k=\pi/l$.
We can also construct theoretical formula for the factor $\mathcal{E}$, for the cases in which the function ϕ_2 is given in a form of Fourier-Bessel integral.

III. Theoretical Formula for Virtual Mass in Each Special Cases

In what follows, the results of calculation, according to the method of previous section, of theoretical value of the factor of virtual mass $\mathcal{E}$ for each special cases (a) to (i) as sketched in Fig. 1 will be given.

Case (a) Here we have

$$F(r)=I_n(kr), \qquad A=-\frac{\omega W}{kI_n'(ka)}$$

$$\phi_2\equiv 0$$

$$\mathcal{E}=\left(\frac{\rho_w}{\rho_m}\right)\left(\frac{a}{h}\right)K \qquad \text{where } K=\frac{I_n(ka)}{kaI_n'(ka)} \qquad k=\pi/l$$

Case (b) Here we have

$$F(r)=K_n(kr)I_n'(kb)-K_n'(kb)I_n(kr)$$

$$A=-\frac{\omega W}{k[K_n'(ka)I_n'(kb)-K_n'(kb)I_n'(ka)]}$$

$$\phi_2\equiv 0$$

$$K=\frac{1}{ka}\cdot\frac{K_n(ka)I_n'(kb)-K_n'(kb)I_n(ka)}{K_n'(ka)I_n'(kb)-K_n'(kb)I_n'(ka)}$$

where $a>b$.

Case (c) Here we have the same formula as in the case (b). The only difference is that here $b>a$.

Case (d) This case has been treated fully by the Author in the previous paper.

Case (e) In this case there exist rigid plane walls at $x=\pm l/2$. The velocity potential ϕ is written in the form $\phi=\phi_1+\phi_2$, where

$$\phi_1=A[I_n'(kb)K_n(kr)-K_n'(kb)I_n(kr)]\cos kx\sin n\theta\sin\omega t.$$

$$A=-\frac{\omega W}{k[K_n'(ka)I_n'(kb)-K_n'(kb)I_n'(ka)]}$$

$$\phi_2=\sum B_t\cosh m_t x\, f_t(r)\sin n\theta\sin\omega t$$

$$f_t(r)=J_n(m_t r)Y_n'(m_t a)-J_n'(m_t a)Y_n(m_t r)=T_n(m_t,r)$$

We must have $\partial\phi_2/\partial r=0$ at $r=a$ and $r=b$. Hence

$$J_n'(m_t b)Y_n'(m_t a)-J_n'(m_t a)Y_n'(m_t b)=0 \tag{11}$$

so that the parameters $m_t a=\lambda_t$ must be so chosen as to satisfy the equation

$$\frac{J_n'(\lambda_t)}{J_n'(\lambda_t E)}=\frac{Y_n'(\lambda_t)}{Y_n'(\lambda_t E)} \tag{12}$$

where $E=b/a$.
Moreover, we must have, at $x=\pm l/2$,

$$\frac{\partial\phi_1}{\partial x} + \frac{\partial\phi_2}{\partial x} = 0,$$

so that

$$\sum B_i m_i \sinh\frac{m_i l}{2}[J_n(m_i r)Y_n'(m_i a) - J_n'(m_i a)Y_n(m_i r)]$$

$$= \sum B_i m_i \sinh\frac{m_i l}{2} S_n(m_i, r) = A[I_n(kb)K_n(kr) - I_n(kr)K_n'(kb)]$$

where we have put

$$S_n(m_i, r) = J_n(m_i r)Y_n'(m_i a) - J_n'(m_i a)Y_n(m_i r)$$

This equation means that the right hand side of this equation is expressed as a form of Fourier-Bessel integral[4] Hence we have

$$B_i m_i \sinh\frac{m_i l}{2} = M_i{\cdot}A\int_a^b ug(g)\, S_n(m_i, u)\, dx \tag{13}$$

where

$$g(u) = I_n'(kb)K_n(kr) - I_n(kr)K_n'(kb)$$

Now we have, by a lengthy but easy calculation;

$$\int_a^b ug(u)\, S_n(\lambda, u)du$$

$$= \frac{\lambda}{k}\frac{1}{k^2+\lambda^2}\Big[Y_n'(\lambda a)J_n'(\lambda b) - J_n'(\lambda a)Y_n'(\lambda b)\Big]$$

$$- \frac{k}{\lambda}\frac{1}{k^2+\lambda^2}\Big[I_n'(kb)K_n'(ka) - K_n'(Kb)I_n'(ka)\Big]$$

so that we have, referring to (11),

$$\int_a^b ug(u)\, S_n(m_i, u)du$$

$$= \frac{k}{m_i}\frac{1}{k^2+m_i^2}\Big[\, I_n'(kb)K_n'(ka) - K_n'(kb)I_n'(ka)\Big]$$

Hence

$$B_i = \frac{M_i}{m_i \sinh\frac{m_i l}{2}} \cdot \frac{k}{k^2+m_i^2}\Big[I_n'(kb)K_n'(ka) - K_n'(kb)I_n'(ka)\Big] \tag{13}$$

Thus, knowing the values of coefficients B_i, the virtual mass-factor K can be calculated from the general formula (10), where

$$\begin{aligned} f_i(a) &= S_n(m_i a) \\ &= J_n(m_i a)Y_n'(m_i a) - J_n'(m_i a)Y_n(m_i a) \\ &= \frac{1}{m_i a} \end{aligned}$$

Note : The function $S_n(m_i, r)$ has similar property as the function

4) See Note below.

$$T_n(m_i,r) = J_n(m_i r)Y_n(m_i a) - J_n(m_i a)Y_n(m_i r)$$

(I. N. Sneddon, Fourier Transforms, 1951, §8.3)

Thus we can show, by actual calculation and repeated application of Lommel integrals, that,

$$\int_a^b Q_n(\lambda,u)\,Q_n(\mu,u)\,u\,du = 0$$

for $\lambda = m_i,\ \mu = m_j,\ \ i \neq j.$

Also we obtain by actual calculation,

$$\int_a^b [\,Q_n(\lambda,u)]^2 u\,du$$

$$= \left(1 - \frac{n^2}{\lambda^2 b^2}\right)\Big[\,Y_n'(\lambda a)J_n(\lambda b) - J_n'(\lambda a)Y_n(\lambda b)\Big]^2 + \left(1 + \frac{n^2}{\lambda^2 a^2}\right)\left(\frac{1}{\lambda a}\right)^2 = M_i$$

where $\lambda = m_i.$

For validity of expansion of a function in the form

$$\sum_i A_i\,Q_n(m_i,r)$$

it is necessary to prove the completeness of the set of functions $Q_n(m_i,r)$ ($i = 1, 2, 3, \cdots$) The above discussion stands on the assumption that this set of functions is complete. (See also, W. B. Ford, Studies on Divergent Series and Summability, 1916, Chap. V, III)

Case (f) Here we have

$$F(r) = K_n(kr), \qquad A = -\frac{\omega W}{kK_n'(ka)}$$

$$\phi_2 \equiv 0,$$

$$\varepsilon = \left(\frac{\rho_w}{\rho_m}\right)\left(\frac{a}{h}\right)K, \text{ where } K = \frac{K_n(ka)}{kaK_n'(ka)}$$

Case (g) In this case, rigid plane walls represented by $x=+l/2$ and $x=-l/2$, extend to infinity. For this case we have to put $\phi=\phi_1+\phi_2$. The radial velocity V_r is given by

$$V_r = \frac{\partial\phi}{\partial r} = \frac{\partial\phi_1}{\partial r} + \frac{\partial\phi_2}{\partial r}$$

Now, let us put

$$\phi_2 = \int_a^\infty sf(s)Q_n(s,r)\,ds\cos kx\sin n\theta\sin\omega t \tag{14}$$

where

$$Q_n(s,r) = J_n(sr)Y_n'(sa) - Y_n(sr)J_n'(sa)$$

Since $Q_n'(s,r)=0$, we have $\partial\phi_2/\partial r=0$ at $r=a$. Therefore $\partial\phi/\partial r=\partial\phi_1/\partial r$ at $r=a$.
Next, along the rigid plane walls we must have

$$V_x = \frac{\partial\phi_1}{\partial x} + \frac{\partial\phi_2}{\partial x} = 0 \quad\text{for}\quad x = \pm\frac{l}{2}.$$

This condition requires that

$$kAK_n'(kr) = \int_a^\infty sf(s)Q_n(s,r)\,ds$$

This is an inversion problem quite similar to that treated by **Sneddon.**

Case (h). For this case, the formal solution is quite the same as in Case (e), only difference being that here we have b<a.

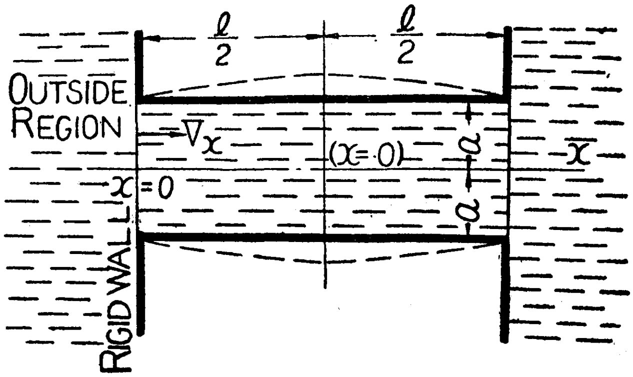

Fig. 2

Case (i) Solution for the case of cylindrical shell with open ends.

By " open ends " we imply that at the end of the shell the water is connected to a water region extending to infinity, as shown in Fig. 2.

In this case, we consider that the water of outside region also makes a vibratory motion with velocity potential ϕ_3.

Taking new origin of coordinates at ends (See Fig. 2) we assume that

$$\phi_3 = \int_0^\infty e^{sx}F(s)\,J_n(sr)ds\,\sin\theta\,\sin\omega t \qquad (15)$$

for the region $x<0$.

For space inside the shell, we take

$$\phi_1 + \phi_2 = A\sin kx\,I_n(kr)\sin n\theta\sin\omega t$$
$$+\sum D_l\cosh m_l\Big(x-\frac{1}{2}l\Big)J_n(m_l r)\,\sin n\theta\sin\omega t \qquad (16)$$

where D_l are unknown constants and m_l the roots of equation $J_n'(ma)=0$, as before. For values of velocity components V_r and V_x at the plane $x=0$, we have two sets of values, the one being given by (15):–

$$(V_x)_0 = \int_0^\infty sF(s)\,J_n(sr)ds\,\sin n\theta\,\sin\omega t dt \qquad (17)$$

$$(V_r)_0 = \int_0^\infty sF(s)\,J_n'(sr)ds\,\sin n\theta\,\sin\omega t dt \qquad (18)$$

while the other set is obtained from (16)

$$(V_x)_0 = kAI_n(kr)\,\sin n\theta\sin\omega t$$
$$+\sum D_l m_l\sinh\Big(-\frac{m_l l}{2}\Big)J_n(m_l r)\,\sin n\theta\sin\omega t \qquad (19)$$

$$(V_r)_0 = \sum D_l m_l\cosh\Big(-\frac{m_l l}{2}\Big)J_n'(m_l r)\,\sin n\theta\sin\omega t \qquad (20)$$

Now, let us compare (17) and (19). We must have

$$\left.\begin{aligned}\int_0^\infty F(s)sJ_n(sr)ds &= kAI_n(kr) - \sum D_i m_i \sinh\frac{m_i l}{2} J_n(m_i r) \\ &\qquad\qquad\qquad \text{for } 0<r<a \\ &= 0, \qquad\qquad \text{for } a<r\end{aligned}\right\} \tag{21}$$

According to a theorem on Fourier-Bessel Integrals* we have in general

$$\begin{aligned}\int_0^\infty ds \int_0^a s\lambda[\Psi(\lambda)] J_n(\lambda s) J_n(sr) d\lambda &= \Psi(r) \quad \text{for } 0<r<a \\ &= 0 \qquad \text{for } a<r\end{aligned}$$

comparing this with (21), we observe that

$$F(s) = \int_0^a \lambda J_n(\lambda s)\Big[kAI_n(k\lambda) - \sum_i D_i m_i \sinh\frac{m_i l}{2} J_n(m_i r) \Big] d\lambda$$

$$\begin{aligned}&= kA \frac{a}{k^2+s^2}[kJ_n(sa) I_n'(ka) - sI_n(ka) J_n'(sa)] \\ &\quad -\sum D_i m_i \sinh\frac{m_i l}{2} \frac{a}{s^2-m_i^2} \{ m_i J_n(sa) J_n'(m_i a) - sJ_n'(sa) J_n(m_i a) \} \\ &= kA \frac{a}{k^2+s^2}[kJ_n(sa) I_n'(ka) - sI_n(ka) J_n'(sa)] \\ &\quad +\sum D_i m_i \sinh\frac{m_i l}{2} \frac{as}{s^2-m_i^2} J_n'(sa) J_n(m_i a)\end{aligned}$$

So that we have by (18), putting $ka=K$,

$$\begin{aligned}(V_r)_0 &= \sin n\theta \sin\omega t\, KA \\ &\quad \int_0^\infty \frac{s}{k^2+s^2}[kJ_n(sa) I_n'(K) - sI_n(K) J_n'(sa)] J_n'(sr)\, ds \\ &\quad + \sum D_i m_i \sinh\frac{m_i l}{2} \int_0^\infty \frac{as^2}{s^2-m_i^2} J_n'(sa) J_n(m_i a) J_n'(sr) ds\end{aligned}$$

and this must coincide with (16). Integrating by r we have

$$\begin{aligned}&\sum D_i \cosh\frac{m_i l}{2} J_n(m_i r) \\ &= KA \int_0^\infty \frac{1}{k^2+s^2}[kJ_n(sa) I_n'(K) - sI_n(K) J_n'(sa)] J_n(sr) ds \\ &\quad + \sum D_i m_i \sinh\frac{m_i l}{2} \int_0^\infty \frac{as}{s^2-m_i^2} J_n'(sa) J_n(m_i a) J_n(sr) ds\end{aligned}$$

Observing that

$$\int_0^a J_n(m_i r) J_n(sr)\, ds = -sJ_n'(sa) J_n(m_i a)$$

we have, by Fourier-Bessel Expansion ;—

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* Gray and Mathews, Bessel Functions, VIII § 3

$$D_i \cosh\frac{m_i l}{2}\cdot\frac{1}{2m_i^2}\left[(m_i^2a^2-n^2)\overline{J_n(m_ia)^2}\right]$$
$$= KA\int_0^\infty \frac{1}{k^2+s^2}\left[kJ_n(sa)I_n'(K)-sI_n(K)J_n'(sa)\right]$$
$$\left[-sJ_n'(sa)J_n(m_ia)\right]ds$$
$$+\sum D_jm_j\sinh\frac{m_jl}{2}\int_0^\infty \frac{as}{s^2+m_j^2}J_n'(sa)J_n(m_ja)$$
$$\left[-sJ_n'(sa)J_n(m_ia)\right]ds \tag{22}$$
$$(i = 1, 2, 3, \cdots\cdots)$$

This being a system of linear simultaneous equations for D_i, these unknown constants are determined.

The amount of total kinetic energy of water can be obtained by taking surface integral of

$$\phi \times (\text{ normal velocity })$$

along all the boundary surface. The rigid plane wall at $x = 0$ ($a<r<\infty$) contributes nothing to the integral, since normal velocity at there is zero. So that the kinetic energy of water is given by the same formula as in previous section, provided we interchange the coefficients B_i by D_i. Therefore, we have, for the effect of virtual mass of water ;—

$$\varepsilon = \frac{T_w}{T_m} = \left(\frac{\rho_w}{\rho_m}\right)\left(\frac{a}{h}\right)\frac{I_n(ka)}{I_n'(ka)\cdot ka}$$
$$\left[1+\sum_i\frac{D_i}{A}\frac{m_ia}{I_n(ka)}\cdot\frac{2k^2\cosh\left(\frac{1}{2}m_il\right)}{(k^2+m_i^2)}\right]$$

For a practical estimation, there remains the question of convergence of series D_i.