On Vibration of a Cylindrical Shell, which is Filled with Water *

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Abstract

The author has made hydrodynamical study on vibratory motion of a cylindrical shell, inside of which water is filled, and has obtained the amount of virtual mass of water for the case in which both ends of the cylinder are fitted with rigid end-plates.

I. Introduction

When a cylindrical shell, inside which water is filled, makes a vibratory motion, the water contained inside the shell also makes vibratory motion. The effect of this vibratory motion of water is to increase the apparent mass of the shell and lower its natural frequency of vibration as compared to values when there is no water. This effect is called the virtual mass of water. In the present paper, the Author gives some results of theoretical estimations of this virtual mass for the case of a vibration of a shell in form of circular cylinder, both ends of which are left free, or are closed with rigid end-plates.

According to the calculation, the effect of rigid end-plates appear as a kind of "end effect", which has significant influence only if the ratio of the length of cylindrical shell to its radius is less than 5.

2. Solution for the Case of a Series of Cylinders

Let us consider a cylindrical shell of radius a and length l, as shown in Fig. 1. We take x-axis along the center line of the shell.

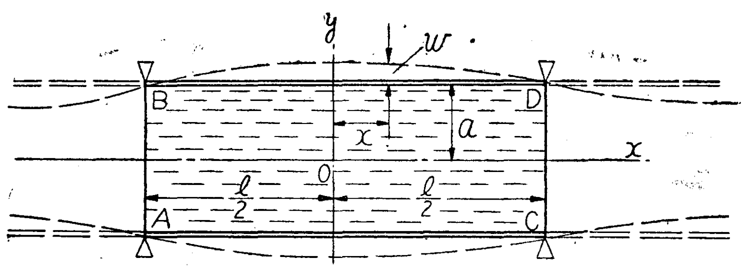

Fig. 1.

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In this section, we treat the case in which there exist a series of cylinders of same dimensions along the x-axis, Fig. 1.showing one of the series.

The shell is supposed to be supported at each ends, that is at $x = \pm l/2$, and make vibration with frequency f. The angular frequency corresponding to it is $\omega = 2\pi f$.

When the water is filled in the shell, the water will also make vibratory motion, with the same angular frequency ω. If the travelling velocity of pressure (or sound) wave in water is c, the wave length is given by $\lambda = c/f$. For example, if $c = 1460$ m/sec. and $f = 50$/sec., then $\lambda = 1460 \div 50 = 29.2$ m. Usually the dimensions $2a$ or l of the shell is small in comparison with the wave length.

Now, the vibratory motion of the water is to be determined from the equation of wave motion ; –

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{1}$$

But, if the wave length λ is very large in comparison to the main dimensions of the shell, we may use the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0\,, \quad \text{or} \quad \nabla^2\phi = 0 \tag{2}$$

instead of the above equation (1), which correspond to neglecting the compressibility of water. ϕ is the velocity potential.

A solution of equation (2) in terms of cylindrical co-ordinates (r, θ, x) may be written

$$\phi_1 = A\cos kx \sin n\theta \; I_n(kr)\sin\omega t \tag{3}$$

Corresponding to this value of ϕ_1 we have

$$\left.\begin{aligned} V_r &= \frac{\partial\phi_1}{\partial r} = kA\cos kx \sin n\theta \; I_n'(kr)\sin\omega t \\ V_x &= \frac{\partial\phi_1}{\partial x} = -kA\sin kx\sin n\theta \; I_n(kr)\sin\omega t \end{aligned}\right\} \tag{4}$$

Now, let us assume that the wall of the shell is making vibration, its radial displacement w being given by

$$w = W\cos kx \sin n\theta \cos\omega t \tag{5}$$

where W is a constant. If the water is vibrating together with the shell, we must have at $r = a$, $V_r = \partial w/\partial t$. Whence we have by (4) : –

$$A = -\frac{\omega W}{kI_n'(kr_0)}$$

In what follows, we shall take up the case in which $k = \pi\, l$, but by slight change of expression, we can estimate the case in which $k = m\pi\, l$, where m is an odd integer.

The kinetic energy at any instant of water contained in one span ($x = -l/2$ to $+l/2$) of the cylinder is given by

$$T_w = -\rho_w \int_{\theta=0}^{2\pi} a\, d\theta \int_{x=0}^{2\pi/k} \left[\phi \frac{\partial w}{\partial t} \right] dx$$

where ρ_w is the density of water.

Or, putting the value of ϕ_1 as given by (3): —

$$T_w = \rho_w \frac{\omega^2 a}{k} W^2 \frac{I_n(ka)}{I_n'(ka)} \frac{\pi^2}{k} \sin^2 \omega t$$

While the amount of kinetic energy of wall of shell, whose thickness is $2h$ and density ρ_m, is given by

$$T_m = \rho_m \int_{\theta=0}^{2\pi} a\, d\theta \int_{x=0}^{2\pi/k} \left(\frac{\partial w}{\partial t} \right)^2 \cdot \frac{2h}{2} \cdot dx$$

$$= \rho_m \omega^2 W^2 ah \frac{\pi^2}{k} \sin^2 \omega t$$

And the ratio T_w/T_m is

$$\varepsilon = \frac{T_w}{T_m} = \left(\frac{\rho_w}{\rho_m} \right) \left(\frac{a}{h} \right) \frac{I_n(K)}{nI_n(K)+KI_{n+1}(K)} \qquad (6)$$

where we have put $K = ka = \pi a/l$.

If the water exist outside of the shell (instead of inside) and extend to infinity, while the inside of the shell is vacant, we should have, instead of (3),

$$\phi_1 = A \cos kx \sin n\theta\, K_n(kr) \sin \omega t \qquad (3')$$

and, by making the similar calculations, we obtain

$$\varepsilon' = \frac{T_w}{T_m} = \left(\frac{\rho_w}{\rho_m} \right) \left(\frac{a}{h} \right) \left[\frac{-K_n(K)}{nK_n(K)-KK_{n+1}(K)} \right] \qquad (7)$$

If the water exist both inside and outside of the shell, we must take the sum $\varepsilon+\varepsilon'$ of the above two values in order to obtain the ratio T_w/T_m. In Table I, some values of the coefficients

$$M_1 = \frac{I_n(K)}{nI_n(K)+KI_{n+1}(K)}$$

$$M_1' = \frac{-K_n(K)}{nK_n(K)-KK_{n+1}(K)}$$

are shown.

Table 1. Values of M_1 and M_1'

		K=0	K=0.2	K=0.4	K=0.6
M_1	n=2	0.5000	0.4993	0.4935	0.4566
	3	0.3333	0.3328	0.3311	0.3284
	4	0.2500	0.2498	0.2490	0.2478
M_1'	n=2	0.5000	0.4952	0.4865	0.4645
	3	0.3333	0.3322	0.3286	0.3242
	4	0.2500	0.2496	0.2483	0.2464

3. Solution for the Case of Cylindrical Shell with rigid Bottom Plates.

With the above mentioned value of ϕ_1, the axial velocity V_x of water at ends $x = \pm l/2$ have some values and are not identically zero. If we wish to obtain the solution for the case of rigid end-plates, we must add to ϕ_1 another potential ϕ_2 such that,

(a) $\nabla^2 \phi_2 = 0$, throughout the inside of the shell

(b) at $r = a$, $\frac{\partial \phi_2}{\partial r} = 0$

(c) at $x = \pm \frac{l}{2}$, $\frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial x} = 0$.

We take

$$\phi_2 = \sum_i B_i \cosh m_i x \sin n\theta \, J_n(m_i r) \sin \omega t \qquad (i=1, 2, 3, \cdots\cdots\cdots) \tag{8}$$

which satisty the equation $\nabla^2 \phi_2 = 0$.

According to the condition (b), we must have

$$J_n'(m_i a) = 0$$

According to the condition (c), we have, at $x = \pm \frac{l}{2}$,

$$- A\, k \sin kx \sin n\theta \, I_n(kr) + \sum_i B_i m_i \sinh m_i x \sin n\theta \, J_n(m_i r) = 0$$

If we put for shortness,

$$B_i m_i \sinh m_i \frac{l}{2} = A \frac{\pi}{l} C_i ,$$

the above equation reduces to: —

$$I_n(\pi \frac{r}{l}) = \sum_i C_i J_n(m_i r) \qquad 0 \leqq r \leqq a \tag{9}$$

which means that, when we expand the function at the left hand side of this equation as a series of Bessel functions $J_n(m_i r)$, the coefficients of expansion are C_i.

Now, according to the known formula *

$$\int_0^a J_n(m_i r) J_n(m_j r)\, r dr = 0 \qquad (j \neq i)$$

$$\int_0^a [J_n(m_i r)]^2\, r dr = \frac{1}{2m_i^2} \left[(m_i^2 a^2 - n^2) \overline{J_n(m_i a)^2} \right]$$

the coefficients C_i can be easily determined thus; —

$$\frac{C_i^2}{2m_i^2} \left[(m_i^2 a^2 - n^2) \overline{J_n(m_i a)^2} \right] = \int_0^a I_n(kr) J_n(m_i r)\, r dr \quad .$$

* H. Lamb, Hydrodynamics, Sect. 191.

$$= \frac{2}{k^2+m_i^2}\Big[kJ_n(m_ia)I_n'(ka) - m_iI_n(ka)J_n'(m_ia)\Big]^*$$

$$= \frac{a}{k^2+m_i^2}\Big[kJ_n(m_ia)I_n'(ka)\Big]$$

Knowing thus the value of constants C_i, the values of B_i can be calculated, and we have

$$B_i = A\frac{2m_ik^2a}{\sinh(\frac{1}{2}m_il)}\cdot\frac{1}{(k^2+m_i^2)(m_i^2a^2-n^2)}\cdot\frac{I_n'(ka)}{J_n(m_ia)}$$

The value of kinetic energy of water contained in the region $x = \pm l\,2$ is,

$$T_w = -\rho_w\int_{\theta=0}^{2\pi} a\,d\theta\int_{x=-l/2}^{x=+l/2}\Big[(\phi_1+\phi_2)\Big]\,dx$$

Putting the values of ϕ_1 and ϕ_2 into this expression, and utilizing the formula

$$\int_{-l/2}^{+l/2}\cos kx\cosh m_ix\,dx$$

$$= \frac{1}{2}\left|\frac{exp(m_ix)[m_i\cos kx + k\sin kx]}{(k^2+m_i^2)}\right|$$

$$= \frac{2k}{k^2+m_i^2}\cosh(\frac{1}{2}m_il)$$

we have at last

$$T_w = \rho_w a\frac{W^2}{k}\frac{I_n(ka)}{I_n'(ka)}\omega^2\pi\cdot$$

$$\left[\frac{2}{l}\sum_i\frac{B_i}{A}\frac{J_n(m_ia)}{I_n(ka)}\cdot\frac{2k\cosh(\frac{1}{2}m_il)}{k^2+m_i^2}\right]$$

Comparing this value with the expression for T_m as obtained in the previous section, we have,

$$\varepsilon = \frac{T_w}{T_m} = \left(\frac{\rho_w}{\rho_m}\right)\left(\frac{a}{h}\right)\frac{I_n(ka)}{I_n'(ka)\,ka}.$$

$$\left[1+\sum_i\frac{B_i}{A}\frac{m_ia}{I_n(ka)}\cdot\frac{2k^2\cosh(\frac{1}{2}m_il)}{(k^2+m_i^2)}\right] \qquad (10)$$

This expression can be transformed to a more convenient form, as follows; —

$$\varepsilon = \frac{\rho_w}{\rho_m}\cdot\frac{a}{h}\cdot M_2$$

where we have

$$M_2 = M_1 + \sum_i\frac{\left(\frac{4l}{\pi a}\right)\xi_i}{\left[1+\left(\xi_i\frac{l}{\pi a}\right)^2\right]^2}\cdot\frac{\coth\frac{\xi_il}{2a}}{(\xi_i^2-n^2)}$$

where M_1 is the value obtained in the previous section. ξ_i are the roots of the equation

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* Mac Lachlan, Bessel Functions for Engineers, p. 115.

$$J_n'(\xi) = 0$$

and have following values: —

$n = 2$	$\xi_1 = 3.04$	$\xi_2 = 6.7$	$\xi_3 = 7.9$
$n = 3$	$\xi_1 = 4.3$	$\xi_2 = 8.0$	$\xi_3 = 11.4$
$n = 4$	$\xi_1 = 5.3$	$\xi_2 = 9.3$	....

The values of M_1 and M_2 for a range of values of l/a up to 5 are shown in Fig. 2.

From the figure we see that for $l/a > 5$, we may take M_1, M_2 to be approximately equal to unity.

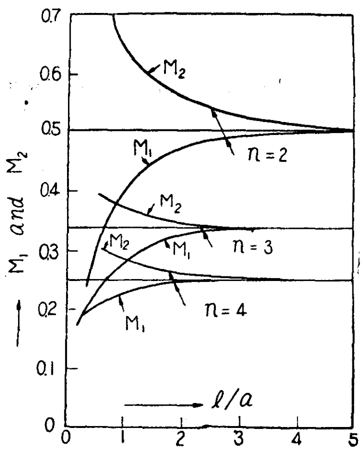

Fig. 2. Values of M_1 and M_2