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Transient Characteristics of Vacuum Tube Oscillators

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Abstract

Van der Pol discussed the mechanism of self sustained oscillation by the equation which is called by his name.

And since van der Pol, a number of researches have been reported on self sustained oscillations expressed by the van der Pol's differential equation. Nothing seems to be added to them.

However, the auther discussed van der Pol's equation in a modified form and have obtained the actual data about the transient as well as the steady characteristics of this oscillation.

I. Preliminary Remarks

There are many cases of electrical oscillatory systems which are led to the following differential equation;

$$LC\frac{d^2V}{dt^2} + RC\frac{dV}{dt} + V = M\frac{di}{dt} \tag{1}$$

where V denotes the oscillatory voltage in the grid side of vacuum tube oscillator. Equation (1) may be held in the cases whether plate tunning or grid tunning oscillator; L, C, R denote the inductance, capacitance and resistance of the oscillatory component respectively, M is the mutual inductance between grid and plate coil and i is the plate current, which can be expressed as follows;

$$i = g_1 V - g_3 V^3 \ , \qquad g_1 > 0 \ , \ g_3 > 0 \tag{2}$$

We can replace i in eq. (1) in terms of V by using eq. (2) and introducting the following new variables

$$\tau = \omega_0 t \qquad V = V_0 x \tag{3}$$

where ω_0 is a natural frequency of the oscillatory system that is $\omega_0^2 = 1/LC$, V_0 is a quantity which has a dimension of Volt and to be determined from eq. (4) such as;

$$3\,\omega_0 g_3 M V_0^2 = 1 \tag{4}$$

Further we introduce a parameter n defined by

$$-n = \omega_0 (RC - g_1 M) \qquad n > 0 \ , \ M > 0 \tag{5}$$

Here we have assumed that $-n < 0$; this condition is very essential otherwise no self excited oscillation would be posible.

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Substituting the relations (2), (3), (4) and (5) into eq, (1) we shall obtain the following equation.

$$\frac{d^2x}{d\tau^2} + (-n + x^2)\frac{dx}{d\tau} + x = 0 \qquad (6)$$

This is the basic equation in our treatment and is one of the modified forms of van der Pol equation. Since time τ does not occur explicitely in eq. (6), it is possible to reduce the equation to one of first order by introducing $dx/d\tau = y$ as a new variable following the well known procedure in the non lineary theory of oscillation.

In this way we obtain the equation

$$\frac{dy}{dx} = n - x^2 - \frac{x}{y} \qquad (7)$$

It is not possible to obtain the solution curves in the x, y-plane by explicit integration, but the approximate solution curves can be drawn with various methods. If they are once determined in sufficient detail, it would be very useful to characterize the feature of the oscillation.

And the qualitative charactors of the solution can be obtained from the characteristic equation even though the solution curves themselves could not be obtained explicitly.

The characteristic equation will be expressed by the following equation from eq. (7)

$$S^2 - nS + 1 = 0 \qquad (8)$$

It is concluded from this equation that the parameter n must be limited as follows:

$$n < 2 \qquad (9)$$

Under this condition the solution curves of eq. (7) are spirals which start from the point near the origin and will approach to a definite closed curve in the x, y-plane which is called "Limit Cycle". Existence of the closed curve, limit cycle, in the x, y-plane shows that eq. (6) has a periodic solution.

If the value of n becomes larger than 2, then relaxation oscillation will take place.

The time τ, which will elapse while y varies from y_1 to y_2 on the solution curve, is expressed by

$$\tau = \int_{y_1}^{y_2} \frac{1}{y}\,dx \qquad (10)$$

Before concluding our discussion something about analytical treatment must be added.

If we introduce polar coordinates in the x, y-plane that is

$$x = K\sin\theta\ , \qquad y = K\cos\theta$$

we obtain the following equation from eq. (6)

$$\frac{dK}{d\tau} = \frac{K}{2}(n - \frac{K^2}{4}) + \frac{nK}{2}\cos 2\theta + \frac{K^3}{8}\cos 4\theta$$

$$\frac{d\theta}{d\tau} = 1 - \frac{1}{2}(n - \frac{K^2}{2})\sin 2\theta - \frac{K^2}{8}\sin 4\theta$$

As a first approximation let us take a mean value terms in the right hand side of these equations, that is

$$\frac{dK}{d\tau} = \frac{K}{2}(n - \frac{K^2}{4}) \tag{11}$$

$$\frac{d\theta}{d\tau} - 1 \tag{12}$$

From these equations K and τ are given by

$$K = \sqrt{\frac{4n}{1 + Ce^{-n\tau}}} \tag{13}$$
$$\theta = \tau$$

After sufficient lapse of time τ, K will approach to the value $2\sqrt{n}$, which represents the amplitude of the oscillation in a steady states and C is a integration constant which is determined from the initial condition $\tau = 0$, $K = K_0$, that is

$$C = \frac{4n}{K_0^2} - 1 \fallingdotseq \frac{4n}{K_0^2} \tag{15}$$

After time τ_0 determined from

$$Ce^{-n\tau_0} = 1$$

K will be built up to the value $\sqrt{2n}$ and this τ_0 serves as a measure of the quickness of building up of the oscillations. Such a time is given from the above relations, that is,

$$\tau_0 = \frac{2}{n}\log\frac{2\sqrt{n}}{K_0} \tag{16}$$

Some times it is convenient to compare the number of waves which exist in the transient duration of the oscillation.

For this purpose it is convenient to use the ratio defined by the following formula,

$$\Lambda = \frac{\tau_1}{T_s} = \frac{4.6}{n\,T_s}(1 + \log_{10}\frac{2\sqrt{n}}{K_0}) \tag{17}$$

where T_s denotes the period of the oscillation in the steady states and τ_1 is determined from

$$Ce^{-n\tau_1} = 0.01$$

In other words after the lapse of time τ_1, K will take a value of 99.5 per cent of the steady states value.

II. Nummerical Calculation and Characteristic Deta of the Oscillations

Since it is impossible, as mentioned above, to obtain the solution curves of eq. (7) by direct integration, the author adopt an approximate integration method as follws:

Coordinates axis are divided into small intervals such as

$$x_{m+1} - x_m = \Delta x_m \quad , \quad y_{m+1} - y_m = \Delta y_m$$

Then tangent of the m-th segment which pass through a point (x_m, y_m) is determined from eq: (7)

$$\frac{\Delta y_m}{\Delta x_m} = - x_m^2 - \frac{x_m}{y_m} = C_m$$

It is quite reasonable to consider that C_m changes its value in an equal difference in the neighbourhood of the considering point:

$$C_{m+1} - C_m = C_m - C_{m-1}$$

or

$$C_{m+1} = 2C_m - C_{m-1}$$

If we extend the m-th segment until it intersects with adjacent interval $x = x_{m+1}$, it should be tangent which is determined by C_{m+1}.

Therefore it is more probable to draw the segment which has the tangent given by the arithmetical mean of C_m and C_{m+1} through the point x_m, y_m that is;

$$\frac{\Delta y_m}{\Delta x_m} = \frac{1}{2}(C_m + C_{m+1}) = \frac{1}{2}(3C_m - C_{m-1})$$

Therefore

$$y_{m+1} = y_m + \frac{1}{2}(3C_m - C_{m-1})\Delta x_m$$

And again from this relation C_{m+1} is determined as follows;

$$\frac{\Delta y_{m+1}}{\Delta x_{m+1}} = n - x_{m+1}^2 - \frac{x_{m+1}}{y_{m+1}} = C_{m+1}$$

and the value of this C_{m+1} must coincide with the value $2C_m - C_{m-1}$ within a sufficient accuracy by taking the intervals small enough.

Thus starting from a point that is given by initial conditions we can construct the approximate, but sufficiently detailed, solution curve in such a way as mentioned above.

Some examples of the calculations are as follows:

Table 1. Example of nummerical calculation

$n = 0,1$

x	y	x^2	x/y	$\Delta y/\Delta x$
0	0.10000	0	0	0.10000
0.001	0.10010	0.00000	0.00999	0.09000
0.002	0.10019	0.00000	0.01996	0.08003
0.003	0.10027	0.00001	0.02992	0.07006

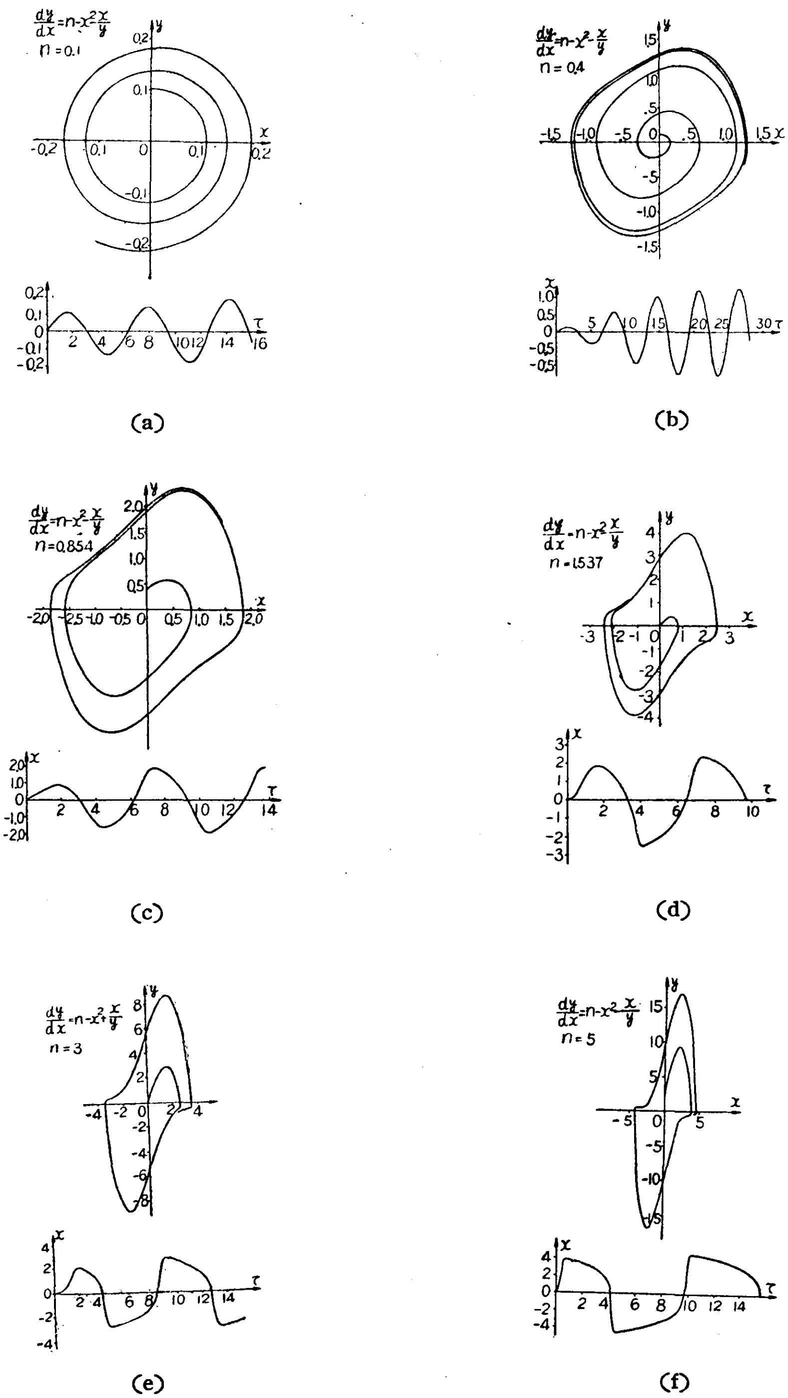

Fig. 1. Solution Curves for various value of *n*

Since τ is expressed by eq. (10), in our approximate calculation τ is given by the area.

$$\Delta\tau = \frac{1}{y}\,\Delta x$$

In the actual treatment scale is so chosen as the area 1 cm² correspond to 0.002 in τ.

One example of these calculation is shown in Table 2.

Table 2. Example of nummerical calculation

$n = 0.4$

x	$\Delta\tau$	τ	x	$\Delta\tau$	τ
0	–	0	0.03	0.0926	0.2850
0.01	0.0976	0.0976	0.04	0.0910	0.3760
0.02	0.0948	0.1948	0.05	0.0904	0.4664

The solution curves and the actual wave forms are shown in Fig. 1 (from a to f), for various values of paramater n, in these cases where $n < 2$ the oscillations are normal and where $n \geqq 2$ they are relaxation oscillations as mentioned in part 1. Next let us consider about the wave forms of these oscillations in a steady states.

As is evident, x is a odd function of τ in the steady states as seen from the figures (Fig. 1 from a to f), so the following relation must be satisfied;

$$x(\tau + \pi) = -x(\tau)$$

Consequently x will be expressed as follows:

$$x = a_1 \sin\tau + a_3 \sin 3\tau + \cdots\cdots\cdots\cdots + b_1 \cos\tau + b_3 \cos 3\tau + \cdots\cdots\cdots\cdots$$

These coefficients of Fourier expansion will be determined approximately by the following method. Dividing a half cycle interval (π) into m sections we can determine a definite value of x corresponding to the mid point of each section,

$$a_j = \frac{2}{m}\sum_{\nu=1}^{m} x_\nu \sin\frac{j\nu\pi}{m}$$

$$b_j = \frac{2}{m}\sum_{\nu=1}^{m} x_\nu \cos\frac{j\nu\pi}{m}$$

Then the distortion factor is defined by

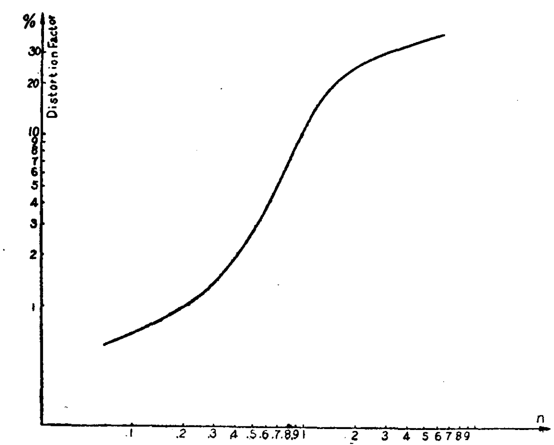

Fig. 2. Distortion factor

$$\text{Distortion factor} = \frac{\sqrt{A_3^2 + A_5^2 + \cdots\cdots + A_j^2 + \cdots\cdots}}{A_1}$$

where $$A_j^2 = a_j^2 + b_j^2$$

In our calculation the interval are divided into fifteen sections ($m = 15$) and higher harmonics are taken into account up to the seventh ($j = 7$).

The calculated results from the above relations with various values of n are shown in Table 3.

Table 3.
Amplitude of harmonics and distortion factors with various values of n

n	A_1	A_3	A_5	A_7	Dist. Fact. %
0,100	0.1957	0.00098	0.00079	0.000437	0.68
0.400	1.2670	0.01689	0.00532	0.000618	1.81
0.854	1.8960	0.12769	0.0550	0.02410	7.49
1.537	2.4400	0.5180	0.1110	0.02640	21.7
3.000	3.5940	0.8647	0.3914	0.20960	27.1
5.000	4.6930	1.5050	0.6956	0.44530	38.2

Distortion facters are also shown in Fig. 2.
Some remarks about the period of the oscillations must be added.
In a steady states of the oscillations period T_s is defined as follows;

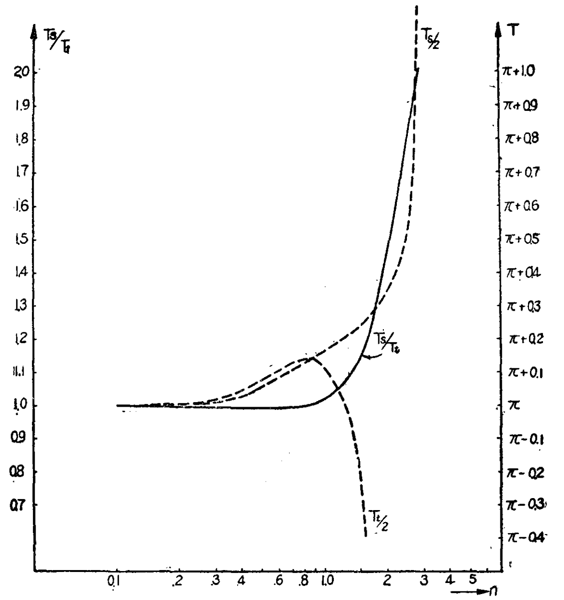

Fig. 3. Transient and steady states half period of van der Pal oscillation

$$x(\tau + T_s) - x(\tau) = 0$$
$$y(\tau + T_s) - y(\tau) = 0$$

But during the transient states of the oscillations there is no period which satisfy the above relations. So, something about the the period in the transient states must be added.

Suppose that the solution curve cut the positive y axis at time τ_j and pass through the positive x axis at the point (K_j, O) then the solution curve cut the negative y axis at time τ_{j+1}. The time interval $\tau_{j+1} - \tau_j$ may be "transient half period" and is expressed by

$T_t/2$, and T_t may be called as a "**transient period**".

It should be noticed here, that this transient periods depend upon the value K_j which denotes the peak value of x in the half cycle.

Calculated values of them are shown in Table 4 for various values of n, in which K_j is taken one tenth of its steady states value determined from the limit cycle.

Table 4. Transient half periods for value of n

n	$T_s/2$	$T_t/2$	T_s/T_t
0.100	π	π	1
0.400	3.168	3.189	0.993
0.854	3.289	3.290	0.999
1.537	3.390	2.900	1.168
3.000	4.437	2.035	2.159
5.000	5.740	1.463	3.922

Figure 3 shows the half period of steady and transient states interpolated from the above calculated value.

It is interesting that the ratio T_s/T_t becomes approximately unity when n takes the value equal to 0.854 (see Fig. 3). Wave numbers Λ during the transient states of the oscillations defind by eq. (17) are shown in Figure (4). The characteristic data of the oscillation expressed by van der Pol's equation have been given in our discussions as mentioned above.

They will be useful in designing the oscillators which are driven by pulsive wave forms.

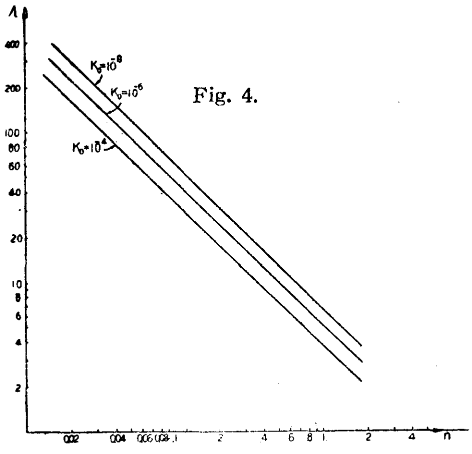

Fig. 4.

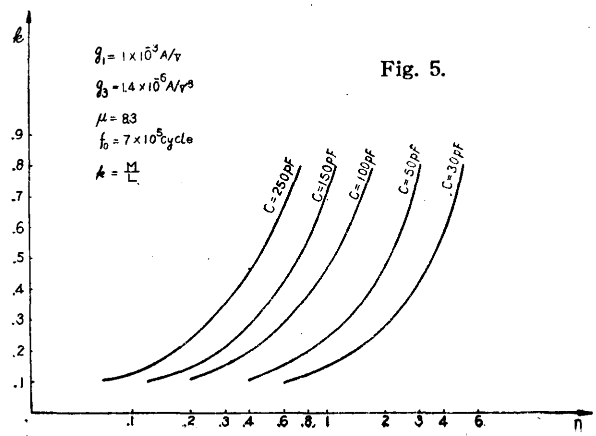

Fig. 5.

Fig. 4. Wave numbers contained in transient duration

Fig. 5. Relations between coupling coefficient k=M/L and parameter n

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