Title	On the lateral vibrations of trapezoidal cantilevers of uniform thickness
Sub Title	
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Publisher	慶應義塾大学藤原記念工学部
Publication year	1951
Jtitle	Proceedings of the Fujihara Memorial Faculty of Engineering Keio University Vol.4, No.12 (1951.) ,p.13(13)- 19(19)
JaLC DOI	
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Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00040012-0013

On the Lateral Vibrations of Trapezoidal Cantilevers of Uniform Thickness

(Received March 4, 1952)

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Abstract

The author has previously made a theoretical analysis for the frequencies of the lateral vibrations of trapezoidal cantilevers of uniform thickness by the Rayleigh's method and examined the variations of the frequencies by the degrees of convergence or divergence expressed by the relation $c = (b_0 - b_1)/b_0$, where b_0 and b_1 being the breadth of the cantilever at the fixed end and the free end respectively.

The author in the present paper has obtained the exact solutions of the same problem by means of hypergeometric functions. The results obtained are compared with the results obtained approximately by the Rayleigh's method for a numerical example to find the both results in close agreements.

I. Introduction

The author previously has made a theoretical analysis on the lateral vibrations of trapezoidal cantilevers of uniform thickness but varying degrees of convergence or divergence expressed by a factor $c = (b_0 - b_1)/b_0$, where b_0 and b_1 represent the breadth of the cantilever at the fixed end and the free end respectively. The analysis was performed with approximate calculations by the Rayleigh's method.[1] The author, further, has conducted some experiments to find the results in close agreement with theoretical results.[2]

In the present paper, the author has solved the problem by means of hypergeometric functions. The results obtained are compared with the results obtained approximately by the Rayleigh's method for a numerical example to find the both results in close agreements. The author wishes to express his thanks to Prof. Toyotaro Suhara who gave him many a valuable suggestion and advice.

II. Theoretical Considerations

Fig. 1 represents the trapezoidal cantilever of uniform thickness h, the breadth

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1) Report published by the "Long Column Research Committee", composed mainly of the members of Faculty of Engineering of Waseda University and Keio University, 1951, pp. 133 / 144.

2) The same as 1). pp. 145 / 150.

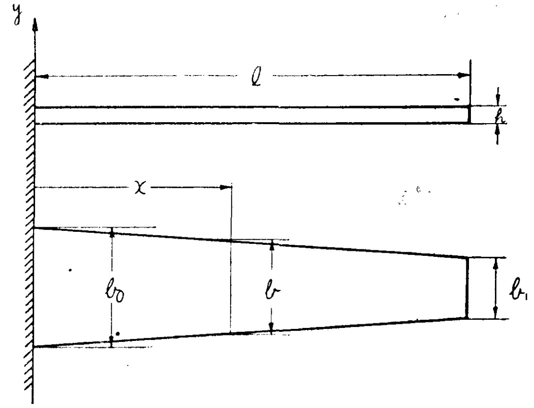

Fig. 1. Trapezoidal cantilever with uniform thickness

of which being b_0 and b_1 at the fixed end and the free end respectively. The equation of the lateral vibration is

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) + \rho A\frac{\partial^2 y}{\partial t^2} = 0 \tag{1}$$

If we put

$$y = u\cos pt\,,\quad b = b_0\left(1 - c\frac{x}{l}\right),\quad c = \frac{b_0 - b_1}{b_0}\,,$$

$$A = h\,b_0\left(1 - c\frac{x}{l}\right),\quad I = \frac{1}{12}bh^3 = \frac{1}{12}h^3 b_0\left(1 - c\frac{x}{l}\right),$$

the above equation yields to

$$\left(1 - \frac{c}{l}x\right)\frac{d^4u}{dx^4} - 2\frac{c}{l}\frac{d^3u}{dx^3} - \rho p^2\frac{12}{Eh^2}\left(1 - \frac{c}{l}x\right)u = 0 \tag{1.1}$$

Further, if we put

$$\frac{c}{l} = \lambda\,,\quad \rho p^2\frac{12}{Eh^2} = B\,,$$

the eq. (1.1) may be written as follows.

$$\left(1 - \lambda x\right)\frac{d^4u}{dx^4} - 2\lambda\frac{d^3u}{dx^3} - B\left(1 - \lambda x\right)u = 0 \tag{1.2}$$

Substituting the relation of $1 - \lambda x = \xi$ in the above equation, we get

$$\xi^4\frac{d^4u}{d\xi^4} + 2\xi^3\frac{d^3u}{d\xi^3} - \frac{B}{\lambda^4}\xi^4 u = 0 \tag{1.3}$$

Denoting $\vartheta_\xi = \xi\frac{d}{d\xi}$, the eq. (1.3) reduces to

$$\vartheta_\xi(\vartheta_\xi - 1)^2(\vartheta_\xi - 2)\,u - \frac{B}{\lambda^4}\xi^4 u = 0 \tag{1.4}$$

Putting $e = B/\lambda^4$, $z = e(\xi^4/4^4)$ in the above equation, we obtain finally

$$\vartheta_z(\vartheta_z - \frac{1}{4})^2(\vartheta_z - \frac{1}{2})\,u - zu = 0 \tag{1.5}$$

The indicial equation of the eq. (1.5) will be

$$\rho\,(\rho-\frac{1}{4})^2(\rho-\frac{1}{2})=0 \qquad (2)$$

from which $\rho = 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}$ may be obtained.

Putting

$$u=\sum_{m=0}^{\infty} c_m z^{\rho+m} = c_0 z^{\rho} + c_1 z^{\rho+1} + \cdots\cdots\cdots + c_m z^{\rho+m} ,$$

the relations between the coefficients reduce to

$$\left.\begin{aligned} c_0 &= c_1(\rho+1)(\rho+1-\frac{1}{4})^2(\rho+1-\frac{1}{2}) \\ c_{m-1} &= c_m(\rho+m)(\rho+m-\frac{1}{4})^2(\rho+m-\frac{1}{2}) \end{aligned}\right\} \qquad (3)$$

Therefore, the first two solutions may be written in the following manner.

$$u_1 = {}_0F_3(1, \frac{3}{4}, \frac{3}{4}, \frac{1}{2}; z) \qquad (4)$$

$$u_2 = z^{\frac{1}{4}}\ {}_0F_3(\frac{5}{4}, 1, 1, \frac{3}{4}; z) \qquad (5)$$

As to the third solution u_3, we proceed as follows.

$$u = \sum_{m=0}^{\infty} c_m z^{\rho+m} = \sum_{m=0} \frac{c_{m-1} z^{\rho+m}}{(\rho+m)(\rho-\frac{1}{4}+m)^2(\rho-\frac{1}{2}+m)}$$

$$= \frac{c_0 z^{\rho+1}}{(\rho+1)(\rho+\frac{1}{4}+1)^2(\rho-\frac{1}{2}+1)} + \frac{c_1 z^{\rho+2}}{(\rho+2)(\rho-\frac{1}{4}+2)^2(\rho-\frac{1}{2}+2)} + \cdots\cdots\cdots$$

$$+ \frac{c_{m-1} z^{\rho+m}}{(\rho+m)(\rho-\frac{1}{4}+m)^2(\rho-\frac{1}{2}+m)} + \cdots\cdots\cdots$$

$$= \sum_{m=0}^{\infty} \frac{c_0 z^{\rho+m}}{\begin{Bmatrix} (\rho+1)(\rho+1-\frac{1}{4})^2(\rho+1-\frac{1}{2}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (\rho+m)(\rho+m-\frac{1}{4})^2(\rho+m-\frac{1}{2}) \end{Bmatrix}} \qquad (6)$$

Hence, we obtain immediately

$$\frac{\partial u}{\partial \rho} = \frac{\partial}{\partial \rho} \sum_{m=0}^{\infty} \frac{\Gamma(\rho+1)\left\{\Gamma(\rho-\frac{1}{4}+1)\right\}^2\Gamma(\rho-\frac{1}{2}+1)}{\Gamma(\rho+m+1)\left\{\Gamma(\rho-\frac{1}{4}+m+1)\right\}^2\Gamma(\rho-\frac{1}{2}+m+1)}\, c_0 z^{\rho+m}$$

If we put further

$$s=\sum_{m=0}^{\infty}\frac{\Gamma(\rho+1)\left\{\Gamma(\rho-\frac{1}{4}+1)\right\}^2\Gamma(\rho-\frac{1}{2}+1)}{\Gamma(\rho+m+1)\left\{\Gamma(\rho-\frac{1}{4}+m+1)\right\}^2\Gamma(\rho-\frac{1}{2}+m+1)}c_0z^{\rho+m},$$

the following two expressions may readily be obtained.

$$\log s=\sum_{m=0}^{\infty}\Big[\log\Gamma(\rho+1)+2\log\Gamma(\rho-\frac{1}{4}+1)+\log\Gamma(\rho-\frac{1}{2}+1)$$
$$-\log\Gamma(\rho+m-1)-2\log\Gamma(\rho-\frac{1}{4}+m+1)-\log\Gamma(\rho-\frac{1}{2}+m+1)$$
$$+\log(c_0z^{\rho+m})\Big],$$

$$\frac{d}{d\rho}\log s=\sum_{m=0}^{\infty}\Big[\Psi(\rho)+2\Psi(\rho-\frac{1}{4})+\Psi(\rho-\frac{1}{2})-\Psi(\rho+m)-2\Psi(\rho-\frac{1}{4}+m)$$
$$-\Psi(\rho-\frac{1}{2}+m)+\log z\Big],$$

where $\Psi(x)=\dfrac{d}{dx}\log\Gamma(1+x)$.

Remembering the relation

$$s\frac{d}{d\rho}\log s=\frac{ds}{d\rho},$$

we have

$$u_3=\left(\frac{\partial u}{\partial\rho}\right)_{\rho\to\frac{1}{4}}=\left(s\frac{d}{d\rho}\log s\right)_{\rho\to\frac{1}{4}}$$

$$=\Bigg|\sum_{m=0}^{\infty}\frac{\Gamma(\rho+1)\left\{\Gamma(\rho-\frac{1}{4}+1)\right\}^2\Gamma(\rho-\frac{1}{2}+1)}{\Gamma(\rho+m+1)\left\{\Gamma(\rho-\frac{1}{4}+m+1)\right\}^2\Gamma(\rho-\frac{1}{2}+m+1)}c_0z^{\rho+m}$$
$$\times\Big\{\Psi(\rho)+2\Psi(\rho-\frac{1}{4})+\Psi(\rho-\frac{1}{2})-\Psi(\rho+m)$$
$$-2\Psi(\rho-\frac{1}{4}+m)-\Psi(\rho-\frac{1}{2}+m)+\log z\Big\}\Bigg|_{\rho\to\frac{1}{4}}$$

Evaluating the above expression further, we get

$$u_3=\frac{\Gamma(\frac{1}{4}+1)\left\{\Gamma(1)\right\}^2\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4}+1)\left\{\Gamma(1)\right\}^2\Gamma(\frac{3}{4})}c_0z^{\frac{1}{4}}$$
$$\cdot\Big\{\Psi(\frac{1}{4})+2\Psi(0)+\Psi(-\frac{1}{4})-\Psi(\frac{1}{4})-2\Psi(0)-\Psi(-\frac{1}{4})+\log z\Big\}$$
$$+\frac{\Gamma(\frac{1}{4}+1)\left\{\Gamma(1)\right\}^2\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4}+2)\left\{\Gamma(2)\right\}^2\Gamma(\frac{3}{4}+1)}c_0z^{\frac{5}{4}}$$

$$\cdot\left\{\Psi(\frac{1}{4})+2\Psi(0)+\Psi(-\frac{1}{4})-\Psi(\frac{1}{4}+1)-2\Psi(0+1)-\Psi(1-\frac{1}{4})\right.$$

$$\left.+\log z\right\}+\cdots\cdots\cdots\cdots$$

$$=c_0(-4.408\,z^{\frac{5}{4}}+z^{\frac{1}{4}}\log z+1.0664\,z^{\frac{5}{4}}\log z) \tag{7}$$

Further, the fourth solution may be written down easily as follows.

$$u_4=z^{\frac{1}{2}}\,{}_0F_3\left(\frac{3}{2},\frac{5}{4},\frac{5}{4},\ 1\ ;z\right) \tag{8}$$

The general solution of eq. (1.5) is, therefore

$$u=C_1u_1+C_2u_2+C_3u_3+C_4u_4\quad, \tag{9}$$

where C_1, C_2, C_3 and C_4 represent the integration constants. Remembering the relations $z=e(\xi^4/4^4)=B\xi^4/256\lambda^4$ and $\xi=1-\lambda x$, the eq. (9) reduces to

$$u=C_1\,{}_0F_3\left(1,\frac{3}{4},\frac{3}{4},\frac{1}{2}\ ;\frac{B}{256\lambda^4}(1-\lambda x)^4\right)$$

$$+C_2\left\{\frac{B}{256\lambda^4}(1-\lambda x)^4\right\}^{\frac{1}{4}}{}_0F_3\left(\frac{5}{4},\ 1,\ 1,\frac{3}{4}\ ;\frac{B}{256\lambda^4}(1-\lambda x)^4\right)$$

$$+C_3\Big[-4.408\,\frac{B^{\frac{5}{4}}}{1024\lambda^5}(1-\lambda x)^5+\frac{B^{\frac{1}{4}}}{4\lambda}(1-\lambda x)\log\left\{\frac{B}{256\lambda^4}(1-\lambda x)^4\right\}$$

$$+1.0664\,\frac{B^{\frac{5}{4}}}{1024\lambda^5}(1-\lambda x)^5\log\left\{\frac{B}{256\lambda^4}(1-\lambda x)^4\right\}\Big]$$

$$+C_4\left\{\frac{B}{256\lambda^4}(1-\lambda x)^4\right\}^{\frac{1}{2}}{}_0F_3\left(\frac{3}{2},\frac{5}{4},\frac{5}{4},\ 1\ ;\frac{B}{256\lambda^4}(1-\lambda x)^4\right) \tag{9.1}$$

If we take the first two terms [3] in the expanding series of the hypergeometric functions, the eq. (9.1) tends to

$$u=C_1\left\{1+\frac{\frac{B}{256\lambda^4}(1-\lambda x)^4}{1\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}}\right\}+C_2\left\{\frac{B}{256\lambda^4}(1-\lambda x)^4\right\}^{\frac{1}{4}}\left\{1+\frac{\frac{B}{256\lambda^4}(1-\lambda x)^4}{\frac{5}{4}\cdot1\cdot1\cdot\frac{3}{4}}\right\}$$

$$+C_3\Big[-4.408\,\frac{B^{\frac{5}{4}}}{1024\lambda^5}(1-\lambda x)^5+\frac{B^{\frac{1}{4}}}{4\lambda}(1-\lambda x)\log\left\{\frac{B}{256\lambda^4}(1-\lambda x)^4\right\}$$

$$+1.0664\,\frac{B^{\frac{5}{4}}}{1024\lambda^5}(1-\lambda x)^5\log\left\{\frac{B}{256\lambda^4}(1-\lambda x)^4\right\}\Big]$$

$$+C_4\left\{\frac{B}{256\lambda^4}(1-\lambda x)^4\right\}^{\frac{1}{2}}\left\{1+\frac{\frac{B}{256\lambda^4}(1-\lambda x)^4}{\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{5}{4}\cdot1}\right\} \tag{10}$$

3) The third term is usually very small compared with the second term in our case.

The boundary conditions of the problem are as follows.

$$\left.\begin{array}{llll} x=0: & y=u\cos pt=0 & \text{or} & u=0 \\ x=0: & \dfrac{dy}{dx}=\dfrac{du}{dx}\cos pt=0 & \text{or} & \dfrac{du}{dx}=0 \\ x=l: & EI\dfrac{d^2y}{dx^2}=0 & \text{or} & \dfrac{d^2u}{dx^2}=0 \\ x=l: & \dfrac{d}{dx}\left(EI\dfrac{d^2y}{dx^2}\right)=0 & \text{or} & \dfrac{d^3u}{dx^3}=0 \end{array}\right\}$$

Applying the eq. (10) and its derivatives to the above conditions, we obtain finally the following frequency equation of the determinant form, in which $\nu=B^{\frac{1}{4}}l$.

$$\begin{vmatrix} 1+\dfrac{\nu^4}{72c^4}, & \dfrac{\nu}{4c}\left(1+\dfrac{\nu^4}{240c^4}\right), & P, & \dfrac{\nu^2}{16c^2}\left(1+\dfrac{\nu^4}{600c^4}\right), \\ \dfrac{\nu^4}{18c^4}, & \dfrac{\nu}{4c}\left(1+\dfrac{\nu^4}{48c^4}\right), & Q, & \dfrac{\nu^2}{16c^2}\left(2+\dfrac{\nu^4}{100c^4}\right), \\ \dfrac{\nu^4}{6c^4}(1-c)^2, & \dfrac{\nu^5}{48c^5}(1-c)^3, & R, & \dfrac{\nu^2}{16c^2}\left\{2+\dfrac{\nu^4}{100c^4}(1-c^4)\right\}+\dfrac{\nu^6}{400c^6}(1-c)^4, \\ \dfrac{\nu^4}{3c^4}(1-c), & \dfrac{\nu^5}{16c^5}(1-c)^2, & S, & \dfrac{\nu^6}{80c^6}(1-c)^3, \end{vmatrix}=0 \qquad (11)$$

where

$$P=-4.408\frac{\nu^5}{1024c^5}+\frac{\nu}{4c}\log\left\{\frac{\nu^4}{256c^4}\right\}+1.0664\frac{\nu^5}{1024c^5}\log\left\{\frac{\nu^4}{256c^4}\right\},$$

$$Q=\frac{\nu}{c}-17.776\frac{\nu^5}{1024c^5}+\left\{\frac{\nu}{4c}+5.332\frac{\nu^5}{1024c^5}\right\}\log\left\{\frac{\nu^4}{256c^4}\right\},$$

$$R=-17.776\frac{\nu^5}{256c^5}(1-c)^3+5.332\frac{\nu^5}{256c^5}(1-c)^3\log\left\{\frac{\nu^4}{256c^4}(1-c)^4\right\}$$
$$+\left\{\frac{\nu}{c(1-c)}+21.328\frac{\nu^5}{1024c^5}(1-c)^3\right\},$$

$$S=-\frac{\nu}{(1-c)^2c}-\frac{\nu^5}{16c^5}(1-c)^2+\frac{\nu^5}{16c^5}(1-c)^2\log\left\{\frac{\nu^4}{256c^4}(1-c)^4\right\}.$$

The frequency equation (11) gives us the relationship between c and ν, and as $\nu=B^{\frac{1}{4}}l=(\rho p^2\,12/Eh^2)^{\frac{1}{4}}\,l$, we are able to obtain the circular frequency p of the lateral vibration from the eq. (11), provided the values of c, ρ, E, h and l are given. Hence, the natural frequency of the lateral vibration of the trapezoidal cantilever f is known by the expression $f=p/2\pi$. If c is constant, it is evident from eq. (11) that f remains constant.

III. Numerical Example

The numerical calculations were made as to the trapezoidal cantilever of duralmin, in which $E = 7220\ kg/mm^2$ and $\gamma = 2.7\ g/cm^3$. We put further $l = 100\ mm$ and $h = 5\ mm$ in our case. The results by eq. (11) concerning to the natural frequency f of the lateral vibration of the trapezoidal cantilever for various values of c showed that, $f = 400/s$ for $c = +0.5$ and $f = 415/s$ for $c = -0.5$ The results for $c = 1$ and $c = 0$ are easily obtainable by the hitherto aquainted expressions, the results by which being $f = 839/s$ for $c = 1$ and $f = 414/s$ for $c = 0$ respectively.

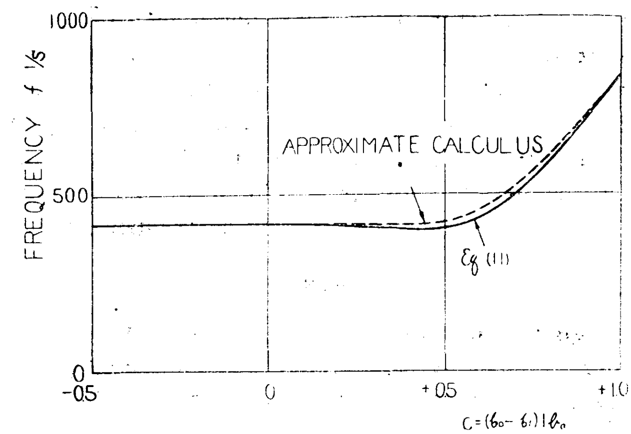

Fig. 2. Relations between f and c.

These results are shown in full line in Fig. 2. The dotted line in the same figure corresponds to the calculated results obtained by the same author by Rayleigh's method. The results by eq. (11) based upon the exact differential equation gives somewhat lower values than that obtained approximately by means of Rayleigh's method. The experimental results in other example showed some lower values of f than that obtained by Rayleigh's method.[4] Perhaps in this example the true frequency may also show closer coincidence with the results by eq. (11). The fact that the difference between the approximate method and the exact method is very small, however, shows that the approximate method due to Rayleigh's method is practically satisfactory in the case considered, except the region of c ranging $+0.5$ to $+1.0$ where the approximation holds no longer.

4) The same as 2).