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Transverse Vibrations of a Rotating Beam with Variable Rigidity under the Influence of Centrifugal Force

(Received March 20, 1952)

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Abstract

In this paper, the author established the fundamental equation of transverse vibrations of a rotating beam with variable rigidity under the influence of centrifugal force. Then he took the flexural rigidity EI and the cross-sectional area A of the beam as the functions of the distance x from its root, such as

$$EI = E_0 I_0 \left(1 - \lambda \frac{x}{l} \right)^m$$

$$A = A_0 \left(1 - \lambda \frac{x}{l} \right)^n ,$$

and showed that in such special cases as

1) $m = 5, \quad n = 2$
2) $m = 6, \quad n = 3$
3) $m = 6, \quad n = 2$
4) $m = 7, \quad n = 3$

the equations are reduced to binomial equation and the solutions are obtained as the sum of hypergeometric series. Therefore the frequencies are determined by the eigen value method.

From these analysis, an interesting result was obtained; if the frequencies of the vibration, which is perpendicular to the plane of rotation, are known, the frequencies of the vibration, which takes place in a plane making an arbitrary angle with the plane of rotation, can also be gained.

So, only the vibration perpendicular to the plane of rotation needes to be studied.

I. Introduction

When a blade rotates around the axis perpendicular to its own axis, the centrifugal force stiffen it to some extent and so its frequencies increase. In such cases, if it vibrates in the direction perpendicular to the plane of rotation, the centrifugal forces act as a system of parallel forces, and if it vibrates in the plane of rotation, the centrifugal forces act as the radial forces. But, if it vibrates in any direction in respect to the plane of rotation, the influence of centrifugal forces differs from the above two cases.

The transverse vibration of a rotating blade, when vibration takes place in a plane perpendicular to the plane of rotation, have been studied by many authors,

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Recently the vibration which takes place in the plane of rotation was analized by R.L.Sutherland[1] and the relationship between the frequencies when vibration is perpendicular to the plane of rotation and in the plane of rotation was pointed out by R.Plunkett[2] and was gained from the experiment by T.Takahasi[3] also.

In this paper, the author, regarding the blade as a beam with variable cross-sectional area and taking the flexural rigidity and the cross-sectional area as algebraic functions, analized the vibration which take place in a plane making an arbitrary angle with the plane of rotation, and discussed the relationship between the direction of vibration which may take place and the frequencies of vibration.

II. Nomenclature

The nomenclature used in this paper is as follows:

E = Young's modulus of the beam,
I = moment of the cross-section of the beam,
A = cross-sectional area of the beam,
w = weight of the beam per unit volume,
l = length of the beam,
R = radius of the rotor,
ω = angular velocity of the rotor,
μ = circular frequency,
g = gravitational acceleration,
t = time,
$x, y; r, \theta$ = rectangular and polar coordinates.,
$E_0 I_0, A_0$ = values of EI and A at the root of the beam.

III. Fundamental Equation

Let the angle between the plane of rotation and the direction of vibration be θ_0 and the coördinates be taken as shown in Fig. 1.

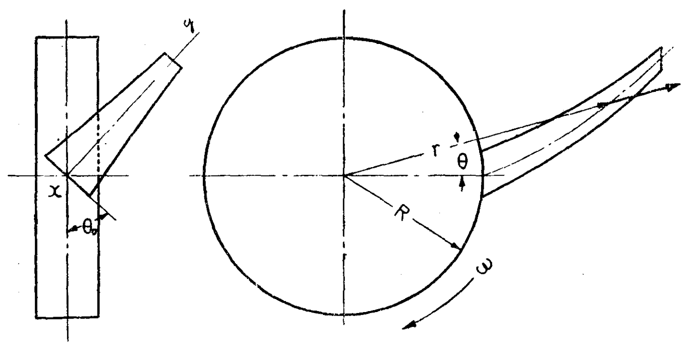


Fig. 1.

1) R.L.Sutherland: J. of App. Mech. **16.** (4), P.389~394 (1950)
2) R.Plunkett: J. of App. Mech. **17.** (2) P.224 (1952).
3) Reported at the annual meeting of the Japan Soc. Mech. Eng., April, 3, 1951.

Then, the following relations are gained;

$$\left.\begin{aligned} y\sin\theta_0 &= r\sin\theta = (x+R)\tan\theta \\ r\cos\theta &= x+R \end{aligned}\right\} \quad (1)$$

By using (1), x – and y – components of the centrifugal force, which act on an element of the beam, length ds, are gained;

x – component $\frac{w}{g} A\, ds\,(R+x)\,\omega^2$

y – component $\frac{w}{g} A\, ds\, y \sin^2\theta_0\,\omega^2$

And, the forces which influence the transverse vibration are

$$\frac{w}{g} A (R+x)\,\omega^2 dy$$

and $\frac{w}{g} Ay\sin^2\theta_0\omega^2 dx$

Therefore, considering the influence of the centrifugal force only, the equation of the bsam is given as follows;

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) + \frac{wA}{g}\cdot\frac{\partial^2 y}{\partial t^2}$$
$$= \frac{\partial}{\partial x}\left[\frac{\partial y}{\partial x}\int_x^l (x+R)\,\omega^2\frac{Aw}{g}\,dx\right] - \frac{\partial}{\partial x}\left[\int_x^l \frac{Aw\omega^2}{g}\, y\sin^2\theta_0 dx\right] \quad (2)$$

In (2), let θ_0 be zero, then the last term of its right hand side dissapears, and it reduces to the equation by T.Suhara, on the vibration perpendicular to the plane of rotation. Let θ_0 be $\frac{\pi}{2}$, then it reduces to the equation on the vibration in the plane of rotation, and this vibration was studied by the author formerly.

IV. The flexural rigidity *EI* and the sectional area A

In this paper, the author gave the flexural rigidity EI and the crosssectional area A of the beam, in terms of the distance x from the root of it, with such algebraic functions as

$$\left.\begin{aligned} EI &= E_0 I_0 \left(1-\lambda\frac{x}{l}\right)^m \\ A &= A_0 \left(1-\lambda\frac{x}{l}\right)^n \end{aligned}\right\} \quad (3)$$

In general, Young's modulus E is a constant value, but considering such cases as the gas turbine blade which has some temperature distribution along its length he gave the flexural rigidity EI variable, particularly.

Solutions of (2)

Substituting (3) into (2)and putting

$y = Ye^{i\mu t}$, $t = T\tau$ and $x = 1/(1-\lambda\frac{x}{l})$, we get

$$\{X^4\overset{''''}{Y}+(12-2m)X^3\overset{'''}{Y}+(2m^2-14m+36)X^2\overset{''}{Y}+(2m^2-14m+24)X\overset{'}{Y}\}$$

$$+\frac{(1-\lambda)^{n+1}wA_0\omega^2l^4}{(n+1)\lambda^4 gE_0I_0}\left\{\frac{1-(n+1)\lambda}{n+2}+\frac{R}{l}\right\}X^{m-4}\overset{''}{Y}$$

$$-\frac{wA_0\omega^2l^4}{\lambda^4gE_0I_0}\left[\frac{1}{n+1}(1+\lambda\frac{R}{l})X^2\overset{''}{Y}+\left\{\frac{1-n}{n+1}-\frac{2\lambda R}{(n+1)l}-\frac{R}{l}\right\}X\overset{'}{Y}\right]X^{m-n-3}$$

$$+\frac{wA_0\omega^2l^4}{\lambda^4gE_0I_0}\left\{\frac{X^2\overset{''}{Y}}{n+2}+\frac{2X\overset{'}{Y}}{n+2}-(\sin^2\theta_0+\frac{\mu^2}{\omega^2T^2})Y\right\}X^{m-n-4}=0 \qquad (4)$$

where Y is the function of x only, and $\overset{'}{Y}, \overset{''}{Y}, \overset{'''}{Y}, \overset{''''}{Y}$ are its differentiation respect to x.

Since (4) is a tetranomial differential equation, to obtain its solution is very toilsome. But, assuming such relations as (1) $m = n + 3$, (2) $m = n + 4$, (3) $m = 6$, (4) $n = 3$, (5) $n = 2$, exist between m and n, then, (4) becomes a trinomial equation and its solution is obtained a little more easily as the sum of power series. These cases were studied by the author previously.

Further, let us consider the following four particular cases.

$$\left.\begin{array}{ll} 1)\ m=5, & n=2; \\ 2)\ m=6, & n=3; \\ 3)\ m=6, & n=2; \\ 4)\ m=7, & n=3. \end{array}\right\} \qquad (5)$$

In these cases, (4) become binomial equations and their solutions are obtained as the sum of hypergeometric series:

$$Y = C_1Y_1 + C_2Y_2 + C_3Y_3 + C_4Y_4 \quad \cdots\cdots \qquad (6)$$

where

$$Y_1 = Z^{-\gamma_0}\,{}_2F_3[\alpha_1-\gamma_0, \alpha_2-\gamma_0; 1+\gamma_0-\gamma_1, 1+\gamma_0-\gamma_2, 1+\gamma_0-\gamma_3; Z]$$

$$Y_2 = Z^{-\gamma_1}\,{}_2F_3[\alpha_1-\gamma_1, \alpha_2-\gamma_1; 1+\gamma_1-\gamma_0' 1+\gamma_1-\gamma_2, 1+\gamma_1-\gamma_3; Z]$$

$$Y_3 = Z^{-\gamma_2}\,{}_2F_3[\alpha_1-\gamma_2, \alpha_2-\gamma_2; 1+\gamma_2-\gamma_0, 1+\gamma_2-\gamma_1, 1+\gamma_2-\gamma_3; Z]$$

$$Y_4 = Z^{-\gamma_3}\,{}_2F_3[\alpha_1-\gamma_3, \alpha_2-\gamma_3; 1+\gamma_3-\gamma_0, 1+\gamma_3-\gamma_1, 1+\gamma_3-\gamma_2; Z]$$

Case (1) $m = 5$, $n = 2$.

γ_0, γ_1, γ_2 and γ_3 are the roots of

$$x\left[x^3-4x^2+\left\{21-\frac{B}{3}(1+\frac{\lambda R}{l})\right\}x-\left\{14-\frac{B}{3}(2+\frac{3\lambda R}{l}+\frac{R}{l})\right\}\right]=0.$$

α_1 and α_2 are the roots of

$$\left\{\frac{(1-\lambda)^3}{3}\left(\frac{1+3\lambda}{4}+\frac{R}{l}\right)+\frac{1}{4}\right\}x^2-\left\{\frac{(1-\lambda)^3}{3}\left(\frac{1+3\lambda}{4}+\frac{R}{l}\right)-\frac{1}{2}\right\}x$$

$$-\left(\sin^2\theta_0+\frac{\mu^2}{\omega^2T^2}\right)=0$$

$$Z=-\frac{B}{X}\left\{\frac{(1-\lambda)^3}{3}\left(\frac{1+3\lambda}{4}+\frac{R}{l}\right)+\frac{1}{4}\right\}$$

$$B = \frac{wA_0\omega^2 l^4}{\lambda^4 g E_0 I_0}$$

Case (2) $m = 6,\quad n = 3.$

γ_0, γ_1, γ_2 and γ_3 are the roots of

$$x\left[x^3 - 6x^2 + \left\{ 55 - \frac{B}{4}\left(1 + \frac{\lambda R}{l}\right) + \frac{B(1-\lambda)^4}{4}\left(\frac{1+4\lambda}{5} + \frac{R}{l}\right)\right\} x + \left\{ \frac{3B}{4}\left(1 + \frac{\lambda R}{l}\right) + \frac{R}{l}B - \frac{B(1-\lambda)^4}{4}\left(\frac{1+4\lambda}{5} + \frac{R}{l}\right) - 32 \right\}\right] = 0$$

α_1 and α_2 are the roots of

$$x^2 + x - \left(\sin^2\theta_0 + \frac{\mu^2}{\omega^2 T^2}\right) = 0$$

$$Z = -\frac{B}{5X}$$

Case (3) $m = 6,\quad n = 2$

γ_0, γ_1, γ_2 and γ_3 are the roots of

$$x^4 + \left[38 + B\left\{ \frac{(1-\lambda)^3}{3}\left(\frac{1+3\lambda}{4} + \frac{R}{l}\right) + \frac{1}{4}\right\}\right] x^3 - \left[109 + B\left\{ \frac{(1-\lambda)^3}{3}\left(\frac{1+3\lambda}{4} + \frac{R}{l}\right) - \frac{1}{4}\right\}\right] x^2 + \left\{ 70 + \frac{2B}{3}\left(\frac{1+3\lambda}{4} + \frac{R}{l}\right)\right\} x - B\left(\sin^2\theta_0 + \frac{\mu^2}{\omega^2 T^2}\right) = 0$$

α_1 and α_2 are the roots of

$$x\left\{ x - \frac{2 + \frac{3\lambda R}{l} + \frac{R}{l}}{1 + \frac{\lambda R}{l}}\right\} = 0$$

$$Z = \frac{B}{3}\left(1 + \frac{\lambda R}{l}\right) X$$

Case (4) $m = 7,\quad n = 3.$

γ_0, γ_1, γ_2 and γ_3 are the roots of

$$x^4 - 8x^3 + \left(53 + \frac{B}{5}\right) x^2 - \left(22 - \frac{B}{5}\right) x - B\left(\sin^2\theta_0 + \frac{\mu^2}{\omega^2 T^2}\right) = 0$$

α_1 and α_2 are the roots of

$$x\left\{ x - \frac{3\left(1 + \frac{\lambda R}{l}\right) + \frac{4R}{l} - (1-\lambda)^4\left(\frac{1+4\lambda}{5} + \frac{R}{l}\right)}{\left(1 + \frac{\lambda R}{l}\right) - (1-\lambda)^4\left(\frac{1+4\lambda}{5} + \frac{R}{l}\right)}\right\} = 0$$

$$Z = \frac{B\left\{\left(1+\frac{\lambda R}{l}\right) - (1-\lambda)^4\left(\frac{1+4\lambda}{5} + \frac{R}{l}\right)\right\}X}{4},$$

VI. Frequencies of the beam

Assuming one end of the beam fixed on the rotor and the other end free, then, the boundary conditions become as follows;

$$(y)_{x=0} = (y')_{x=0} = (y'')_{x=l} = (y''')_{x=l} = 0.$$

That is,

$$(Y)_{x=0} = (Y')_{x=0} = (Y'')_{x=l} = (Y''')_{x=0} = 0.$$

Applying these bounbary conditious to the solutions of (4), ge get

$$\begin{vmatrix} (Y_1)_{x=0} & (Y_2)_{x=0} & (Y_3)_{x=0} & (Y_4)_{x=0} \\ (Y_1')_{x=0} & (Y_2')_{x=0} & (Y_3')_{x=0} & (Y_4')_{x=0} \\ (Y_1'')_{x=l} & (Y_2'')_{x=l} & (Y_3'')_{x=l} & (Y_4'')_{x=l} \\ (Y_1''')_{x=l} & (Y_2''')_{x=l} & (Y_3''')_{x=l} & (Y_4''')_{x=l} \end{vmatrix} = 0$$

From this determinant the frequencies μ are determined. For example, putting $m = 5$, $n = 2$, $\lambda = 0.4$, $B = 3\times10^4$ and $R/l = 2.5$, the fundamental frequency μ when vibration is in the plane of rotation is gained as $6.97\ \omega$.

VII. The relationship between frequencies and direction of vibration

From tho equation established, a relationship between frequency of vibration μ, which is in a plane making angle θ_0 with the plane of rotation, and the frequency of the vibration $\bar{\mu}$, which is perpendicular to the plane of rotation, was found as

$$\frac{\mu^2}{\omega^2 T^2} = \sin^2\theta_0 + \frac{\bar{\mu}^2}{\omega^2 T^2}$$

Further, the author pointed out that in Southwell's formula the coefficient K, when vibration is in a plane making angle θ_0 with the plane of rotation, is $\sin^2\theta_0$ less than $\bar{K}$, when vibration is perpendicular to the plane of rotation:

$$K = \bar{K} - \sin^2\theta_0$$

The experiment by T.Takahashi showed the same formula of K.

VIII. Conclusions

The author established the equation of the vibration of the beam, whicch vibrates in any direction in respect to the plane of rotation, and showed that the frequencies can be determined for the beam with a given form.

Further, he pointed out that the frequencies are related in a very simple manner to the frequencies when vibration is perpendicular to the plane of rotation.

IX. Acknowledgement

The author wishes to thank Prof. T.Suhara of Keio University, who offered continuous and detailed advice, and Assist. Prof. T.Takahashi of Waseda University who made many helpful suggestions.

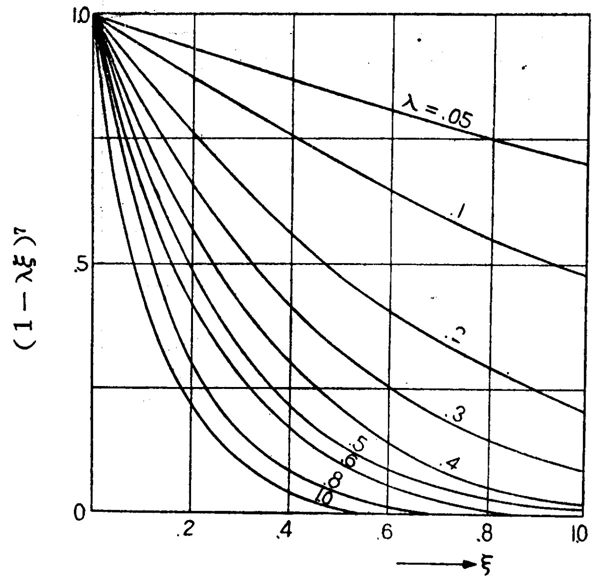


Fig. 2

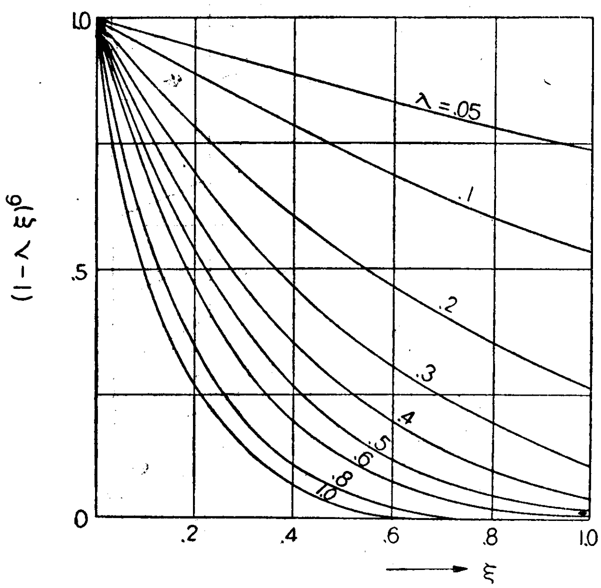
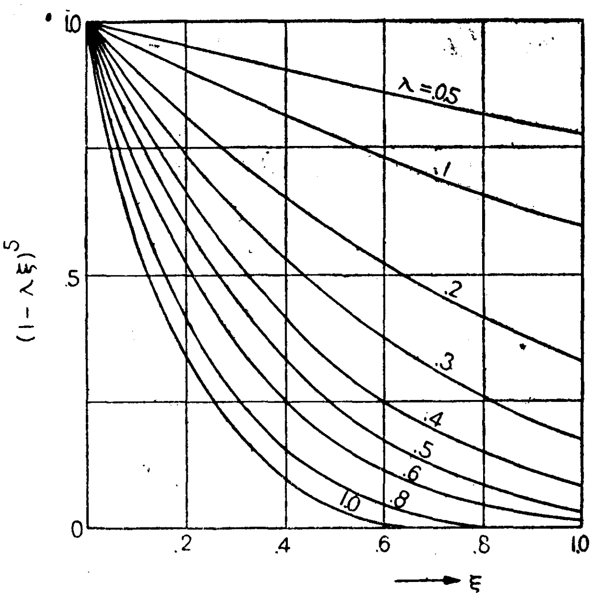


Fig. 3.


1.0
.5
0
$(1-\lambda\xi)^5$
λ=.05
.1
.2
.3
.4
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.6
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.4
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ξ

Fig. 4.

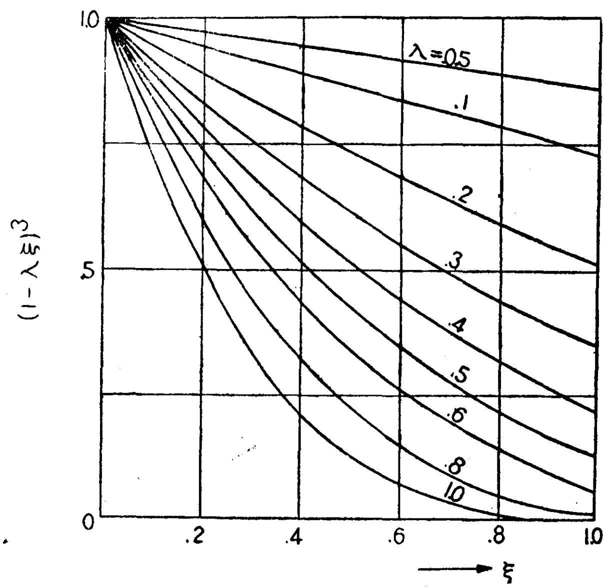
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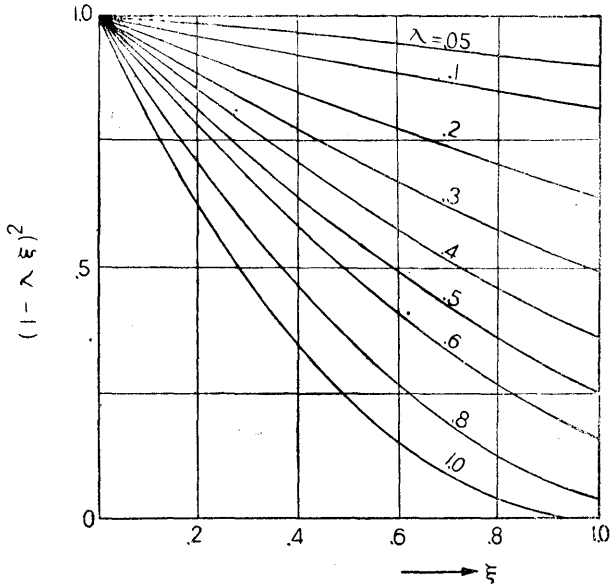
Fig. 5.



Fig, 6.