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Author	栖原, 豊太郎(Suhara, Toyotaro)
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Thermal and Centrifugal Stresses in a Rotating Disk of Variable Thickness (I)

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Toyotaro SUHARA *

Abstract

In this paper, thermal and centrifugal stresses in a rotating circular disk of small thickness but of arbitrary profiles of certain types are analysed for given radial distributions of temperature in the disk, taking the coefficients of elasticity and of thermal expansion as functions of the radial distance from the center, corresponding to the given temperature distributions.

Introduction

With the present tendency of utilizing high temperatures in gas and steam turbine operations, thermal stresses in the working parts such as turbine rotors have become as important as centrifugal stresses.

In order to make analyses for these stresses closely at elevated temperatures it is necessary to introduce the effects of the changes of material properties in analytical forms into the basic equations of elasticity. In this paper analyses are made on the substantially same lines as my former paper[1] where the coefficients of elasticity and thermal expansion were taken as functions of temperature or functions of co-ordinates and Poisson's ratio alone was considered constant for all temperatures. The analyses hold only for the elastic state strained below yield point of the material.

In the analyses, all variables are assumed to be symmetrical about the axis of rotation and a plane (mid-plane of the disk) perpendicular to the axis, and the disk is considered thin as compared with its diameter so that all the axial stresses may be neglected.

Nomenclature

The following nomenclature is used in the paper:

r, θ, z = cylindrical co-ordinates

r, r_1, r_2 = variable radius, inner and outer radii of the disk

Z, Z_0, Z_1, Z_2 = thicknesses of the disk

$\sigma_r, \sigma_{r0}, \sigma_{r1}, \sigma_{r2}$ = radial components of stress

$\sigma_\theta, \sigma_{\theta 0}, \sigma_{\theta 1}, \sigma_{\theta 2}$ = tangential components of stress

u, u_1, u_2 = radial displacements

T, T_0, T_1, T_2 = temperatures

* Dr. Eng., Professor of Mechanical Engineering, Keiō University

1) " Elasticity of Steel strained by Unequal Heating " by Toyotaro Suhara, Journal of the Soc. Mech. Engrs., Japan; Vol. XXI, No. 50, Aug. 1918

α, α_0, α_1, α_2 = coefficients of thermal expansion
E, E_0, E_1, E_2 = Young's moduli
Subscripts 0, 1 and 2 refer to the center, the inner periphery and the outer periphery of the disk, respectively
λ_a, λ_e, λ_σ, λ'_σ, λ_t, λ_z, ν, $N(=1/\nu)$
= arbitrary constants used in expressions (6), (14), etc.
a, ε, ζ, s, s', τ = arbitrary positive integers in (16), (14), etc.

$\mu = \lambda_z/\lambda_e$, $\mu' = \lambda_z/\lambda_a$, $\mu'' = \lambda_z/\lambda_t$, $j = \lim_{\lambda_z \to 0,\, \zeta\to\infty} \lambda_z\zeta$,
$\mu_0 = \lambda_\sigma/\lambda_e$, $\mu_1 = \lambda_\sigma/\lambda_a$, $\mu_2 = \lambda_\sigma/\lambda_t$,
$\phi_0 = j/\lambda_e$, $\phi_1 = j/\lambda_a$, $\phi_2 = j/\lambda_t$, $j' = \lim_{\lambda_e\to 0,\, \varepsilon\to\infty} \lambda_e\varepsilon$

$x = \lambda_z r^\nu$ or $\lambda_\sigma r^\nu$, $\xi = jr^\nu$
$B = \sigma_r/E$ a constant
σ = Poisson's ratio
w = weight of the material of disk per unit volume
g = gravitational acceleration
n = number of revolutions of the disk per second
ω = angular velocity of the disk
C, C', C_0, C_1, C_2 = constants of integration

Part I

Expressions for the Stresses when Z, T, α and E are given as Functions of r

I Basic Equations

The basic equations are obtained on the assumptions that the shape of disk is symmetrical with respect to the axis Z of rotation and the plane $Z = 0$, that all the stresses and strains are also symmetrical with respect to the same axis and the plane, and that the thickness Z of the disk is supposed to vary continuously along the radius and is considered small in the sense that the stresses in the axial direction are disregarded without much error.

The expressions for radial and tangential stresses are

$$\sigma_r = \frac{E}{1-\sigma^2}\left\{\frac{du}{dr} + \frac{\sigma u}{r} - (1+\sigma)\alpha T\right\} \qquad (1)$$

and

$$\sigma_\theta = \frac{E}{1-\sigma^2}\left\{\frac{\sigma du}{dr} + \frac{u}{r} - (1+\sigma)\alpha T\right\} \qquad (2)$$

The stress equation of equilibrium for the disk rotating with an angular velocity ω may be shown to be

$$\frac{d}{dr}(Zr\sigma_r) - Z\sigma_\theta + \frac{w\omega^2}{g}Zr^2 = 0 \qquad (3)$$

In these equations Z, T, α and E are all considered to be function of r, and Poisson's ratio σ alone is supposed independent of temperature and constant throughout the disk.

Eliminating u and σ_θ from the equations (1), (2) and (3), we get a differential equation for the radial stress σ_r in the rotating disk of variable thickness as follows :

$$\frac{d^2}{dr^2}(r\sigma_r)+\left(\frac{1}{r}+\frac{d}{dr}\lg\frac{Z}{E}\right)\frac{d}{dr}(r\sigma_r)$$
$$+\left\{\frac{d^2}{dr^2}\lg Z+\left(\frac{1+\sigma}{r}-\frac{d}{dr}\lg E\right)\frac{d}{dr}\lg Z-\frac{1}{r^2}+\frac{\sigma}{r}\frac{d}{dr}\lg E\right\}(r\sigma_r)$$
$$=-\frac{w\omega^2}{g}r^2\left(\frac{3+\sigma}{r}-\frac{d}{dr}\lg E\right)-E\frac{d}{dr}(\alpha T) \qquad (4)$$

Eliminating σ_r and σ_θ from the same equations we get a differential equation for the radial displacement u as follows:

$$\frac{d^2u}{dr^2}+\left(\frac{1}{r}+\frac{d}{dr}\lg ZE\right)\frac{du}{dr}-\left(\frac{1}{r^2}-\frac{\sigma}{r}\frac{d}{dr}\lg ZE\right)u$$
$$=-(1+\sigma)\left\{(1-\sigma)\frac{w\omega^2 r}{gE}+\frac{d}{dr}\lg ZE\cdot(\alpha T)+\frac{d}{dr}(\alpha T)\right\} \qquad (5)$$

II Disks of Algebraic Profile

We assume the variations of

the thickness of disk of the type $Z=Z_0(1+\lambda_z r^\nu)^{-\zeta}=Z_0(1+x)^{-\zeta}$

the temperature of disk of the type $T=T_0(1+\lambda_t r^\nu)^\tau=T_0(1+x/\mu'')^\tau$

the coefficient of thermal expansion of the type

$\alpha=\alpha_0(1+\lambda_a r^\nu)^a=\alpha_0(1+x/\mu')^a$

Young's modulus of the type $E=E_0(1+\lambda_e r^\nu)^{-\varepsilon}=E_0(1+x/\mu)^{-\varepsilon}$ (6)

Here Z_0, T_0, α_0 and E_0 denote the hypothetical values at $r=0$ in case of a disk with central hole.

Through these expressions (6) we may practically deal with a rotating circular disk of any shape and of any thermal condition. Substituting (6) in eq. (4) and using the new independent variable x in place of r, we get

$$\frac{d^2}{dx^2}(r\sigma_r)+\left(\frac{1}{x}-\frac{\zeta}{1+x}+\frac{\varepsilon}{\mu+x}\right)\frac{d}{dx}(r\sigma_r)$$
$$-\left\{\frac{1}{\nu^2x^2}-\frac{\zeta}{(1+x)^2}+\frac{\{(1+\varepsilon)\zeta+(\varepsilon+\zeta)N\sigma\}x+\mu\zeta+(\varepsilon+\mu\zeta)N\sigma}{x(1+x)(\mu+x)}\right\}(r\sigma_r)$$
$$=-\frac{w\omega^2N}{g\lambda_z^{3N}}x^{3N-2}\left\{\frac{3+\sigma}{\nu}+\frac{\varepsilon x}{\mu+x}\right\}$$
$$-\frac{\alpha_0T_0E_0N\mu^\varepsilon}{\lambda_z^N\mu'^a\mu''^\tau}\cdot\frac{(\mu'+x)^{a-1}(\mu''+x)^{\tau-1}}{x^{1-N}(\mu+x)^\varepsilon}\left\{(a\mu''+\tau\mu')+(a+\tau)x\right\} \qquad (4a)$$

We take the following set of the complementary function and a particular integral η_0 as a complete primitive of eq. (4a):

$$(r\sigma_r)=x^N(1+x)^\zeta\{C_1\eta_1(x)+C_2\eta_2(x)\}+\eta_0(x) \qquad (7)$$

in which $\eta_1(x)=\sum_{n=0}^{\infty}a_nx^n,\quad x=\lambda_z r^\nu$ (8)

and the relation connecting three consecutive a's is

$$\mu(m+1)(m+1+2N)a_{m+1}$$
$$+[(\mu+1)m(m+2N)+\{m+N(1-\sigma)\}(\mu\zeta+\varepsilon)]a_m$$
$$+[(m-1)(m-1+2N)+\{m-1+N(1-\sigma)\}(\zeta+\varepsilon)]a_{m-1}=0$$

with $$a_1=-\frac{(1-\sigma)N(\mu\zeta+\varepsilon)}{(1+2N)\mu}a_0 \qquad (8a)$$

and $$\eta_2(x) = x^{-2N}\sum_{n=0}^{\infty} b_n x^n \quad (N \neq 1/2, 1, 2, 3, \cdots) \tag{9}$$

and the relation connnecting three consecutive b's is

$$\mu(m+1)(m+1-2N)b_{m+1} + [(\mu+1)m(m-2N) + \{m - N(1+\sigma)\}(\mu\zeta + \varepsilon)]b_m + [(m-1)(m-1-2N) + \{m-1-N(1+\sigma)\}(\zeta+\varepsilon)]b_{m-1} = 0 \tag{9a}$$

with $$b_1 = \frac{(1+\sigma)N(\varepsilon+\mu\zeta)}{(1-2N)\mu} b_0$$

A particular integral is

$$\eta_0(x) = x^N(1+x)(\mu+x)^{-\varepsilon+1}\{\sum_{n=1}^{\infty} c_n x^n + x^{2N-1}\sum_{n=1}^{\infty} d_n x^n\} \tag{10}$$

The relations connecting three consecutive c's and d's are

$$\left.\begin{aligned}
&\mu(m+1)(m+1+2N)c_{m+1} \\
&+[(\mu+1)(m+1)(m+1+2N) - \{m+1+N(1+\sigma)\}(\varepsilon+\mu\zeta)]c_m \\
&+[(m+1)(m+1+2N) - \{m+1+N(1+\sigma)\}(\varepsilon+\zeta)]c_{m-1} \\
&= -W_m \quad \text{for} \quad m \leqq a+\tau-1 \\
&= 0 \quad \text{for} \quad m \geqq a+\tau
\end{aligned}\right.$$

and $$c_1 = \frac{-W_0}{(1+2N)\mu} = -\frac{\alpha_0 T_0 E_0 N \mu^{\varepsilon-1}}{(1+2N)\lambda_z^N}\left(\frac{a}{\mu'} + \frac{\tau}{\mu''}\right) \tag{10a}$$

$$W_m = \frac{\alpha_0 T_0 E_0 N \mu^{\varepsilon}}{\lambda_z^N}\sum_{k=0}^{m+1}\frac{a!\,\tau!\,(m+1)}{k!\,(a-k)!\,(m+1-k)!\,(\tau-m-1+k)!}$$

$$W_0 = \frac{\alpha_0 T_0 E_0 N \mu^{\varepsilon}}{\lambda_z^N}\left\{\frac{a}{\mu'} + \frac{\tau}{\mu''}\right\}$$

$$W_{a+\tau-1} = \frac{\alpha_0 T_0 E_0 N \mu^{\varepsilon}(a+\tau)}{\lambda_z^N \mu'^a \mu''^\tau}$$

$$\left.\begin{aligned}
&\mu(m+2N)(m+4N)d_{m+1} \\
&+[(\mu+1)(m+2N)(m+4N) - \{m+N(3+\sigma)\}(\varepsilon+\mu\zeta)]d_m \\
&+[(m+2N)(m+4N) - \{m+N(3+\sigma)\}(\varepsilon+\zeta)]d_{m-1} \\
&= -G_m \quad \text{for} \quad m \leqq \varepsilon \\
&= 0 \quad \text{for} \quad m \geqq \varepsilon+1
\end{aligned}\right. \tag{10b}$$

and $$d_1 = -w\omega^2(3+\sigma)\mu^{\varepsilon-1}/(8g\lambda_z^{3N})$$

$$G_0 = w\omega^2 N^2(3+\sigma)\mu^{\varepsilon}/(g\lambda_z^{3N})$$

$$G_m = w\omega^2\{N^2(3+\sigma) + Nm\}\mu^{\varepsilon-m}\cdot\varepsilon!/\{g\lambda_z^{3N}\cdot m!(\varepsilon-m)!\}$$

$$G_\varepsilon = w\omega^2\{N^2(3+\sigma) + N\varepsilon\}/(g\lambda_z^{3N})$$

The constants C_1 and C_2 in (7) may be determined by the conditions at the inner and outer peripheries of the disk.

σ_θ is obtained from (3) and (7) as follows:

$$\sigma_\theta = \frac{d}{dr}\lg Z\cdot(r\sigma_r) + \frac{d}{dr}(r\sigma_r) + \frac{w\omega^2}{g}r^2$$

or $$r\sigma_\theta = x^N(1+x)^\zeta[C_1\{\eta_1(x) + \nu x\eta_1'(x)\} + C_2\{\eta_2(x) + \nu x\eta_2'(x)\}] + \nu x\{\eta_0'(x) - \zeta(1+x)^{-1}\eta_0(x)\} + w\omega^2 r^2/g \tag{11}$$

and u from (1), (2), (7) and (11)

$$Eu = x^N(1+x)^\zeta[C_1\{(1-\sigma)\eta_1(x)+\nu x\eta_1'(x)\}+C_2\{(1-\sigma)\eta_2(x)+\nu x\eta_2'(x)\}]-\{\nu\zeta x(1+x)^{-1}+\sigma\}\eta_0(x)+\nu x\eta_0'(x)+w\omega^2r^3/g+r\alpha TE \quad (12)$$

where $'$ denotes d/dx.

By the conditions $(\sigma_r)_{r=r1}=\sigma_{r1}$ and $(\sigma_r)_{r=r2}=\sigma_{r2}$,

$$\left.\begin{aligned} C_1 &= \frac{1}{\eta_{11}\eta_{22}-\eta_{12}\eta_{21}}\left\{\frac{(r_1\sigma_{r1}+\eta_{01})\eta_{22}}{x_1^N(1+x_1)^\zeta}-\frac{(r_2\sigma_{r2}+\eta_{02})\eta_{21}}{x_2^N(1+x_2)^\zeta}\right\} \\ C_2 &= \frac{1}{\eta_{11}\eta_{22}-\eta_{12}\eta_{21}}\left\{\frac{(r_1\sigma_{r1}+\eta_{01})\eta_{12}}{x_1^N(1+x_1)^\zeta}-\frac{(r_2\sigma_{r2}+\eta_{02})\eta_{21}}{x_2^N(1+x_2)^\zeta}\right\}\end{aligned}\right\} \quad (13)$$

in which $x_1=\lambda_z r_1^\nu$ and $x_2=\lambda_z r_2^\nu$; and $\eta_{11}, \eta_{12}, \cdots$ denote $\eta_1(x_1)=\eta_1(\lambda_z r_1^\nu)$, $\eta_1(x_2)=\eta_1(\lambda_z r_2^\nu)$, ...

A Special case

When $\nu=2$, $N=1/2$ and $\zeta=\mathcal{E}=a=\tau=1$, $x=\lambda_z r^2$ (6) becomes

$$\left.\begin{aligned} Z &= Z_o(1+\lambda_z r^2)^{-1} \\ T &= T_o(1+\lambda_t r^2) \\ \alpha &= \alpha_o(1+\lambda_\alpha r^2) \\ E &= E_o(1+\lambda_e r^2)^{-1}\end{aligned}\right\} \quad (14)$$

and the solution (7) takes the form of

$$r\sigma_r = \sqrt{x}(1+x)\{C_1\eta_1(x)+C_2\eta_2(x)+\eta_o(x)\} \quad (15)$$

in which $\eta_1(x)=\sum_{n=0}^{\infty}a_n x^n$ (16)

and

$$\left.\begin{aligned} &\lambda_z(n+2)(n+3)a_{n+2} \\ &+(\lambda_e+\lambda_z)\{(n+2)(n+3)-\tfrac{1}{2}(2n+5+\sigma)\}a_{n+1} \\ &+\lambda_e\{(n+2)(n+3)-(2n+5+\sigma)\}a_n=0 \\ &a_1=-\tfrac{1}{4}(1-\sigma)\{1+(\lambda_e/\lambda_z)\}a_0 \\ &a_2=\tfrac{1}{48}(1-\sigma)[(7-\sigma)\{1+(\lambda_e/\lambda_z)\}^2-(8\lambda_e/\lambda_z)]a_1\end{aligned}\right\} \quad (16a)$$

$$\eta_2(x)=\eta_1(x)\lg x+b_{-1}x^{-1}+\sum_{n=1}^{\infty}b_n x^n \quad (17)$$

$$\left.\begin{aligned} &\lambda_z(n+2)(n+3)b_{n+2}+\lambda_z(2n+5)a_{n+2} \\ &+(\lambda_e+\lambda_z)\{(n+2)(n+3)-\tfrac{1}{2}(2n+5+\sigma)\}b_{n+1}+(\lambda_e+\lambda_z)(2n+4)a_{n+1} \\ &+\lambda_e\{(n+2)(n+3)-(2n+5+\sigma)\}b_n+\lambda_e(2n+3)a_n=0 \\ &b_{-1}=\frac{2\lambda_z a_o}{(1+\sigma)(\lambda_e+\lambda_z)},\quad b_1=\left\{\frac{\lambda_e}{\lambda_e+\lambda_z}-\frac{(5+3\sigma)(\lambda_e+\lambda_z)}{8\lambda_z}\right\}a_0\end{aligned}\right\} \quad (17a)$$

$$\eta_0(x)=\sum_{n=1}^{\infty}c_n x^n \quad (18)$$

$$\left.\begin{aligned} &\lambda_z(n+2)(n+3)c_{n+2} \\ &+(\lambda_e+\lambda_z)\{(n+2)(n+3)-\tfrac{1}{2}(2n+5+\sigma)\}c_{n+1} \\ &+\lambda_e\{(n+2)(n+3)-(2n+5+\sigma)\}c_n=0 \\ &c_1=-\frac{w\,\omega^2(3+\sigma)}{8g\lambda_z^{3/2}}-\frac{\alpha_0T_0E_0(\lambda_\alpha+\lambda_t)}{4\lambda_z^{3/2}}\end{aligned}\right\} \quad (18a)$$

$$c_2 = \frac{w\,\omega^2}{96g\lambda_z^{5/2}}\left\{(1-\sigma^2)\lambda_e + (7-\sigma)(3+\sigma)\lambda_z\right\}$$
$$+\frac{\alpha_0 T_0 E_0}{48\lambda_z^{5/2}}\left\{(7-\sigma)(\lambda_e+\lambda_z)(\lambda_a+\lambda_t) - 8\lambda_z\lambda_t\right\}$$

III Disks of Exponential Profile

In the expression for the disk thickness Z in (6), if we make

$$\lambda_z \to 0,\quad \zeta \to \infty \quad \text{and} \quad \text{Lim}\ \lambda_z\zeta = j\ (\text{ a finite value })$$

we get the exponential expression for Z, thus

$$Z = Z_0\,\text{Lim}\,(1+\lambda_z r^\nu)^{-\zeta} = Z_0\,e^{-jr^\nu} = Z_0\,e^{-\xi} \qquad (19)$$

where $\xi = jr^\nu$. The expressions for T, α and E in terms of ξ are

$$\begin{aligned} T &= T_0(1+\lambda_t r^\nu)^\tau = T_0(1+\xi/\phi_2)^\tau \\ \alpha &= \alpha_0(1+\lambda_a r^\nu)^a = \alpha_0(1+\xi/\phi_1)^a \\ E &= E_0(1+\lambda_e r^\nu)^{-\varepsilon} = E_0(1+\xi/\phi_0)^{-\varepsilon} \end{aligned} \qquad (20)$$

The solution for this case may be considered to be a limiting case of the preceding general expressions. The following expression of σ_r is the primitive of Eq. (4) for a special value of exponents

$$\tau = a = \varepsilon = 1$$

in (20):

$$(r\sigma_r) = j^{1/\nu}\exp(jr^\nu)\cdot\{C_1 r\sum_{n=0}^{\infty} a_n (jr^\nu)^n + C_2 r^{-1}\sum_{n=0}^{\infty} b_n (jr^\nu)^n\}$$
$$+\ \xi^{3N}\sum_{n=0}^{\infty} c_n^{\mathrm{I}}\xi^n + \xi^{3N+1}\sum_{n=0}^{\infty} c_n^{\mathrm{II}}\xi^n + \xi^{N+1}\sum_{n=0}^{\infty} c_n^{\mathrm{III}}\xi^n + \xi^{N+2}\sum_{n=0}^{\infty} c_n^{\mathrm{IV}}\xi^n \qquad (21)$$

in which coefficients c^{I}, c^{II}, c^{III} and c^{IV} are given by

$$\begin{aligned} &\phi_0(m+1+2N)(m+1+4N)\,c_{m+1}^{\mathrm{I}} \\ &+[(m+1+2N)(m+1+4N)-(1+\phi_0)\{m+1+N(3+\sigma)\}]c_m^{\mathrm{I}} \\ &\qquad -\{m+1+N(3+\sigma)\}\,c_{m-1}^{\mathrm{I}} = 0 \end{aligned} \qquad (21a)$$

and $\quad c_0^{\mathrm{I}} = -\,w\omega^2(3+\sigma)/(8gj^{3N})$

$$\begin{aligned} &\phi_0(m+2+2N)(m+2+4N)\,c_{m+1}^{\mathrm{II}} \\ &+[(m+2+2N)(m+2+4N)-(1+\phi_0)\{m+2+N(3+\sigma)\}]\,c_m^{\mathrm{II}} \\ &\qquad -\{m+2+N(3+\sigma)\}\,c_{m-1}^{\mathrm{II}} = 0 \end{aligned} \qquad (21b)$$

$$c_0^{\mathrm{II}} = -\,w\omega^2(3+\sigma+\nu)/\{gj^{3N}(2+\nu)(4+\nu)\phi_0\}$$

$$\begin{aligned} &\phi_0(m+2)(m+2+2N)\,c_{m+1}^{\mathrm{III}} \\ &+[(m+2)(m+2+2N)-(1+\phi_0)\{m+2+N(1+\sigma)\}]\,c_m^{\mathrm{III}} \\ &\qquad -\{m+2+N(1+\sigma)\}\,c_{m-1}^{\mathrm{III}} = 0 \\ &c_0^{\mathrm{III}} = -\,\alpha_0 T_0 E_0(\phi_1+\phi_2)/\{j^N(2+\nu)\phi_1\phi_2\} \end{aligned} \qquad (21c)$$

$$\begin{aligned} &\phi_0(m+3)(m+3+2N)\,c_{m+1}^{\mathrm{IV}} \\ &+[(m+3)(m+3+2N)-(1+\phi_0)\{m+3+N(1+\sigma)\}]\,c_m^{\mathrm{IV}} \\ &\qquad -\{m+3+N(1+\sigma)\}\,c_{m-1}^{\mathrm{IV}} = 0 \\ &c_0^{\mathrm{IV}} = -\,\alpha_0 T_0 E_0/\{2j^N(1+\nu)\phi_1\phi_2\} \end{aligned} \qquad (21d)$$

Further, if we take $\nu = 2$ *Eq.* (21) simplifies a little as follows:

$$\sigma_r = j^{1/2} exp (jr^2) \{ C_1 \sum_{n=0}^{\infty} a_n (jr^2)^n + C_2 r^{-2} \sum_{n=0}^{\infty} b_n (jr^2)^n \}$$
$$+ j^{3/2} r^2 \sum_{n=0}^{\infty} (c_n^{\mathrm{I}} + c_n^{\mathrm{III}}) (jr^2)^n$$
$$+ j^{5/2} r^4 \sum_{n=0}^{\infty} (c_n^{\mathrm{II}} + c_n^{\mathrm{IV}}) (jr^2)^n \qquad (22)$$

$$\left.\begin{aligned} &\phi_0 (m + 2) (m + 3) c_{m+1}^{\mathrm{I}} \\ &+ [(m + 2) (m + 3) - \tfrac{1}{2} (1 + \phi_0) (2m + 5 + \sigma)] c_m^{\mathrm{I}} \\ &\qquad - \tfrac{1}{2} (2m + 5 + \sigma) c_{m-1}^{\mathrm{I}} = 0 \\ &c_0^{\mathrm{I}} = - w \omega^2 (3 + \sigma) / (8 g j^{3/2}) \end{aligned}\right\} \qquad (22a)$$

$$\left.\begin{aligned} &\phi_0 (m + 3) (m + 4) c_{m+1}^{\mathrm{II}} \\ &+ [(m + 3) (m + 4) - \tfrac{1}{2} (1 + \phi_0) (2m + 7 + \sigma)] c_m^{\mathrm{II}} \\ &\qquad - \tfrac{1}{2} (2m + 7 + \sigma) c_{m-1}^{\mathrm{II}} = 0 \\ &c_0^{\mathrm{II}} = - w \omega^2 (5 + \sigma) / (24 g j^{3/2} \phi_0) \end{aligned}\right\} \qquad (22b)$$

c^{III} satisfies the first expression of (22a) with

$$c_0^{\mathrm{III}} = - \alpha_0 T_0 E_0 (\phi_1 + \phi_2) / (4 j^{1/2} \phi_1 \phi_2) \qquad (22c)$$

c^{IV} satisfies the first expression of (22b)with

$$c_0^{\mathrm{IV}} = - \alpha_0 T_0 E_0 / (6 j^{1/2} \phi_1 \phi_2) \qquad (22d)$$

De Laval Profile

When σ_r, T, α and E are all constant,
Eq. (4) reduces to

$$\frac{d^2}{dr^2} (\lg Z) + \frac{2 + \sigma}{r} \cdot \frac{d}{dr} (\lg Z) = - \frac{w \omega^2 (3 + \sigma)}{g \sigma_r}$$

the primitive of which is

$$Z = C_1 exp (- C_2 r^{-1-\sigma}) + exp (- w \omega^2 r^2 / 2 g \sigma_r)$$

The last term is a particular integral giving well known De Laval profile.

IV Disk of Uniform thickness

When the disk is of uniform thickness *Eq.* (4) becomes

$$\frac{d^2}{dr^2} (r\sigma_r) + \left(\frac{1}{r} - \frac{d}{dr} \lg E \right) \frac{d}{dr} (r\sigma_r) - \left(\frac{1}{r^2} - \frac{\sigma}{r} \frac{d}{dr} \lg E \right) (r\sigma_r)$$
$$= - \frac{w \omega^2 r^2}{g} \left(\frac{3 + \sigma}{r} - \frac{d}{dr} \lg E \right) - E \frac{d}{dr} (\alpha T) \qquad (23)$$

We assume the variation of

$$\left.\begin{aligned} &\text{Young's modulus} && E = E_0 (1 - \lambda_e^2 r^4) \\ &\text{disk temperature} && T = T_0 (1 + \lambda_t r^2) \\ &\text{coefficient of thermal expansion} && \alpha = \alpha_0 (1 + \lambda_\alpha r^2) \end{aligned}\right\} \qquad (24)$$

Substituting (24) in *Eq.* (23) we get

$$\frac{d^2}{dr^2}(r\sigma_r)+\left(\frac{1}{r}+\frac{4\lambda_e^2r^3}{1-\lambda_e^2r^4}\right)\frac{d}{dr}(r\sigma_r)-\left(\frac{1}{r^2}+\frac{4\sigma\lambda_e^2r^2}{1-\lambda_e^2r^4}\right)(r\sigma_r)$$

$$=\frac{-w\omega^2r^2}{g}\left(\frac{3+\sigma}{r}+\frac{4\lambda_e^2r^3}{1-\lambda_e^2r^4}\right)-E\frac{d}{dr}(\alpha T) \qquad (25)$$

the complete primitive (divided by r) of which is

$$\sigma_r = C_1F(A_1,A_2;3/2;\lambda_e^2r^4)+C_2(\lambda_er^2)^{-1}F(A_1-\tfrac{1}{2},A_2-\tfrac{1}{2};1/2;\lambda_e^2r^4)$$

$$+\sum_{n=1}^{\infty}a_nr^{2n} \qquad (26)$$

where $A_1, A_2 = -\frac{1}{4}\pm\frac{1}{4}\sqrt{5-4\sigma}$

$$\left.\begin{aligned}
\text{and}\quad a_1 &= \frac{1}{2!}\left\{-\frac{3+\sigma}{4}\cdot\frac{w\omega^2}{g}-\frac{1}{2}(\lambda_a+\lambda_t)\alpha_0T_0E_0\right\}\\
a_2 &= \frac{1}{3!}\left\{-\lambda_a\lambda_t\alpha_0T_0E_0\right\}\\
a_3 &= \frac{1}{4!}\lambda_e^2\left\{\frac{1-\sigma^2}{4}\cdot\frac{w\omega^2}{g}+\frac{1}{2}(5-\sigma)(\lambda_a+\lambda_t)\alpha_0T_0E_0\right\}\\
a_4 &= \frac{1}{5!}\lambda_e^2\left\{(11-\sigma)\lambda_a\lambda_t\alpha_0T_0E_0\right\}\\
a_5 &= \frac{1}{6!}\lambda_e^4\left\{\frac{(5+\sigma)(1-\sigma^2)}{4}\cdot\frac{w\omega^2}{g}+\frac{1}{2}(1-\sigma^2)(\lambda_a+\lambda_t)\alpha_0T_0E_0\right\}\\
a_6 &= \frac{1}{7!}\lambda_e^4\left\{(1-\sigma^2)\lambda_a\lambda_t\alpha_0T_0E_0\right\}
\end{aligned}\right\} \qquad (26a)$$

and for $a_7, a_8, \cdots$

$$a_n=\left(1-\frac{5+\sigma}{n}-\frac{1-\sigma}{n+1}\right)\lambda_e^2a_{n-2}, \qquad n=7,8,9,\cdots$$

The tangential stress is

$$\sigma_\theta = C_1\{F(A_1,A_2;3/2;\lambda_e^2r^4)-\tfrac{2}{3}(1-\sigma)\lambda_e^2r^4F(A_1+1,A_2+1;5/2;\lambda_e^2r^4)\}$$

$$+C_2\{-(\lambda_er^2)^{-1}F(A_1-\tfrac{1}{2},A_2-\tfrac{1}{2};1/2;\lambda_e^2r^4)$$

$$+2(1+\sigma)\lambda_er^2F(A_1+\tfrac{1}{2},A_2+\tfrac{1}{2};3/2;\lambda_e^2r^4)\}$$

$$+\sum_{n=1}^{\infty}(2n+1)a_nr^{2n}+w\omega^2r^2/g \qquad (27)$$

in which a's are given by (26a).

The displacement component u is

$$u=C_1(1-\sigma)E_0^{-1}rF(A_1+1,A_2+1;3/2;\lambda_e^2r^4)$$

$$-(C_2/\lambda_e)(1+\sigma)E_0^{-1}r^{-1}F(A_1+\tfrac{1}{2};A_2+\tfrac{1}{2};1/2;\lambda_e^2r^4)+r\sum_{n=1}^{\infty}c_{2n+1}r^{2n} \qquad (28)$$

where c's are given by

$$c_3=\frac{1}{2!}\left\{-\frac{1-\sigma^2}{4}\cdot\frac{w\omega^2}{gE_0}-\frac{1}{2}(1+\sigma)(\lambda_a+\lambda_t)\alpha_0T_0\right\}$$

$$c_5 = \frac{1}{3!}(1+\sigma)(\lambda_e^2 - \lambda_a \lambda_t)\alpha_0 T_0$$

$$c_7 = \frac{1}{4!}\lambda_e^2\left\{-\frac{(5+\sigma)}{4}(1-\sigma^2)\cdot\frac{w\omega^2}{gE_0} - \frac{5}{2}(1+\sigma)(\lambda_a+\lambda_t)\alpha_0 T_0\right\}$$

$$c_9 = \frac{1}{5!}\lambda_e^2\left\{(11+\sigma)\lambda_e^2 + (1-\sigma)\lambda_a\lambda_t\right\} \qquad (28a)$$

and for $n = 5, 6, 7, \cdots$

$$c_{2n+1} = \left(1 - \frac{1-\sigma}{n} - \frac{1+\sigma}{n-1}\right)c_{2n-1}$$

The constants C_1 and C_2 in (26) may be determined on the conditions

$$(\sigma_r)_{r=r1} = \sigma_{r1}, \qquad (\sigma_r)_{r=r2} = \sigma_{r2},$$

$$C_1 = \frac{\left\{\begin{array}{l} r_1^2(\sigma_{r1} - \Sigma_{n=1}^{\infty} a_n r_1^{2n})F(A_1-\frac{1}{2}, A_2-\frac{1}{2}; 1/2; \lambda_e^2 r_2^4) \\ - r_2^2(\sigma_{r2} - \Sigma_{n=1}^{\infty} a_n r_2^{2n})F(A_1-\frac{1}{2}, A_2-\frac{1}{2}; 1/2; \lambda_e^2 r_1^4)\end{array}\right\}}{\left\{\begin{array}{l} r_1^2 F(A_1-\frac{1}{2}, A_2-\frac{1}{2}; 1/2; \lambda_e^2 r_2^4)\cdot F(A_1, A_2; 3/2; \lambda_e^2 r_1^4) \\ - r_2^2 F(A_1-\frac{1}{2}, A_2-\frac{1}{2}; 1/2; \lambda_e^2 r_1^4)\cdot F(A_1, A_2; 3/2; \lambda_e^2 r_2^4)\end{array}\right\}}$$

$$C_2/\lambda_e = \frac{r_1^2 r_2^2\left\{\begin{array}{l}(\sigma_{r2} - \Sigma_{n=1}^{\infty} a_n r_2^{2n})F(A_1, A_2; 3/2; \lambda_e^2 r_1^4) \\ -(\sigma_{r1} - \Sigma_{n=1}^{\infty} a_n r_1^{2n})F(A_1, A_2; 3/2; \lambda_e^2 r_2^4)\end{array}\right\}}{\text{same denominator as above}} \qquad (29)$$

Complete Disk of Uniform Thickness

Putting $r_1 = 0$ in expressions (26), (27) and (28) we get σ_r, σ_θ and u for a complete disk as follows:

$$\sigma_r = C_1 F(A_1, A_2; 3/2; \lambda_e^2 r^4) + \sum_{n=1}^{\infty} a_n r^{2n}$$

$$\sigma_\theta = C_1\{F(A_1, A_2; 3/2; \lambda_e^2 r^4) - \tfrac{2}{3}(1-\sigma)\lambda_e^2 r^4 F(A_1+1, A_2+1; 5/2; \lambda_e^2 r^4)\} + w\omega^2 r^2/g + \sum_{n=1}^{\infty}(2n+1)a_n r^{2n}$$

$$u = C_1(1-\sigma)E_0^{-1} r F(A_1+1, A_2+1; 3/2; \lambda_e^2 r^4) + r\sum_{n=1}^{\infty} c_{2n+1} r^{2n}$$

Taking the radial stress at the circumference

$$(\sigma_r)_{r=r2} = \sigma_{r2}$$

we get $C_1 = (\sigma_{r2} - \sum_{n=1}^{\infty} a_n r_2^{2n}) / F(A_1, A_2; 3/2; \lambda_e^2 r_2^4)$

and

$$\sigma_r = (\sigma_{r2} - \sum_{n=1}^{\infty} a_n r_2^{2n})\frac{F(A_1, A_2; 3/2; \lambda_e^2 r^4)}{F(A_1, A_2; 3/2; \lambda_e^2 r_2^4)} + \sum_{n=1}^{\infty} a_n r^{2n} \qquad (30)$$

$$\sigma_\theta = (\sigma_{r2} - \sum_{n=1}^{\infty} a_n r_2^{2n})\frac{F(A_1, A_2; 3/2; \lambda_e^2 r^4) - \frac{2}{3}(1-\sigma)\lambda_e^2 r^4 F(A_1+1, A_2+1; 5/2; \lambda_e^2 r^4)}{F(A_1, A_2; 3/2; \lambda_e^2 r_2^4)} + \frac{w\omega^2 r^2}{g} + \sum_{n=1}^{\infty}(2n+1)a_n r^{2n} \qquad (31)$$

$$u = (\sigma_{r2} - \sum_{n=1}^{\infty} a_n r_2^{2n})\frac{(1-\sigma) r F(A_1+1, A_2+1; 3/2; \lambda_e^2 r^4)}{E_0 F(A_1, A_2; 3/2; \lambda_e^2 r_2^4)} + r\sum_{n=1}^{\infty} c_{2n+1} r^{2n} \qquad (32)$$

If the condition be $(\sigma_r)_{r=0} = \sigma_{r0}$ we get $C_1 = \sigma_{r0}$, and σ_r, σ_θ and u in (30), (31) and (32), with the last series in each expression in expanded form, may be written as follows:

$$\sigma_r = \sigma_{ro} F(A_1, A_2; 3/2; \lambda_e^2 r^4)$$
$$-(w\omega^2/8g)\{(3+\sigma)r^2 - \tfrac{1}{12}(1-\sigma^2)\lambda_e^2 r^6 - \tfrac{1}{360}(5+\sigma)(1-\sigma^2)\lambda_e^4 r^{10} - \cdots\}$$
$$-\tfrac{1}{4}\alpha_0 T_0 E_0 \{(\lambda_\alpha + \lambda_t)r^2 + \tfrac{2}{3}\lambda_\alpha\lambda_t r^4 - \tfrac{1}{12}(5-\sigma)\lambda_e^2(\lambda_\alpha+\lambda_t)r^6$$
$$-\tfrac{1}{30}(11-\sigma)\lambda_e^2\lambda_\alpha\lambda_t r^8 - \tfrac{1}{360}(1-\sigma^2)\lambda_e^4(\lambda_\alpha+\lambda_t)r^{10} - \cdots\} \quad (33)$$

$$\sigma_\theta = \sigma_{rc}\{F(A_1, A_2; 3/2; \lambda_e^2 r^4)$$
$$-\tfrac{2}{3}(1-\sigma)\lambda_e^2 r^4 F(A_1+1, A_2+1; 5/2; \lambda_e^2 r^4)\}$$
$$-(w\omega^2/8g)\{(1+3\sigma)r^2 - \tfrac{7}{12}(1-\sigma^2)\lambda_e^2 r^6 - \cdots\}$$
$$-\tfrac{1}{4}\alpha_0 T_0 E_0\{3(\lambda_\alpha+\lambda_t)r^2 + \tfrac{10}{3}\lambda_\alpha\lambda_t r^4 - \tfrac{7}{12}(5-\sigma)\lambda_e^2(\lambda_\alpha+\lambda_t)r^6 - \tfrac{3}{10}(11-\sigma)\lambda_e^2\lambda_\alpha\lambda_t r^8 - \cdots\}$$
$$(34)$$

$$u = \sigma_{ro}\sqrt{\lambda_e}\, rF(A_1+1, A_2+1; 3/2; \lambda_e^2 r^4)$$
$$-(w\omega^2/8gE_0)(1-\sigma^2)\{r^3 + \tfrac{1}{12}(5+\sigma)\lambda_e^2 r^7 + \cdots\}$$
$$+\tfrac{1}{4}\alpha_0 T_0(1+\sigma)[-(\lambda_\alpha+\lambda_t)r^3 + \tfrac{2}{3}(\lambda_e^2 - \lambda_\alpha\lambda_t)r^5 + \tfrac{1}{12}(1-\sigma)\lambda_e^2(\lambda_\alpha+\lambda_t)r^7$$
$$+\tfrac{1}{30}\lambda_e^2\{(1-\sigma)\lambda_\alpha\lambda_t + (11+\sigma)\lambda_e^2\}r^9 + \cdots] \quad (35)$$

The first term containing hypergeometric function of the right hand side of (33) denotes the stress due to the radial static pressure at the circumference ($r = r_2$) of disk. The second term denotes the stress due to rotation and the third term due to thermal expansion. The radial stress σ_θ and the displacement u expressed by (34) and (35) are also made of 3 similar items as the above expression (33).

When α, T and E are all constant in the disk with a central hole, (26), (27) and (28) reduce to, with $\lambda_\alpha = \lambda_t = \lambda_e = 0$,

$$\sigma_r = \sigma_{r2}\frac{(r^2-r_1^2)r_2^2}{(r_2^2-r_1^2)r^2} + \sigma_{r1}\frac{(r_2^2-r^2)r_1^2}{(r_2^2-r_1^2)r^2} + \frac{(3+\sigma)w\omega^2}{8g}\frac{(r_2^2-r^2)(r^2-r_1^2)}{r^2} \quad (36)$$

$$\sigma_\theta = \sigma_{r2}\frac{(r^2+r_1^2)r_2^2}{(r_2^2-r_1^2)r^2} - \sigma_{r1}\frac{(r_2^2+r^2)r_1^2}{(r_2^2-r_1^2)r^2} + \frac{(3+\sigma)w\omega^2}{8g}\cdot\frac{(r_2^2+r^2)(r^2+r_1^2)}{r^2} - \frac{(1+\sigma)w\omega^2 r^2}{2g} \quad (37)$$

$$u = \frac{1-\sigma}{E_0}\cdot\frac{r_2^2\sigma_{r2} - r_1^2\sigma_{r1} + (3+\sigma)(w\omega^2/8g)(r_2^4 - r_1^4)}{r_2^2 - r_1^2}\, r$$
$$+\frac{1+\sigma}{E_0}\cdot\frac{r_1^2 r_2^2\{\sigma_{r2} - \sigma_{r1} + (3+\sigma)(w\omega^2/8g)(r_2^2 - r_1^2)\}}{r_2^2 - r_1^2}\cdot\frac{1}{r} - \frac{1-\sigma^2}{8}\cdot\frac{w\omega^2}{gE_0}r^3 \quad (38)$$

For complete disk we get

$$\sigma_r = \sigma_{r2} + \{(3+\sigma)w\omega^2/8g\}(r_2^2 - r^2) \quad (39)$$
$$\sigma_\theta = \sigma_{r2} + \{(3+\sigma)w\omega^2/8g\}\{r_2^2 - r^2(1+3\sigma)/(3+\sigma)\} \quad (40)$$
$$u = \frac{1-\sigma}{E_0}\left\{\sigma_{r2} + \frac{(3+\sigma)w\omega^2}{8g}\left(r_2^2 - \frac{1+\sigma}{3+\sigma}r^2\right)\right\}r \quad (41)$$

(to be continued)