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# Thermal and Centrifugal Stresses in a Rotating Disk of Variable Thickness ( I )

( Received March 1, 1950 )

Toyotaro SUHARA \*

## Abstract

In this paper, thermal and centrifugal stresses in a rotating circular disk of small thickness but of arbitrary profiles of certain types are analysed for given radial distributions of temperature in the disk, taking the coefficients of elasticity and of thermal expansion as functions of the radial distance from the center, corresponding to the given temperature distributions.

## Introduction

With the present tendency of utilizing high temperatures in gas and steam turbine operations, thermal stresses in the working parts such as turbine rotors have become as important as centrifugal stresses.

In order to make analyses for these stresses closely at elevated temperatures it is necessary to introduce the effects of the changes of material properties in analytical forms into the basic equations of elasticity. In this paper analyses are made on the substantially same lines as my former paper<sup>1)</sup> where the coefficients of elasticity and thermal expansion were taken as functions of temperature or functions of co-ordinates and Poisson's ratio alone was considered constant for all temperatures. The analyses hold only for the elastic state strained below yield point of the material.

In the analyses, all variables are assumed to be symmetrical about the axis of rotation and a plane ( mid-plane of the disk ) perpendicular to the axis, and the disk is considered thin as compared with its diameter so that all the axial stresses may be neglected.

## Nomenclature

The following nomenclature is used in the paper :

$r, \theta, z$	= cylindrical co-ordinates
$r, r_1, r_2$	= variable radius, inner and outer radii of the disk
$Z, Z_0, Z_1, Z_2$	= thicknesses of the disk
$\sigma_r, \sigma_{r0}, \sigma_{r1}, \sigma_{r2}$	= radial components of stress
$\sigma_\theta, \sigma_{\theta0}, \sigma_{\theta1}, \sigma_{\theta2}$	= tangential components of stress
$u, u_1, u_2$	= radial displacements
$T, T_0, T_1, T_2$	= temperatures

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1 ) " Elasticity of Steel strained by Unequal Heating " by Toyotaro Suhara, Journal of the Soc. Mech. Engrs., Japan ; Vol. XXI, No. 50, Aug. 1918

$\alpha, \alpha_0, \alpha_1, \alpha_2$  = coefficients of thermal expansion

$E, E_0, E_1, E_2$  = Young's moduli

Subscripts 0, 1 and 2 refer to the center, the inner periphery and the outer periphery of the disk, respectively

$\lambda_a, \lambda_e, \lambda_\sigma, \lambda'_\sigma, \lambda_t, \lambda_z, \nu, N (= 1/\nu)$

= arbitrary constants used in expressions ( 6 ), ( 14 ), etc.

$a, \varepsilon, \zeta, s, s', \tau$  = arbitrary positive integers in ( 16 ), ( 14 ), etc.

$\mu = \lambda_z/\lambda_e, \quad \mu' = \lambda_z/\lambda_a, \quad \mu'' = \lambda_z/\lambda_t, \quad j = \text{Lim}_{\lambda_z \rightarrow 0, \zeta \rightarrow \infty} \lambda_z \zeta,$

$\mu_0 = \lambda_\sigma/\lambda_e, \quad \mu_1 = \lambda_\sigma/\lambda_a, \quad \mu_2 = \lambda_\sigma/\lambda_t,$

$\phi_0 = j/\lambda_e, \quad \phi_1 = j/\lambda_a, \quad \phi_2 = j/\lambda_t, \quad j' = \text{Lim}_{\lambda_e \rightarrow 0, \varepsilon \rightarrow \infty} \lambda_e \varepsilon$

$x = \lambda_r r^\nu \quad \text{or} \quad \lambda_\sigma r^\nu, \quad \xi = j r^\nu$

$B = \sigma_r/E$  a constant

$\sigma$  = Poisson's ratio

$w$  = weight of the material of disk per unit volume

$g$  = gravitational acceleration

$n$  = number of revolutions of the disk per second

$\omega$  = angular velocity of the disk

$C, C', C_0, C_1, C_2$  = constants of integration

### Part I

#### Expressions for the Stresses when $Z, T, \alpha$ and $E$ are given as Functions of $r$

##### I Basic Equations

The basic equations are obtained on the assumptions that the shape of disk is symmetrical with respect' to the axis  $Z$  of rotation and the plane  $Z = 0$ , that all the stresses and strains are also symmetrical with respect to the same axis and the plane, and that the thickness  $Z$  of the disk is supposed to vary continuously along the radius and is considered small in the sense that the stresses in the axial direction are disregarded without much error.

The expressions for radial and tangential stresses are

$$\sigma_r = \frac{E}{1 - \sigma^2} \left\{ \frac{du}{dr} + \frac{\sigma u}{r} - (1 + \sigma) \alpha T \right\} \quad (1)$$

and

$$\sigma_\theta = \frac{E}{1 - \sigma^2} \left\{ \frac{\sigma du}{dr} + \frac{u}{r} - (1 + \sigma) \alpha T \right\} \quad (2)$$

The stress equation of equilibrium for the disk rotating with an angular velocity  $\omega$  may be shown to be

$$\frac{d}{dr} ( Z r \sigma_r ) - Z \sigma_\theta + \frac{w \omega^2}{g} Z r^2 = 0 \quad (3)$$

In these equations  $Z, T, \alpha$  and  $E$  are all considered to be function of  $r$ , and Poisson's ratio  $\sigma$  alone is supposed independent of temperature and constant throughout the disk.

Eliminating  $u$  and  $\sigma_\theta$  from the equations ( 1 ), ( 2 ) and ( 3 ), we get a differential equation for the radial stress  $\sigma_r$  in the rotating disk of variable thickness as

follows :

$$\begin{aligned} & \frac{d^2}{dr^2} (r\sigma_r) + \left( \frac{1}{r} + \frac{d}{dr} \lg \frac{Z}{E} \right) \frac{d}{dr} (r\sigma_r) \\ & + \left\{ \frac{d^2}{dr^2} \lg Z + \left( \frac{1+\sigma}{r} - \frac{d}{dr} \lg E \right) \frac{d}{dr} \lg Z - \frac{1}{r^2} + \frac{\sigma}{r} \frac{d}{dr} \lg E \right\} (r\sigma_r) \\ & = - \frac{w\omega^2}{g} r^2 \left( \frac{3+\sigma}{r} - \frac{d}{dr} \lg E \right) - E \frac{d}{dr} (\alpha T) \end{aligned} \quad (4)$$

Eliminating  $\sigma_r$  and  $\sigma_\theta$  from the same equations we get a differential equation for the radial displacement  $u$  as follows:

$$\begin{aligned} & \frac{d^2 u}{dr^2} + \left( \frac{1}{r} + \frac{d}{dr} \lg ZE \right) \frac{du}{dr} - \left( \frac{1}{r^2} - \frac{\sigma}{r} \frac{d}{dr} \lg ZE \right) u \\ & = - (1 + \sigma) \left\{ (1 - \sigma) \frac{w\omega^2 r}{gE} + \frac{d}{dr} \lg ZE (\alpha T) + \frac{d}{dr} (\alpha T) \right\} \end{aligned} \quad (5)$$

## II Disks of Algebraic Profile

We assume the variations of

the thickness of disk of the type  $Z = Z_0 (1 + \lambda_z r^\nu)^{-\zeta} = Z_0 (1 + x)^{-\zeta}$   
the temperature of disk of the type  $T = T_0 (1 + \lambda_t r^\nu)^\tau = T_0 (1 + x/\mu')^\tau$   
the coefficient of thermal expansion of the type

$$\alpha = \alpha_0 (1 + \lambda_a r^\nu)^a = \alpha_0 (1 + x/\mu')^a$$

Young's modulus of the type  $E = E_0 (1 + \lambda_e r^\nu)^{-\varepsilon} = E_0 (1 + x/\mu)^{-\varepsilon}$

Here  $Z_0$ ,  $T_0$ ,  $\alpha_0$  and  $E_0$  denote the hypothetical values at  $r = 0$  in case of a disk with central hole.

Through these expressions (6) we may practically deal with a rotating circular disk of any shape and of any thermal condition. Substituting (6) in eq. (4) and using the new independent variable  $x$  in place of  $r$ , we get

$$\begin{aligned} & \frac{d^2}{dx^2} (r\sigma_r) + \left( \frac{1}{x} - \frac{\zeta}{1+x} + \frac{\varepsilon}{\mu+x} \right) \frac{d}{dx} (r\sigma_r) \\ & - \left\{ \frac{1}{\nu^2 x^2} - \frac{\zeta}{(1+x)^2} + \frac{\{(1+\varepsilon)\zeta + (\varepsilon+\zeta)N\sigma\}x + \mu\zeta + (\varepsilon + \mu\zeta)N\sigma}{x(1+x)(\mu+x)} \right\} (r\sigma_r) \\ & = - \frac{w\omega^2 N}{g\lambda_z^3 N} x^{3N-3} \left\{ \frac{3+\sigma}{\nu} + \frac{\varepsilon x}{\mu+x} \right\} \\ & - \frac{\alpha_0 T_0 E_0 N \mu^\varepsilon}{\lambda_z N \mu'^a \mu''^\tau} \cdot \frac{(\mu'+x)^{a-1} (\mu''+x)^{\tau-1}}{x^{1-N} (\mu+x)^\varepsilon} \left\{ (a\mu'' + \tau\mu') + (a+1)x \right\} \end{aligned} \quad (4a)$$

We take the following set of the complementary function and a particular integral  $\eta_0$  as a complete primitive of eq. (4a):

$$(r\sigma_r) = x^N (1+x)^\xi \{ C_1 \eta_1(x) + C_2 \eta_2(x) \} + \eta_0(x) \quad (7)$$

$$\text{in which} \quad \eta_1(x) = \sum_{n=0}^{\infty} a_n x^n, \quad x = \lambda_e r^\nu \quad (8)$$

and the relation connecting three consecutive  $a$ 's is

$$\begin{aligned} & \mu(m+1)(m+1+2N)a_{m+1} \\ & + [(\mu+1)m(m+2N) + \{m+N(1-\sigma)\}(\mu\zeta + \varepsilon)]a_m \\ & + [(m-1)(m-1+2N) + \{m-1+N(1-\sigma)\}(\zeta + \varepsilon)]a_{m-1} = 0 \end{aligned} \quad (8a)$$

$$\text{with} \quad a_1 = - \frac{(1-\sigma)N(\mu\zeta + \varepsilon)}{(1+2N)\mu} a_0$$

$$(22)$$

$$\text{and } \eta_2(x) = x^{-2N} \sum_{n=0}^{\infty} b_n x^n \quad (N \neq 1/2, 1, 2, 3, \dots) \quad (9)$$

and the relation connecting three consecutive b's is

$$\begin{aligned} & \mu(m+1)(m+1-2N)b_{m+1} \\ & + [(\mu+1)m(m-2N) + \{m-N(1+\sigma)\}(\mu\zeta + \varepsilon)]b_m \\ & + (m-1)(m-1-2N) + \{m-1-N(1+\sigma)\}(\zeta + \varepsilon)]b_{m-1} = 0 \end{aligned} \quad (9a)$$

$$\text{with } b_1 = \frac{(1+\sigma)N(\varepsilon + \mu\zeta)}{(1-2N)\mu} b_0$$

A particular integral is

$$\eta_0(x) = x^N(1+x)(\mu+x)^{-\varepsilon+1} \left\{ \sum_{n=1}^{\infty} c_n x^n + x^{2N-1} \sum_{n=1}^{\infty} d_n x^n \right\} \quad (10)$$

The relations connecting three consecutive c's and d's are

$$\left\{ \begin{aligned} & \mu(m+1)(m+1+2N)c_{m+1} \\ & + [(\mu+1)(m+1)(m+1+2N) - \{m+1+N(1+\sigma)\}(\varepsilon + \mu\zeta)]c_m \\ & + [(m+1)(m+1+2N) - \{m+1+N(1+\sigma)\}(\varepsilon + \zeta)]c_{m-1} \\ & = -W_m \quad \text{for } m \leq a + \tau - 1 \\ & = 0 \quad \text{for } m \geq a + \tau \end{aligned} \right\}$$

$$\text{and } c_1 = \frac{-W_0}{(1+2N)\mu} = -\frac{\alpha_0 T_0 E_0 N \mu^{\varepsilon-1}}{(1+2N)\lambda_z^N} \left( \frac{a}{\mu'} + \frac{\tau}{\mu''} \right) \quad (10a)$$

$$W_m = \frac{\alpha_0 T_0 E_0 N \mu^{\varepsilon}}{\lambda_z^N} \sum_{k=0}^{m+1} \frac{a! \tau! (m+1)}{k! (a-k)! (m+1-k)! (\tau-m-1+k)!}$$

$$W_0 = \frac{\alpha_0 T_0 E_0 N \mu^{\varepsilon}}{\lambda_z^N} \left\{ \frac{a}{\mu'} + \frac{\tau}{\mu''} \right\}$$

$$W_{a+\tau-1} = \frac{\alpha_0 T_0 E_0 N \mu^{\varepsilon} (a + \tau)}{\lambda_z^N \mu'^a \mu''^{\tau}}$$

$$\left\{ \begin{aligned} & \mu(m+2N)(m+4N)d_{m+1} \\ & + [(\mu+1)(m+2N)(m+4N) - \{m+N(3+\sigma)\}(\varepsilon + \mu\zeta)]d_m \\ & + [(m+2N)(m+4N) - \{m+N(3+\sigma)\}(\varepsilon + \zeta)]d_{m-1} \\ & = -G_m \quad \text{for } m \leq \varepsilon \\ & = 0 \quad \text{for } m \geq \varepsilon + 1 \end{aligned} \right\} \quad (10b)$$

$$\text{and } d_1 = -w\omega^2(3+\sigma)\mu^{\varepsilon-1}/(8g\lambda_z^{3N})$$

$$G_0 = w\omega^2 N^3(3+\sigma)\mu^{\varepsilon}/(g\lambda_z^{3N})$$

$$G_m = w\omega^2 \{N^3(3+\sigma) + Nm\} \mu^{\varepsilon-m} \varepsilon! / \{g\lambda_z^{3N} m! (\varepsilon - m)!\}$$

$$G_{\varepsilon} = w\omega^2 \{N^3(3+\sigma) + N\varepsilon\} / (g\lambda_z^{3N})$$

The constants  $C_1$  and  $C_2$  in (7) may be determined by the conditions at the inner and outer peripheries of the disk.

$\sigma_{\theta}$  is obtained from (3) and (7) as follows:

$$\sigma_{\theta} = \frac{d}{dr} \lg Z \cdot (r\sigma_r) + \frac{d}{dr} (r\sigma_r) + \frac{w\omega^2}{g} r^2$$

$$\text{or } r\sigma_{\theta} = x^N(1+x)^{\zeta} [C_1 \{ \eta_1(x) + \nu x \eta_1'(x) \} + C_2 \{ \eta_2(x) + \nu x \eta_2'(x) \}] + \nu x \{ \eta_0'(x) - \zeta(1+x)^{-1} \eta_0(x) \} + w\omega^2 r^2 / g \quad (11)$$

and  $u$  from (1), (2), (7) and (11)

$$Eu = x^N(1+x) \left[ C_1 \{ (1-\sigma) \eta_1(x) + \nu x \eta_1'(x) \} + C_2 \{ (1-\sigma) \eta_2(x) + \nu x \eta_2'(x) \} \right] - \{ \nu \zeta x (1+x)^{-1} + \sigma \} \eta_0(x) + \nu x \eta_0'(x) + w \omega^2 r^3 / g + r \alpha T E \quad (12)$$

where ' denotes  $d/dx$ .

By the conditions  $(\sigma_r)_{r=r_1} = \sigma_{r_1}$  and  $(\sigma_r)_{r=r_2} = \sigma_{r_2}$ ,

$$\left. \begin{aligned} C_1 &= \frac{1}{\eta_{11} \eta_{22} - \eta_{12} \eta_{21}} \left\{ \frac{(r_1 \sigma_{r_1} + \eta_{01}) \eta_{22}}{x_1^N (1+x_1)^\zeta} - \frac{(r_2 \sigma_{r_2} + \eta_{02}) \eta_{21}}{x_2^N (1+x_2)^\zeta} \right\} \\ C_2 &= \frac{1}{\eta_{11} \eta_{22} - \eta_{12} \eta_{21}} \left\{ \frac{(r_1 \sigma_{r_1} + \eta_{01}) \eta_{12}}{x_1^N (1+x_1)^\zeta} - \frac{(r_2 \sigma_{r_2} + \eta_{02}) \eta_{21}}{x_2^N (1+x_2)^\zeta} \right\} \end{aligned} \right\} \quad (13)$$

in which  $x_1 = \lambda_z r_1^\nu$  and  $x_2 = \lambda_z r_2^\nu$ ; and  $\eta_{11}, \eta_{12}, \dots$  denote  $\eta_1(x_1) = \eta_1(\lambda_z r_1^\nu)$ ,  $\eta_1(x_2) = \eta_1(\lambda_z r_2^\nu), \dots$ .

### A Special case

When  $\nu = 2$ ,  $N = 1/2$  and  $\zeta = \varepsilon = a = \tau = 1$ ,  $x = \lambda_z r^2$  (6) becomes

$$\left. \begin{aligned} Z &= Z_o (1 + \lambda_z r^2)^{-1} \\ T &= T_o (1 + \lambda_t r^2) \\ \alpha &= \alpha_o (1 + \lambda_a r^2) \\ E &= E_o (1 + \lambda_e r^2)^{-1} \end{aligned} \right\} \quad (14)$$

and the solution (7) takes the form of

$$r \sigma_r = \sqrt{x} (1+x) \{ C_1 \eta_1(x) + C_2 \eta_2(x) + \eta_0(x) \} \quad (15)$$

$$\text{in which } \eta_1(x) = \sum_{n=0}^{\infty} a_n x^n \quad (16)$$

and

$$\left. \begin{aligned} &\lambda_z (n+2) (n+3) a_{n+2} \\ &+ (\lambda_e + \lambda_z) \{ (n+2) (n+3) - \frac{1}{2} (2n+5+\sigma) \} a_{n+1} \\ &+ \lambda_e \{ (n+2) (n+3) - (2n+5+\sigma) \} a_n = 0 \\ &a_1 = -\frac{1}{4} (1-\sigma) \{ 1 + (\lambda_e/\lambda_z) \} a_0 \\ &a_2 = \frac{1}{48} (1-\sigma) [ (7-\sigma) \{ 1 + (\lambda_e/\lambda_z) \}^2 - (8\lambda_e/\lambda_z) ] a_1 \end{aligned} \right\} \quad (16a)$$

$$\eta_2(x) = \eta_1(x) \lg x + b_{-1} x^{-1} + \sum_{n=1}^{\infty} b_n x^n \quad (17)$$

$$\left. \begin{aligned} &\lambda_z (n+2) (n+3) b_{n+2} + \lambda_z (2n+5) a_{n+2} \\ &+ (\lambda_e + \lambda_z) \{ (n+2) (n+3) - \frac{1}{2} (2n+5+\sigma) \} b_{n+1} + (\lambda_e + \lambda_z) (2n+4) a_{n+1} \\ &+ \lambda_e \{ (n+2) (n+3) - (2n+5+\sigma) \} b_n + \lambda_e (2n+3) a_n = 0 \end{aligned} \right\} \quad (17a)$$

$$b_{-1} = \frac{2\lambda_z a_0}{(1+\sigma)(\lambda_e + \lambda_z)}, \quad b_1 = \left\{ \frac{\lambda_e}{\lambda_e + \lambda_z} - \frac{(5+3\sigma)(\lambda_e + \lambda_z)}{8\lambda_z} \right\} a_0$$

$$\eta_0(x) = \sum_{n=1}^{\infty} c_n x^n \quad (18)$$

$$\left. \begin{aligned} &\lambda_z (n+2) (n+3) c_{n+2} \\ &+ (\lambda_e + \lambda_z) \{ (n+2) (n+3) - \frac{1}{2} (2n+5+\sigma) \} c_{n+1} \\ &+ \lambda_e \{ (n+2) (n+3) - (2n+5+\sigma) \} c_n = 0 \end{aligned} \right\} \quad (18a)$$

$$c_1 = -\frac{w \omega^2 (3+\sigma)}{8g\lambda_z^{3/2}} - \frac{\alpha_0 T_0 E_0 (\lambda_a + \lambda_t)}{4\lambda_z^{3/2}} \quad (24)$$

$$c_2 = \frac{w \omega^2}{96g\lambda_z^{5/2}} \left\{ (1-\sigma^2) \lambda_e + (7-\sigma)(3+\sigma) \lambda_z \right\} \\ + \frac{\alpha_0 T_0 E_0}{48\lambda_z^{5/2}} \left\{ (7-\sigma)(\lambda_e + \lambda_z)(\lambda_e + \lambda_z) - 8\lambda_e \lambda_z \right\}$$

### III Disks of Exponential Profile

In the expression for the disk thickness  $Z$  in (6), if we make

$$\lambda_z \rightarrow 0, \quad \zeta \rightarrow \infty \quad \text{and} \quad \text{Lim } \lambda_z \zeta = j \quad (\text{a finite value})$$

we get the exponential expression for  $Z$ , thus

$$Z = Z_0 \text{Lim} (I + \lambda_e r^\nu)^{-\zeta} = Z_0 e^{-jr^\nu} = Z_0 e^{-\xi} \quad (19)$$

where  $\xi = jr^\nu$ . The expressions for  $T$ ,  $\alpha$  and  $E$  in terms of  $\xi$  are

$$T = T_0 (I + \lambda_e r^\nu)^\tau = T_0 (I + \xi/\phi_2)^\tau \\ \alpha = \alpha_0 (I + \lambda_e r^\nu)^\alpha = \alpha_0 (I + \xi/\phi_1)^\alpha \\ E = E_0 (I + \lambda_e r^\nu)^{-\varepsilon} = E_0 (I + \xi/\phi_0)^{-\varepsilon} \quad (20)$$

The solution for this case may be considered to be a limiting case of the preceding general expressions. The following expression of  $\sigma_r$  is the primitive of Eq. (4) for a special value of exponents

$$\tau = \alpha = \varepsilon = 1$$

in (20):

$$(r\sigma_r) = j^{1/\nu} \exp(jr^\nu) \cdot \left\{ C_1 r \sum_{n=0}^{\infty} a_n (jr^\nu)^n + C_2 r^{-1} \sum_{n=0}^{\infty} b_n (jr^\nu)^n \right\} \\ + \xi^{3N} \sum_{n=0}^{\infty} c_n^I \xi^n + \xi^{3N+1} \sum_{n=0}^{\infty} c_n^{II} \xi^n + \xi^{N+1} \sum_{n=0}^{\infty} c_n^{III} \xi^n + \xi^{N+2} \sum_{n=0}^{\infty} c_n^{IV} \xi^n \quad (21)$$

in which coefficients  $c^I$ ,  $c^{II}$ ,  $c^{III}$  and  $c^{IV}$  are given by

$$\phi_0 (m+1+2N)(m+1+4N) c_{m+1}^I \\ + [ (m+1+2N)(m+1+4N) - (1+\phi_0) \{ m+1+N(3+\sigma) \} ] c_m^I \\ - \{ m+1+N(3+\sigma) \} c_{m-1}^I = 0 \quad (21a)$$

and  $c_0^I = -w\omega^2(3+\sigma)/(8gj^{2N})$

$$\phi_0 (m+2+2N)(m+2+4N) c_{m+1}^{II} \\ + [ (m+2+2N)(m+2+4N) - (1+\phi_0) \{ m+2+N(3+\sigma) \} ] c_m^{II} \\ - \{ m+2+N(3+\sigma) \} c_{m-1}^{II} = 0 \quad (21b)$$

$$c_0^{II} = -w\omega^2(3+\sigma+\nu) / \{ gj^{2N}(2+\nu)(4+\nu)\phi_0 \}$$

$$\phi_0 (m+2)(m+2+2N) c_{m+1}^{III} \\ + [ (m+2)(m+2+2N) - (1+\phi_0) \{ m+2+N(1+\sigma) \} ] c_m^{III} \\ - \{ m+2+N(1+\sigma) \} c_{m-1}^{III} = 0 \quad (21c)$$

$$c_0^{III} = -\alpha_0 T_0 E_0 (\phi_1 + \phi_2) / \{ j^{N(2+\nu)} \phi_1 \phi_2 \}$$

$$\phi_0 (m+3)(m+3+2N) c_{m+1}^{IV} \\ + [ (m+3)(m+3+2N) - (1+\phi_0) \{ m+3+N(1+\sigma) \} ] c_m^{IV} \\ - \{ m+3+N(1+\sigma) \} c_{m-1}^{IV} = 0 \quad (21d)$$

$$c_0^{IV} = -\alpha_0 T_0 E_0 / \{ 2 j^{N(1+\nu)} \phi_1 \phi_2 \}$$

(25.)

Further, if we take  $\nu = 2$  Eq. (21) simplifies a little as follows:

$$\begin{aligned} \sigma_r = & j^{1/2} \exp(jr^2) \left\{ C_1 \sum_{n=0}^{\infty} a_n (jr^2)^n + C_2 r^{-2} \sum_{n=1}^{\infty} b_n (jr^2)^n \right\} \\ & + j^{3/2} r^2 \sum_{n=0}^{\infty} (c_n^I + c_n^{II}) (jr^2)^n \\ & + j^{5/2} r^4 \sum_{n=0}^{\infty} (c_n^{II} + c_n^{IV}) (jr^2)^n \end{aligned} \quad (22)$$

$$\left. \begin{aligned} & \phi_0 (m+2)(m+3) c_{m+1}^I \\ & + \left[ (m+2)(m+3) - \frac{1}{2}(1+\phi_0)(2m+5+\sigma) \right] c_m^I \\ & \quad - \frac{1}{2}(2m+5+\sigma) c_{m-1}^I = 0 \\ & c_0^I = -w\omega^2(3+\sigma)/(8gj^{3/2}) \end{aligned} \right\} \quad (22a)$$

$$\left. \begin{aligned} & \phi_0 (m+3)(m+4) c_{m+1}^{II} \\ & + \left[ (m+3)(m+4) - \frac{1}{2}(1+\phi_0)(2m+7+\sigma) \right] c_m^{II} \\ & \quad - \frac{1}{2}(2m+7+\sigma) c_{m-1}^{II} = 0 \\ & c_0^{II} = -w\omega^2(5+\sigma)/(24gj^{3/2}\phi_0) \end{aligned} \right\} \quad (22b)$$

$c^{III}$  satisfies the first expression of (22a) with

$$c_0^{III} = -\alpha_0 T_0 E_0 (\phi_1 + \phi_2) / (4j^{1/2} \phi_1 \phi_2) \quad (22c)$$

$c^{IV}$  satisfies the first expression of (22b) with

$$c_0^{IV} = -\alpha_0 T_0 E_0 / (6j^{1/2} \phi_1 \phi_2) \quad (22d)$$

### De Laval Profile

When  $\sigma_r$ ,  $T$ ,  $\alpha$  and  $E$  are all constant, Eq. (4) reduces to

$$\frac{d^2}{dr^2} (\lg Z) + \frac{2+\sigma}{r} \cdot \frac{d}{dr} (\lg Z) = -\frac{w\omega^2(3+\sigma)}{g\sigma_r}$$

the primitive of which is

$$Z = C_1 \exp(-C_2 r^{-1-\sigma}) + \exp(-w\omega^2 r^2 / 2g\sigma_r)$$

The last term is a particular integral giving well known De Laval profile.

### IV Disk of Uniform thickness

When the disk is of uniform thickness Eq. (4) becomes

$$\begin{aligned} & \frac{d^2}{dr^2} (r\sigma_r) + \left( \frac{1}{r} - \frac{d}{dr} \lg E \right) \frac{d}{dr} (r\sigma_r) - \left( \frac{1}{r^2} - \frac{\sigma}{r} \frac{d}{dr} \lg E \right) (r\sigma_r) \\ & = -\frac{w\omega^2 r^2}{g} \left( \frac{3+\sigma}{r} - \frac{d}{dr} \lg E \right) - E \frac{d}{dr} (\alpha T) \end{aligned} \quad (23)$$

We assume the variation of

$$\left. \begin{aligned} \text{Young's modulus} & E = E_0 (1 - \lambda_c r^4) \\ \text{disk temperature} & T = T_0 (1 + \lambda_t r^2) \\ \text{coefficient of thermal expansion} & \alpha = \alpha_0 (1 + \lambda_\alpha r^2) \end{aligned} \right\} \quad (24)$$

$$(26)$$



Substituting (24) in Eq. (23) we get

$$\begin{aligned} \frac{d^3}{dr^3} (r\sigma_r) + \left( \frac{1}{r} + \frac{4\lambda_e^2 r^3}{1-\lambda_e^2 r^4} \right) \frac{d}{dr} (r\sigma_r) - \left( \frac{1}{r^2} + \frac{4\sigma\lambda_e^2 r^2}{1-\lambda_e^2 r^4} \right) (r\sigma_r) \\ = -\frac{w\omega^2 r^2}{g} \left( \frac{3+\sigma}{r} + \frac{4\lambda_e^2 r^3}{1-\lambda_e^2 r^4} \right) - E \frac{d}{dr} (\alpha T) \end{aligned} \quad (25)$$

the complete primitive (divided by  $r$ ) of which is

$$\begin{aligned} \sigma_r = C_1 F(A_1, A_2; 3/2; \lambda_e^2 r^4) + C_2 (\lambda_e r^2)^{-1} F(A_1 - \frac{1}{2}, A_2 - \frac{1}{2}; 1/2; \lambda_e^2 r^4) \\ + \sum_{n=1}^{\infty} a_n r^{2n} \end{aligned} \quad (26)$$

where  $A_1, A_2 = -\frac{1}{4} \pm \frac{1}{4} \sqrt{5-4\sigma}$

$$\begin{aligned} \text{and } a_1 = \frac{1}{2!} \left\{ -\frac{3+\sigma}{4} \cdot \frac{w\omega^2}{g} - \frac{1}{2} (\lambda_a + \lambda_t) \alpha_0 T_0 E_0 \right\} \\ a_2 = \frac{1}{3!} \left\{ -\lambda_a \lambda_t \alpha_0 T_0 E_0 \right\} \\ a_3 = \frac{1}{4!} \lambda_e^2 \left\{ \frac{1-\sigma^2}{4} \cdot \frac{w\omega^2}{g} + \frac{1}{2} (5-\sigma) (\lambda_a + \lambda_t) \alpha_0 T_0 E_0 \right\} \\ a_4 = \frac{1}{5!} \lambda_e^2 \left\{ (11-\sigma) \lambda_a \lambda_t \alpha_0 T_0 E_0 \right\} \\ a_5 = \frac{1}{6!} \lambda_e^4 \left\{ \frac{(5+\sigma)(1-\sigma^2)}{4} \cdot \frac{w\omega^2}{g} + \frac{1}{2} (1-\sigma^2) (\lambda_a + \lambda_t) \alpha_0 T_0 E_0 \right\} \\ a_6 = \frac{1}{7!} \lambda_e^4 \left\{ (1-\sigma^2) \lambda_a \lambda_t \alpha_0 T_0 E_0 \right\} \end{aligned} \quad (26a)$$

and for  $a_7, a_8, \dots$

$$a_n = \left( 1 - \frac{5+\sigma}{n} - \frac{1-\sigma}{n+1} \right) \lambda_e^2 a_{n-2}, \quad n = 7, 8, 9, \dots$$

The tangential stress is

$$\begin{aligned} \sigma_\theta = C_1 \{ F(A_1, A_2; 3/2; \lambda_e^2 r^4) - \frac{2}{3} (1-\sigma) \lambda_e^2 r^4 F(A_1+1, A_2+1; 5/2; \lambda_e^2 r^4) \} \\ + C_2 \{ -(\lambda_e r^2)^{-1} F(A_1 - \frac{1}{2}, A_2 - \frac{1}{2}; 1/2; \lambda_e^2 r^4) \\ + 2(1+\sigma) \lambda_e r^2 F(A_1 + \frac{1}{2}, A_2 + \frac{1}{2}; 3/2; \lambda_e^2 r^4) \} \\ + \sum_{n=1}^{\infty} (2n+1) a_n r^{2n} + w\omega^2 r^2/g \end{aligned} \quad (27)$$

in which  $a$ 's are given by (26a).

The displacement component  $u$  is

$$\begin{aligned} u = C_1 (1-\sigma) E_0^{-1} r F(A_1+1, A_2+1; 3/2; \lambda_e^2 r^4) \\ - (C_2/\lambda_e) (1+\sigma) E_0^{-1} r^{-1} F(A_1 + \frac{1}{2}, A_2 + \frac{1}{2}; 1/2; \lambda_e^2 r^4) + r \sum_{n=1}^{\infty} c_{2n+1} r^{2n} \end{aligned} \quad (28)$$

where  $c$ 's are given by

$$c_3 = \frac{1}{2!} \left\{ -\frac{1-\sigma^2}{4} \cdot \frac{w\omega^2}{gE_0} - \frac{1}{2} (1+\sigma) (\lambda_a + \lambda_t) \alpha_0 T_0 \right\} \quad (27)$$

$$\begin{aligned}
c_5 &= \frac{1}{3!} (1 + \sigma) (\lambda_e^2 - \lambda_a \lambda_t) \alpha_0 T_0 \\
c_7 &= \frac{1}{4!} \lambda_e^2 \left\{ - (5 + \sigma) \frac{(1 - \sigma^2)}{4} \frac{w\omega^2}{gE_0} - \frac{5}{2} (1 + \sigma) (\lambda_a + \lambda_t) \alpha_0 T_0 \right\} \\
c_9 &= \frac{1}{5!} \lambda_e^2 \left\{ (11 + \sigma) \lambda_e^2 + (1 - \sigma) \lambda_a \lambda_t \right\}
\end{aligned} \tag{28a}$$

and for  $n = 5, 6, 7, \dots$

$$c_{2n+1} = \left( 1 - \frac{1 - \sigma}{n} - \frac{1 + \sigma}{n - 1} \right) c_{2n-1}$$

The constants  $C_1$  and  $C_2$  in (26) may be determined on the conditions

$$\begin{aligned}
&(\sigma_r)_{r=r_1} = \sigma_{r_1}, \quad (\sigma_r)_{r=r_2} = \sigma_{r_2}, \\
&\left\{ \begin{aligned} &r_1^2 (\sigma_{r_1} - \sum_{n=1}^{\infty} a_n r_1^{2n}) F(A_1 - \frac{1}{2}, A_2 - \frac{1}{2}; 1/2; \lambda_e^2 r_1^4) \\ &- r_2^2 (\sigma_{r_2} - \sum_{n=1}^{\infty} a_n r_2^{2n}) F(A_1 - \frac{1}{2}, A_2 - \frac{1}{2}; 1/2; \lambda_e^2 r_1^4) \end{aligned} \right\} \\
C_1 &= \frac{\left\{ \begin{aligned} &r_1^2 F(A_1 - \frac{1}{2}, A_2 - \frac{1}{2}; 1/2; \lambda_e^2 r_2^4) \cdot F(A_1, A_2; 3/2; \lambda_e^2 r_1^4) \\ &- r_2^2 F(A_1 - \frac{1}{2}, A_2 - \frac{1}{2}; 1/2; \lambda_e^2 r_1^4) \cdot F(A_1, A_2; 3/2; \lambda_e^2 r_2^4) \end{aligned} \right\}}{\left\{ \begin{aligned} &(\sigma_{r_2} - \sum_{n=1}^{\infty} a_n r_2^{2n}) F(A_1, A_2; 3/2; \lambda_e^2 r_1^4) \\ &- (\sigma_{r_1} - \sum_{n=1}^{\infty} a_n r_1^{2n}) F(A_1, A_2; 3/2; \lambda_e^2 r_2^4) \end{aligned} \right\}} \\
C_2/\lambda_e &= \frac{\left\{ \begin{aligned} &(\sigma_{r_2} - \sum_{n=1}^{\infty} a_n r_2^{2n}) F(A_1, A_2; 3/2; \lambda_e^2 r_1^4) \\ &- (\sigma_{r_1} - \sum_{n=1}^{\infty} a_n r_1^{2n}) F(A_1, A_2; 3/2; \lambda_e^2 r_2^4) \end{aligned} \right\}}{\text{same denominator as above}}
\end{aligned} \tag{29}$$

### Complete Disk of Uniform Thickness

Putting  $r_1 = 0$  in expressions (26), (27) and (28) we get  $\sigma_r$ ,  $\sigma_\theta$  and  $u$  for a complete disk as follows:

$$\begin{aligned}
\sigma_r &= C_1 F(A_1, A_2; 3/2; \lambda_e^2 r^4) + \sum_{n=1}^{\infty} a_n r^{2n} \\
\sigma_\theta &= C_1 \left\{ F(A_1, A_2; 3/2; \lambda_e^2 r^4) - \frac{2}{3} (1 - \sigma) \lambda_e^2 r^4 F(A_1 + 1, A_2 + 1; 5/2; \lambda_e^2 r^4) \right\} \\
&\quad + w\omega^2 r^2 / g + \sum_{n=1}^{\infty} (2n + 1) a_n r^{2n} \\
u &= C_1 (1 - \sigma) E_0^{-1} r F(A_1 + 1, A_2 + 1; 3/2; \lambda_e^2 r^4) + r \sum_{n=1}^{\infty} c_{2n+1} r^{2n}
\end{aligned}$$

Taking the radial stress at the circumference

$$\begin{aligned}
&(\sigma_r)_{r=r_2} = \sigma_{r_2} \\
\text{we get } C_1 &= (\sigma_{r_2} - \sum_{n=1}^{\infty} a_n r_2^{2n}) / F(A_1, A_2; 3/2; \lambda_e^2 r_2^4) \\
\text{and } \sigma_r &= (\sigma_{r_2} - \sum_{n=1}^{\infty} a_n r_2^{2n}) \frac{F(A_1, A_2; 3/2; \lambda_e^2 r^4)}{F(A_1, A_2; 3/2; \lambda_e^2 r_2^4)} + \sum_{n=1}^{\infty} a_n r^{2n}
\end{aligned} \tag{30}$$

$$\begin{aligned}
\sigma_\theta &= (\sigma_{r_2} - \sum_{n=1}^{\infty} a_n r_2^{2n}) \frac{F(A_1, A_2; 3/2; \lambda_e^2 r^4) - \frac{2}{3} (1 - \sigma) \lambda_e^2 r^4 F(A_1 + 1, A_2 + 1; 5/2; \lambda_e^2 r^4)}{F(A_1, A_2; 3/2; \lambda_e^2 r_2^4)} \\
&\quad + \frac{w\omega^2 r^2}{g} + \sum_{n=1}^{\infty} (2n + 1) a_n r^{2n}
\end{aligned} \tag{31}$$

$$u = (\sigma_{r_2} - \sum_{n=1}^{\infty} a_n r_2^{2n}) \frac{(1 - \sigma) r F(A_1 + 1, A_2 + 1; 3/2; \lambda_e^2 r^4)}{E_0 F(A_1, A_2; 3/2; \lambda_e^2 r_2^4)} + r \sum_{n=1}^{\infty} c_{2n+1} r^{2n} \tag{32}$$

If the condition be  $(\sigma_r)_{r=0} = \sigma_{r_0}$  we get  $C_1 = \sigma_{r_0}$ , and  $\sigma_r$ ,  $\sigma_\theta$  and  $u$  in (30), (31) and (32), with the last series in each expression in expanded form, may be

(28)

written as follows :

$$\begin{aligned} \sigma_r &= \sigma_{r0} F( A_1, A_2; 3/2; \lambda_e^2 r^4 ) \\ &- (w\omega^2/8g) \left\{ (3 + \sigma) r^2 - \frac{1}{12} (1 - \sigma^2) \lambda_e^2 r^6 - \frac{1}{360} (5 + \sigma) (1 - \sigma^2) \lambda_e^4 r^{10} - \dots \right\} \\ &- \frac{1}{4} \alpha_0 T_0 E_0 \left\{ (\lambda_\alpha + \lambda_t) r^2 + \frac{2}{3} \lambda_\alpha \lambda_t r^4 - \frac{1}{12} (5 - \sigma) \lambda_e^2 (\lambda_\alpha + \lambda_t) r^6 \right. \\ &\quad \left. - \frac{1}{30} (11 - \sigma) \lambda_e^2 \lambda_\alpha \lambda_t r^8 - \frac{1}{360} (1 - \sigma^2) \lambda_e^4 (\lambda_\alpha + \lambda_t) r^{10} - \dots \right\} \quad (33) \end{aligned}$$

$$\begin{aligned} \sigma_\theta &= \sigma_{r0} \left\{ F( A_1, A_2; 3/2; \lambda_e^2 r^4 ) \right. \\ &\quad \left. - \frac{2}{3} (1 - \sigma) \lambda_e^2 r^4 F( A_1 + 1, A_2 + 1; 5/2; \lambda_e^2 r^4 ) \right\} \\ &- (w\omega^2/8g) \left\{ (1 + 3\sigma) r^2 - \frac{7}{12} (1 - \sigma^2) \lambda_e^2 r^6 - \dots \right\} \\ &- \frac{1}{4} \alpha_0 T_0 E_0 \left\{ 3 (\lambda_\alpha + \lambda_t) r^2 + \frac{10}{3} \lambda_\alpha \lambda_t r^4 - \frac{7}{12} (5 - \sigma) \lambda_e^2 (\lambda_\alpha + \lambda_t) r^6 - \frac{3}{10} (11 - \sigma) \lambda_e^2 \lambda_\alpha \lambda_t r^8 - \dots \right\} \quad (34) \end{aligned}$$

$$\begin{aligned} u &= \sigma_{r0} \sqrt{\lambda_e} r F( A_1 + 1, A_2 + 1; 3/2; \lambda_e^2 r^4 ) \\ &- (w\omega^2/8gE_0) (1 - \sigma^2) \left\{ r^3 + \frac{1}{12} (5 + \sigma) \lambda_e^2 r^7 + \dots \right\} \\ &+ \frac{1}{4} \alpha_0 T_0 (1 + \sigma) \left[ - (\lambda_\alpha + \lambda_t) r^3 + \frac{2}{3} (\lambda_e^2 - \lambda_\alpha \lambda_t) r^5 + \frac{1}{12} (1 - \sigma) \lambda_e^2 (\lambda_\alpha + \lambda_t) r^7 \right. \\ &\quad \left. + \frac{1}{30} \lambda_e^2 \{ (1 - \sigma) \lambda_\alpha \lambda_t + (11 + \sigma) \lambda_e^2 \} r^9 + \dots \right] \quad (35) \end{aligned}$$

The first term containing hypergeometric function of the right hand side of (33) denotes the stress due to the radial static pressure at the circumference ( $r = r_2$ ) of disk. The second term denotes the stress due to rotation and the third term due to thermal expansion. The radial stress  $\sigma_\theta$  and the displacement  $u$  expressed by (34) and (35) are also made of 3 similar items as the above expression (33).

When  $\alpha$ ,  $T$  and  $E$  are all constant in the disk with a central hole,

(26), (27) and (28) reduce to, with  $\lambda_\alpha = \lambda_t = \lambda_e = 0$ ,

$$\sigma_r = \sigma_{r2} \frac{(r^2 - r_1^2) r_2^2}{(r_2^2 - r_1^2) r^2} + \sigma_{r1} \frac{(r_2^2 - r^2) r_1^2}{(r_2^2 - r_1^2) r^2} + \frac{(3 + \sigma) w\omega^2}{8g} \frac{(r_2^2 - r^2)(r^2 - r_1^2)}{r^2} \quad (36)$$

$$\begin{aligned} \sigma_\theta &= \sigma_{r2} \frac{(r^2 + r_1^2) r_2^2}{(r_2^2 - r_1^2) r^2} - \sigma_{r1} \frac{(r_2^2 + r^2) r_1^2}{(r_2^2 - r_1^2) r^2} + \frac{(3 + \sigma) w\omega^2}{8g} \frac{(r_2^2 + r^2)(r^2 + r_1^2)}{r^2} \\ &\quad - \frac{(1 + \sigma) w\omega^2 r^2}{2g} \quad (37) \end{aligned}$$

$$\begin{aligned} u &= \frac{1 - \sigma}{E_0} \frac{r_2^2 \sigma_{r2} - r_1^2 \sigma_{r1} + (3 + \sigma) (w\omega^2/8g) (r_2^4 - r_1^4)}{r_2^2 - r_1^2} r \\ &+ \frac{1 + \sigma}{E_0} \frac{r_1^2 r_2^2 \{ \sigma_{r2} - \sigma_{r1} + (3 + \sigma) (w\omega^2/8g) (r_2^2 - r_1^2) \}}{r_2^2 - r_1^2} \cdot \frac{1}{r} - \frac{1 - \sigma^2}{8} \frac{w\omega^2}{gE_0} r^3 \quad (38) \end{aligned}$$

For complete disk we get

$$\sigma_r = \sigma_{r2} + \{ (3 + \sigma) w\omega^2 / 8g \} (r_2^2 - r^2) \quad (39)$$

$$\sigma_\theta = \sigma_{r2} + \{ (3 + \sigma) w\omega^2 / 8g \} \{ r_2^2 - r^2 (1 + 3\sigma) / (3 + \sigma) \} \quad (40)$$

$$u = \frac{1 - \sigma}{E_0} \left\{ \sigma_{r2} + \frac{(3 + \sigma) w\omega^2}{8g} \left( r_2^2 - \frac{1 + \sigma}{3 + \sigma} r^2 \right) \right\} r \quad (41)$$

( to be continued )