Study on Relaxation and Sinusoidal Oscillations

(Received Oct. 10, 1951)

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Abstracts

A relaxation and a sinusoidal oscillation are expressed by the same type differential equations. But the mechanism of the oscillations are different entirely. We can find the distinction of them.

I. Introduction

Differential equations concerning vacuum tube oscillators with linear characteristics of tubes, are constant co-efficient linear type. The simplest case is the second order differential equation,

$$\ddot{x} + 2h_0\dot{x} + \omega x_0^2 = 0 \qquad \cdots\cdots \quad (1, 1)$$

where dots denote time derivatives. The constants h_0 and ω_0 depend on the circuit elements. The integral of (1, 1) is

$$x = K_1 e^{q_1 t} + K_2 e^{q_2 t} = A e^{-h_0 t} \sin(\sqrt{h_0^2 - \omega_0^2}\, t + \varphi) \qquad \cdots\cdots \quad (1, 2)$$

where K_1, K_2, A and φ are arbitrary constants and q_1, q_2, are roots of characteristic equation of (1. 1)

However, (1, 2) can not be applied to real oscillators which have non-linear characteristic in vacuum tubes. (1, 2) does not mean constant amplitude oscillations except when $h_0 = 0$. Even if $h_0 = 0$, the the amplitude depends on arbitrary constant A which is decided from initial conditions. On the other hand, in actual case the amplitude always approaches asymptotically to a constant value independent of initial conditions. This is one of reasons that non-linear differential equation can not be solved easily. Namely, exact solution for non-conservative system should not have such arbitrary constants that are related linearly with function of time as (1, 2). One example is perturbation method to get a solution of non-linear differential equation sligntly differed frcm (1, 1). By the methnd, the right side of (1, 2) is corrected successively and arbitrary constant are not related linearly with the functlon of time.

Now let us try to correct the left side of (1, 2). For some non-linear differential equations, exactly or approximately the solution may be often

$$f(x) = K_1 e^{q_1 t} + K_2 e^{q_2 t} \qquad \cdots\cdots \quad (1, 3)$$

Differential equation for (1, 3) is

$$\ddot{x} + \left\{ \frac{f''(x)}{f'(x)} \dot{x} - (q_1 + q_2) \right\} \dot{x} + q_1 q_2 \frac{f(x)}{f'(x)} = 0 \qquad \cdots\cdots \quad (1, 4)$$

where dash denote differentiation respect to dependent variable x. The following is based on (1, 3) and (1, 4).

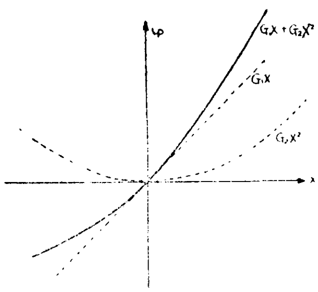

Fig. 1. Unsymmetric characteristc curve for the equation (2. 1)

II. Wave distortion in unsymmetric characteristic.

As the first example of (1, 3) and (1, 4), we shall examine a plate tuned oscillator which has unsymmetric characteristic of vacuum tube. (See Fig. 1.)

Here, we do not take account of saturation charactristic, then its characteristic is expressed by

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$i_g = G_1 (e_g + e_p/\mu) + G_2 (G_g + e_p/\mu)^2$ (2,1)

The circuit and its notations are given by Fig. 2. i_p, e_g, e_p are alternating components of plate current, grid voltage and plate voltage respectively. G_1 and μ is mutual conductance and amplification factor of tube. If grid current is neleglected, the equations corresponding to our scheme are

$$\left.\begin{aligned} -e_p &= L\frac{di}{dt} \\ &= (1/C)\int (i_p - i)\, dt \\ e_g &= M\frac{di}{dt} \end{aligned}\right\} \cdots (2, 2)$$

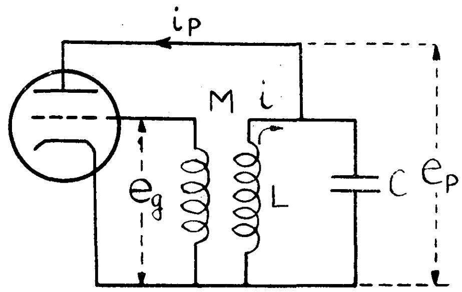

Fig. 2. Plate tuned oscillator circuit expressed by the differential equation (2,3).

From (2, 2), we can get the second order differential equation of current i.

$$\frac{d^2i}{dt^2} + \left\{ - \frac{G_2 (M - L/\mu)^2}{LC} \frac{di}{dt} - \frac{G_1 (M - L / \mu)}{LC} \right\} \frac{di}{dt} + \frac{1}{LC} i = 0 \cdots\cdots (2, 3)$$

we simplify the notations as follows;

$$i = x \qquad\qquad 1/LC = \omega^2$$

$$- \frac{G_2(M - L/\mu)^2}{LC} = 2\gamma \qquad\qquad - \frac{G_1 (M - L/\mu)}{LC} = 2h$$

Then (2, 3) is

$$\frac{d^2x}{dt^2} + \left\{ 2\gamma\frac{dx}{dt} + 2h \right\} \frac{dx}{dt} + \omega^2x = 0 \qquad\cdots\cdots (2, 4)$$

If h is small and negative, the oscillation is so hard that x is small and its wave form is distorted. Then we can assume γ and x is small.

The approximate solution of (2, 4) is

$$\begin{aligned} xe^{\gamma x} &= K_1e^{(-h + \sqrt{h^2 - \omega^2})t} + K_2e^{(-h \sqrt{h^2 - \omega^2})t} \\ &= Ae^{-ht} \sin (\sqrt{h^2 - \omega^2}\, t + \varphi) \end{aligned} \qquad\cdots\cdots (2,5)$$

(2, 5) is easily understood by the aid of following substitution to (1, 4),

$$f (x) = xe^{\gamma x}$$

Then

$$\frac{f (x)}{f' (x)} = \frac{x}{1 + \gamma x} = x - \gamma x^2 + \gamma^2x^3 - \cdots\cdots \doteqdot x$$

and

$$\frac{f'' (x)}{f' (x)} = \frac{2\gamma + \gamma^2x}{1 + \gamma x} = 2\gamma - \gamma^2x \cdots\cdots \cdots\cdots \doteqdot 2\gamma$$

In order to know the wave form, we shall examine the curve in x-plane representing a circle, the center of which is at origin in z-plane where z means $xe^{\gamma x}$.

We take comlex coordinate

$$\left.\begin{aligned} z &= u + jv \\ x &= \xi + j\eta \end{aligned}\right\} \qquad\cdots\cdots (2,6)$$

By applyiug (2, 6) to $z = x\, e^{\gamma x}$

$$\left.\begin{aligned} u &= e^{\gamma\xi}(\xi\cos\gamma\eta - \eta\sin\gamma\eta) \\ v &= e^{\gamma\xi}(\eta\cos\gamma\eta + \xi\sin\gamma\eta) \end{aligned}\right\} \qquad\cdots\cdots (2,7)$$

The circle in z-plane, the radius of which is r and whose center is at origin, is

$$u^2 + v^2 = r^2 \qquad \cdots\cdots (2, 8)$$

from (2, 7) and (2, 8), we obtain

$$\eta^2 = r^2 e^{-2\gamma\xi} - \xi^2 \qquad \cdots\cdots (2, 9)$$

Fig. 3. (a) shows curves of $r^2\, e^{-2\gamma\xi}$ and $-\xi^2$. Therefore, a closed curve in (b) represents (2, 9). (c) and (d) are the wave forms of current i and voltage e_p, which is slightly different from sine wave.

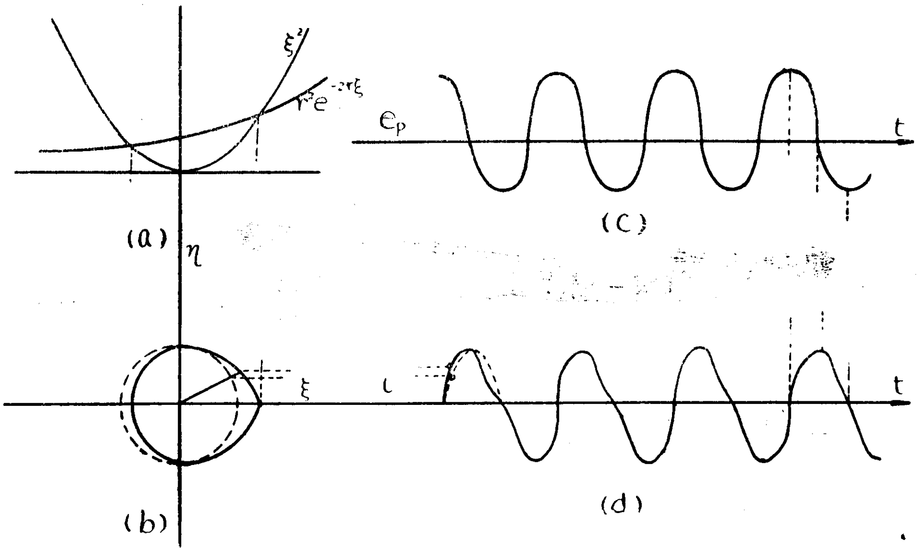

Fig. 3. (a) Curves expressing the equation (2. 9).
(b) Closed curve representing a circle of (2. 8).
(c) Wave form of current i that flows in the inductance L in Fig. 2.
(d) Wave form of plate voltage got from (c) by differentiation.

III. Relaxation oscillation

Now, we shall discuss a symmetric multivibrator which is the most well-known relaxation oscillator. The circuit is shown in Fig. 4.

Neglecting the grid current, we obtain the equations (3, 1) by Kirchhoff's low.

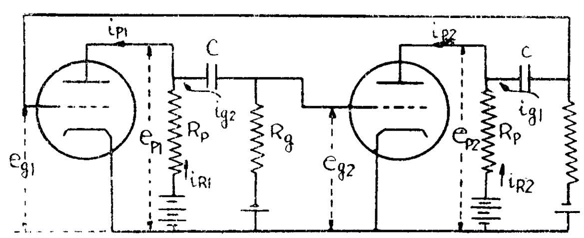

Fig. 4. Multivibrator expressed by the differential equation(3. 3)

$$i_{p1} = i_{r1} + i_{g2}$$
$$-e_{p1} = i_{r1} R_p + (1/c)\int i_{g2}\, dt \quad \cdots (3, 1)$$
$$-e_{1g} = i_{g1} R_p$$

where currents and voltages do not contain D. C. component. The vacuum tubes are assumed to have symmetric saturated characteristics and shown approximately by the third degree polynomial (3, 2)

$$i_{p1} = G_1 (e_{g1} + e_{p1}/\mu) + G_3 (e_{g1} + e_{p1} /\mu)^3 \qquad \cdots\cdots (3, 2)$$

As the circuit is symmetric, we have

$$i_{g2} = - i_{g1}$$

Then we can obtain the differential equation

$$\frac{dx}{dt} = \frac{a_0 x + a_2 x^3}{b_0 + b_2 x^2} \qquad \cdots\cdots (3, 3)$$

where $x = (e_{g1} + e_{p1}/\mu)$

$$\begin{aligned} a_0 &= -(1/R_p + G_1/\mu)/C \\ b_0 &= (1 + R_g/R_p) - G_1R_g(1 - 1/\mu) \\ a_2 &= -G_3/\mu C \\ b_2 &= -3G_3R_g(1 - 1/\mu) \end{aligned} \qquad \cdots\cdots\cdots (3,4)$$

(3, 3) can be integrated easily as following

$$x(x^2 + a_0/a_2)^{\frac{1}{2}\left(\frac{a_0 b_1}{b_0 a_2} - 1\right)} = Ke^{\frac{b_0}{a_0}t} \qquad \cdots\cdots (3,5)$$

Where K is arbitrary integral constant. We set $z =$ (the left side of (3, 5).). We may understand the mechanism of the oscillation by a curve in z-piane representing real axis of z-plane, as a_0/b_0 is real.

Before we reseach the representation, we must notice the case in which we put self-inductancee in series to the condensers. Then the differential equation is second order

$$\frac{d^2x}{dt^2} + \left\{ -\frac{b_0 + b_2x^2}{a_0 + 3a_2x^2}\frac{1}{LC} + \frac{6a_2x}{a_0 + 3a_2x^2}\frac{dx}{dt} \right\}\frac{dx}{dt} + \frac{a_0x + a_2x^3}{a_0 + 3a_2x^2}\frac{1}{LC} = 0 \cdots (3,6)$$

In usual circuit,

$$3a_2/a_0 \doteqdot b_2/b_0$$

Then comparing (3, 6) and (3, 5) with (1, 3) (1, 4), we obtain the approximate solution (3, 7)

$$x(x^2 + a_0/a_2)^{\gamma} = K_1e^{q_1t} + K_2e^{q_2t} \qquad \cdots\cdots (3,7)$$

when $\gamma = \frac{1}{2}\left(\frac{a_0 b_2}{b_0 a_2} - 1\right)$

$$q_1 = (b_0/2a_0LC) + \sqrt{(b_0/2a_0LC)^2 - 1/LC}$$

$$q_2 = (b_0/2a_0LC) - \sqrt{(b_0/2a_0LC)^2 - 1/LC}$$

When q_1 q_2 are real, mechanism is almost the same as the case in (3, 5).

We transform x and z into polar coodinate

$$x = re^{j\theta}$$

$$z = Re^{j\Theta}$$

After simple calculaticn, we obtain

$$R = r\{(r\cos\theta - \sqrt{a_0/a_2}j)^2 + r^2\sin^2\theta\}^{\frac{\gamma}{2}}\{(r\cos\theta + \sqrt{a_0/a_2}j)^2 + r^2\sin^2\theta\}^{\frac{\gamma}{2}} \cdots (4,1)$$

$$\Theta = \theta + \gamma\tan^{-1}\frac{r\sin\theta}{r\cos\theta - \sqrt{a_0/a_2}j} + \gamma\tan^{-1}\frac{r\sin\theta}{r\cos\theta + \sqrt{a_0/a_2}j} \qquad \cdots\cdots (4,2)$$

Positive real axis of z-plane is

$$\Theta = 0 \qquad \cdots\cdots (4,3)$$

Setting the second and third term on (4, 2) to φ_A and φ_B respectively, these are the angles around the points $\pm\sqrt{a_0/a_2}j$ to a representative point p. (See Fig.5)

We can prove easily that the trajectory for $\varphi_A + \varphi_B = k$ (const.) is an orthogonal hyperbola which approaches to the axis inclined at $-k/2$. (See Fig, 6.)

The intersected point of the hyperbola and $\theta = -k\gamma$, satisfies the group of curves for (4, 3) in Fig.7.

If γ is rational number, the number of curves is finite, and if not, it is infinite. But vacuum tube is not so stable that we need not to pay any attention to this matter.

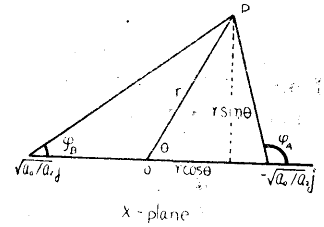

Fig. 5.

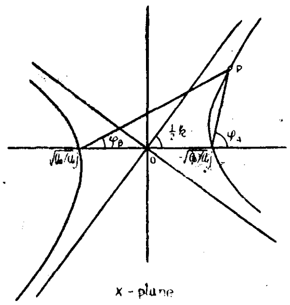

Fig. 6.

We should not take in consideration of imaginary axis in x-plane from a physical point of view.

When small x is given at the initial condition, the representntive point moves along the real axis, as time t increaces. After representativ point reaches the point of $dR/dr = 0$ ($x = \pm\sqrt{b_0/b_2 j}$), it can not go forward physically. As $dR/dr = 0$ means $dx/dt = \infty$, x jumps here. Condition of the jump is decided by the continuity of energy containd in the reactance of the circuit.

That is

$$\frac{1}{2}\frac{1}{C}\left\{\int i_g\, dt\right\}^2 + \frac{1}{2} L\, i_g^2 = E \qquad \cdots\cdots (4,4)$$

Where E is constant. Then, ... (4, 5)

$$\int i_g\, dt = \pm\sqrt{2EC - LCi_g^2} \qquad \cdots\cdots (4,6)$$

The solution of integral equation (4, 5)

$$\pm\, i_g / \sqrt{2EC} = \sin (2t / \sqrt{LC})$$

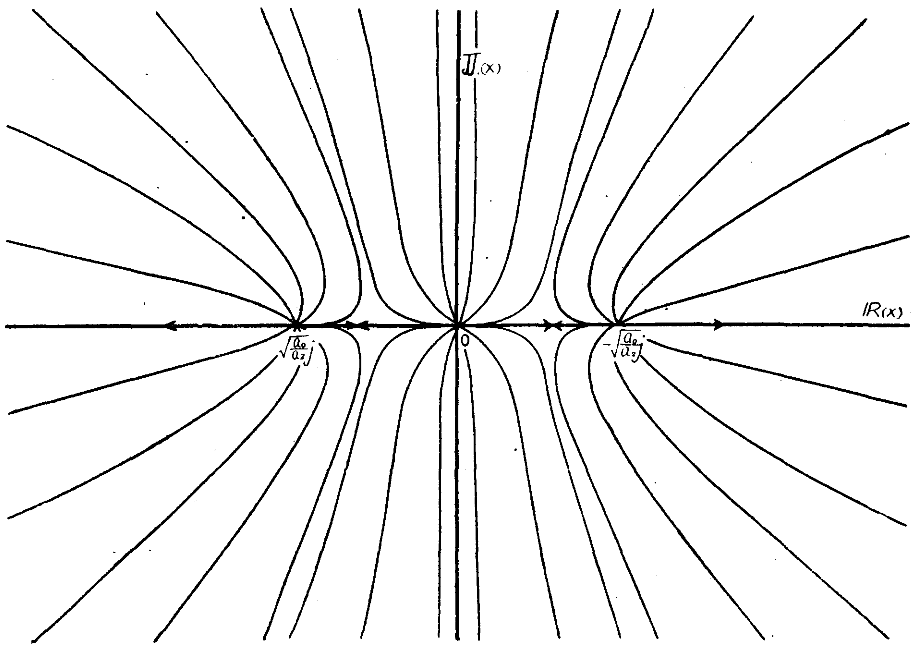

Fig. 7. Curves in x-plane representing the real axis in z-plane.

from (3, 1) (3, 4) and (4, 6), we obtain the rélation betweon di_p/dx and x.

For small L;

$$\frac{di_p}{dx} = -\frac{\left(\frac{1}{R_p} + \frac{1}{R_g}\right) - \frac{9L}{4R_pR_g^2(1+\mu)}\left(1 \pm \sqrt{b_0/b_2}\, jx\right)}{\left(1 + \frac{1}{\mu}\right) - \frac{9L}{4\mu CR_g^2(1+\mu)}\left(1 \pm \sqrt{b_0/b_2}\, jx\right)} \qquad \cdots\cdots (4.7)$$

When there is no inductance, di_p/dx is constant. When the circuit has a small inductance, di_p/dx takes the same constant value at the initial instant of jump, when $x = \pm\sqrt{b_0/b_2}\, j$. The path of the jump in x-i_p plane is shown Fig. 8. Doted line is the path when $L = o$, Fig. 9. is the wave form of i_p.

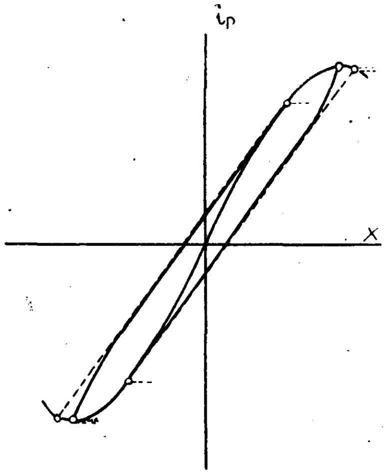

Fig. 8. Path of the representative point in i_p-x plane when the oscillation is relaxation.

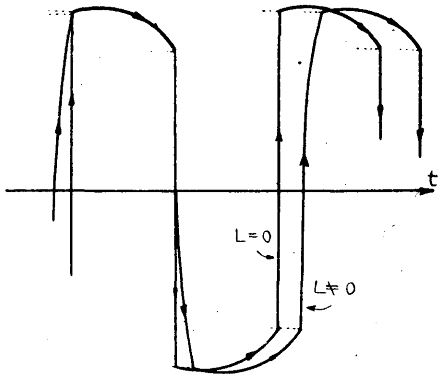

Fig. 9. Relaxation wave form of plate current i_p got from Fig. 8.

V. Sinusoidal oscillation

In this section, we shall study on the case when the right side of (3, 7) is oscillatory. If q_1, q_2 are imaginary, z executes an increaced oscillatory process. So we should examine a curve in x-plane representing a circle in z-plane the center of which is at origin.

Then we can say, from (4, 1)

$$r\{(r\cos\theta-\sqrt{a_0/a_2}\,j)^2+r^2\sin^2\theta\}^{\frac{\nu}{2}}\ \{(r\cos\theta+\sqrt{a_0/a_2}\,j)^2+r^2\sin^2\theta\}^{\frac{\nu}{2}} = \rho \;\cdots\cdots\; (5,1)$$

where ρ is a constant radius of a circle in z-plane. Facters in the brackets of (5,1) is square of the distance from $\pm\sqrt{a_0/a_2}\,j$ to a representative point. (See Fig. 10.) Then we obtain closed curves for various ρ in (5, 1) shown in Fig. 11.

At the begining of oscillation, representative point goes along inner closed curves, Then gradually it moves to the outer closed curves. This osillatory case is caused when G_1 is negative, known as transitron character, (Fig, 12).

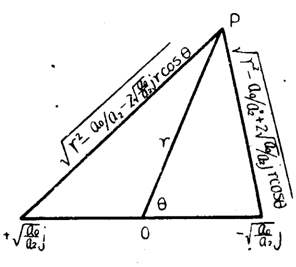

Fig. 10.

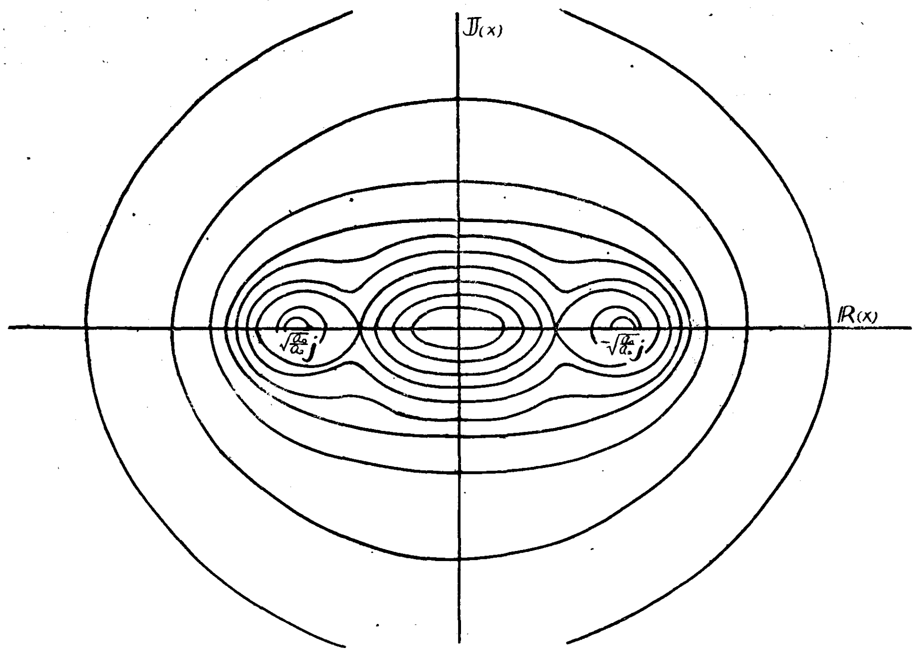

Fig. 11. Closed curves representing the circles in z-plane.

We can recognize the following distinctions, period and initial transient phenomena, between relaxation and sinusoidal oscillation.

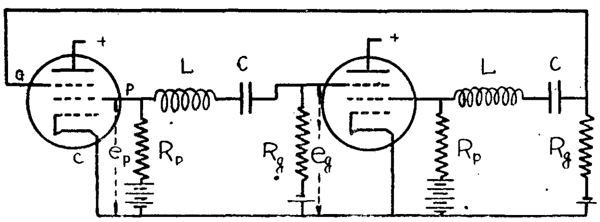

Fig. 12. Sinsoidal osillator corresponding to the multivibrator.

(1) Period,

In (3, 5) period of relaxation oscillation is proportional to b_0 / a_0 and the proportional constant is decided from nonlinearity. In (3, 7), we can find almost equal period to (3, 5).

But sinusoidal period is $2\pi / \sqrt{(b_0/2a_0LC)^2 - 1/LC}$ which is extremely short than the relaxation period. The circuits of both cases have same circuit elements except G_1, G_2 and μ.

(2) Initial transient phenomena,

Relaxation oscillatlon build up rapidly in one period. (See Photo. 1) While sinusoidal oscillation does gradually. (See Photo. 2.)

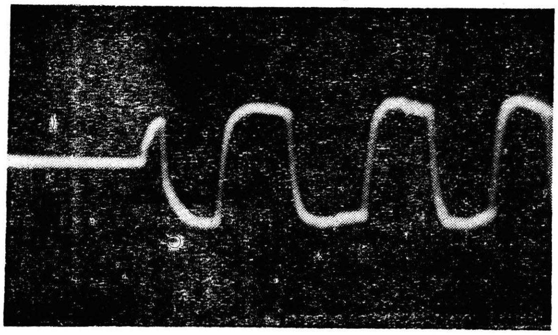

Photo. 1. Initial wave form of multivibrator.

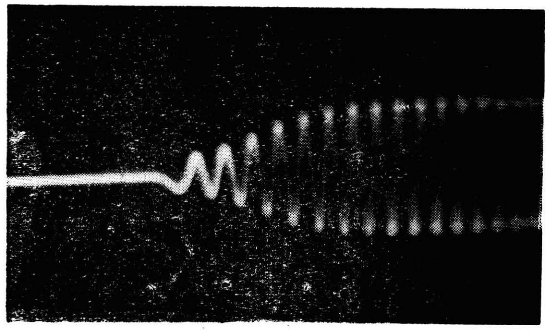

Photo. 2. Initial wave form of sinusoidal oscillator.

Appendix.

B. van der Pol showed[1] that the symmetric multivibrator was equivalent to transitron relaxation oscillator. (See Fig. 13.) Sinusoidal oscillator corresponding to Fig 13. is shown in Fig. 14.

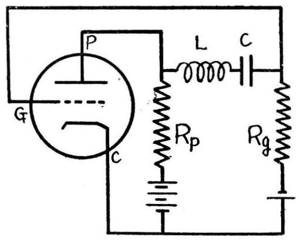

Fig. 13, Equivalent sinusoidal oscillator circuit to Fig. 12.

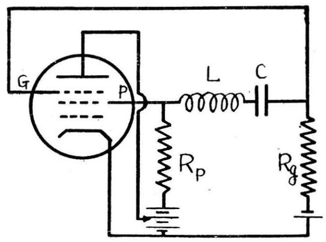

Fig. 14. Transitron relaxation oscillator corresponding to Fig. 13.

(1) B. van der Pol ; Phil. Mag. 2 (1926)