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On a Property of Linear Ordinary Differential Equation which relates to "End Effect" in Theory of Curved Shells *

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Fumiki Kito **

Abstract

Let there be given a linear ordinary differential equation of the form:

$$A_0\frac{d^ny}{dx^n} + A_1\frac{d^{n-1}y}{dx^{n-1}} + \cdot\cdot\cdot\cdot\cdot + A_{n-1}\frac{dy}{dx} + A_ny = 0$$

where first s coefficients $A_0, A_1, \text{----} A_s$ have their magnitudes very small compa with the remaining coefficients $A_{s+1}, \text{----} A_n$. Such a circumstance occurs frequen in the theory of thin curved shells.

In the present paper, it is shown that under such a circumstance there contained in the general solution just (s + 1) functions of very rapidly vary nature as the independent variable x varies. These (s + 1) functions will repres so called " Edge Effect " in the theory of thin curved shells. The argument based on Volterra's theory of Composition of Functions.

I Introduction

The theory of thin curved shell is frequently required in connection with st gth of machine parts etc. An example of which is seen in the problem of stre gth of spiral casings of water turbines and pumps. The Author has previou made some considerations on deformations of thin shell whose middle surface i form of anchor ring, with the object of throwing some light on the strength those spiral casings of hydraulic machinery.[1] Suppose we take up a thin shel elastic material whose middle surface is in from of a surface of revolution.

To find its state of equilibrium (or of vibration) when it is acted upon by axi symmetrical loads, and under some edge conditions which are also axially s metrical, is one of difficult problems in theory of elasticity.

One reason of complexity of calculation may be attributed to phenomenon o called " Edge Effect " , which appear in the neibourhood of edge, but soon di pear at some distance from the edge. When we examine the differential equt which is satisfied by the components of displacement (or some combinations them, known usually as Variable of Reissner-Meissner) we observe that coeffice

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(1) This Journal, Vol. 1. No. 3, 1948. F. Kito, A Theoretical Study on the Strength of a S in Form of Torus. (1)

of this differential equation contain certain parameter

$$\mu = \frac{1}{12(1-\nu^2)}\left(\frac{h}{R}\right)^2$$

where ν = Poisson's ratio, R = representative Radius, h = representative thickness, of the shell. In case of thin shell, the value of this parameter μ is very small (for ex., $\mu = 0.3\times10^{-4}$ for a cylindrical shell of 1 m dia. and 10 mm thick.)

If we take $\mu = 0$, this means that we take the shell to be a membrane, and there exist no "Edge Effect".

In connection with the above statement, we consider the following problem in theory of ordinary linear differential equations; there being given a linear ordinary differential equation of the form

$$A_0\frac{d^n y}{dx^n} + A_1\frac{d^{n-1}y}{dx^{n-1}} + \cdot\cdot\cdot\cdot + A_{n-1}\frac{dy}{dx} + A_n y = 0$$

where A_0, A_1, …… A_n are given functions of the independent variable x, first s+1 coefficients A_0, A_1,……A_s among them have their magnitudes very small as compared with the remaining coefficients A_{s+1}……A_n. In such a case, it is required to see how the general solution is characterized by this fact.

In what follows, some results of study made by the Author concerning this problem will be given. The variables contained in our discussion is confined to real domain, in order to simplify thoughts.

II Preliminary Remark on a Property of Algebraic Equations.

The following Lemma concerning the property of algebraic equations, is mentioned here, as it is closely connected with our problem.

Given an algebraic equation of degree n:-

$$\mu\,[1+A_1x+A_2x^2] + A_3x^3 + \cdots\cdots + A_nx^n = 0 \qquad \cdots\cdots (1)$$

the numerical factor μ is supposed to be of very small magnitude. In such a case the roots of this equation (1) can be expressed in following form of power series in μ and ν, where we put

$$\nu = \sqrt[3]{\mu} \qquad \text{and} \qquad \omega = \frac{1}{2}(-1+\sqrt{3}\,i)$$

$$x_1 = -\frac{1}{\sqrt[3]{A_3}}\nu + \alpha_{12}\nu^2 + \alpha_{13}\nu^3 + \cdots\cdots$$

$$x_2 = -\frac{1}{\sqrt[3]{A_3}}\omega\nu + \alpha_{22}\nu^2 + \alpha_{23}\nu^3 + \cdots\cdots$$

$$x_3 = -\frac{1}{\sqrt[3]{A_3}}\omega^2\nu + \alpha_{32}\nu^2 + \alpha_{33}\nu^3 + \cdots\cdots$$

$$x_4 = \alpha_4 + \alpha_{41}\mu + \alpha_{42}\mu^2 + \cdots\cdots$$

$$\cdots\cdots \qquad \cdots\cdots \qquad \cdots\cdots$$

$$x_n = \alpha_n + \alpha_{n1}\mu + \alpha_{n2}\mu^2 + \cdots\cdots$$

When $\mu\to0$, these roots become

$$x_1 = x_2 = x_3 = 0, \qquad x_4 = \alpha_4, \cdots\cdots x_n = \alpha_n$$

Similar remark can be made regarding the algebraic equation of the from

$$\mu[1 + A_1x + A_2x_2 + \cdots\cdots\cdots + A_mx^m] + A_{m+1}x^{m+1} + \cdots\cdots\cdots + A_nx^n = 0 \cdots (2)$$

III Operational Solution of Linear Ordinary Differential Equation under a given Initial Condition.

Let us consider a linear ordinary differential equation of order n :-

$$\mu[p_0(t)\frac{d^n u}{dt^n}+p_1(t)\frac{d^{n-1}u}{dt^{n-1}}+p_2(t)\frac{d^{n-2}u}{dt^{n-2}}]$$
$$+p_3(t)\frac{d^{n-3}u}{dt^{n-3}}+\quad\cdots\cdot\quad+p_{n-1}(t)\frac{du}{dt}+p_n(t)u=q(t),\qquad\cdots\cdots(3)$$

where $p_1(t)\cdots\cdots p_n(t)$ and $q(t)$ are given functions of independent variable t. (they are assumed to be continuous functions, of bounded values, of t) u is the unknown function to be determined. The parameter μ is introduced to represent very small number. The first coefficient $p_0(t)$ may be assumed to be equal to 1, without loss of generality. Instead of regarding u to be function of t, we regard it to be function of $\xi=t-\tau$, which means that the initial state is taken at the point $t=\tau$, τ being an arbitrary (but fixed) number. In accordance to that, we also regard $p_1(t),\cdots\cdots$ as functions of $\xi=t-\tau$, which means thet we put $\xi+\tau$ instead of t in expression for $p_1(t)$, etc. According to the notation of Volterra[2], the so-called "Composition" of two functions $f(x,y)$, $g(x,y)$ is defined by

$$\int_x^y f(x,\xi)\,g(\xi,y)\,d\xi$$

and the resultant of composition is denoted by $f^*g^*(x,y)$ or briefly by f^*g^*. When two functions $f(x,y)$ and $g(x,y)$ are such that $f^*g^*\equiv g^*f^*$ they are said to be permutable to each other. Two functions of the form $f(x-y), g(x-y)$ are always permutable to each other.

Now, put

$$\frac{d^n u}{dt^n}=U_n,\qquad\frac{d^{n-1}u}{dt^{n-1}}=U_{n-1},\cdots\cdots\frac{du}{dt}=U_1,\qquad u=U_0\quad.$$

Integrating the first relation between the limits $t=\tau$ to $t=t$ we have

$$\int_\tau^t\frac{d^n u}{dt^n}\,dt=U_{n-1}-A_{n-1}$$

where A_{n-1} is the value of U_{n-1} at $t=\tau$ [or $\xi=0$] Using the Calculus of Composition, this relation may be written,

$$1^*U_n{}^*=U_{n-1}-A_{n-1}$$

In this way, we have the following system of equations written in style of Calculus of Composition, the last equation being obtained by integration of both sides of equation (3) between the limits from $t=\tau$ to $t=t$.

$$\left.\begin{aligned}&1^*U_n{}^*=U_{n-1}-A_{n-1}\\&1^*U_{n-1}{}^*=U_{n-2}-A_{n-2}\\&\cdots\quad\cdots\quad\cdots\\&1^*U_1{}^*=U_0-A_0\\&\mu[1^*U_n{}^*+p_1{}^*U_{n-1}{}^*+p_2{}^*U_{n-2}{}^*]+p_3{}^*U_{n-3}{}^*\\&+\cdots\cdots+p_{n-1}{}^*U_1{}^*+p_n{}^*U_0{}^*=1^*q^*\end{aligned}\right\}\qquad\cdots\cdot(5)$$

Further we put

$$p_i{}^*[\xi]U^*\equiv\Pi_i{}^*[\xi]1^*U^*\qquad\cdots\cdot\ (6)$$

(2) V. Volterra.—J.Pérès, Lecons sur la Composition et les fonctions permutables, 1924.

which means that, we determine Π_i, associated with p_i, so that the effect of operation of comosition $\Pi_i{*}1^*$ upon U is the same as that of p_i^*. Put (6) into (5), and write X^*, instead of 1^*, in order to emphasize the operation of 1^*. The system of equations thus obtained has the form of simultaneous algebraic linear equation, in which the usual multiplication is here replaced by operation of composition. Since functions contained therein are functions of $t-\tau$ or ξ, the operation of composition are all permutable ones. Therefore wc can solve the system of equatios (5) and obtain unknown quantities $U_0, U_1, \cdots\cdots U_n$, treating this system of equtions as if it was ordinary algebraic equations. Thus we obtain;-

$$U_0 = \frac{X^*N}{X^*D} \qquad \cdots\cdots (7)$$

Where

$$N = \begin{vmatrix} \mu & \mu\Pi_1^* & \mu\Pi_2^* & \Pi_3^* & \cdots\cdots & \Pi_{n-1}^* & q^* \\ X^* & -1 & 0 & 0 & \cdots\cdots & 0 & -A_{n-1} \\ 0 & X^* & -1 & 0 & \cdots\cdots & 0 & -A_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots\cdots & X^* & -A_0 \end{vmatrix} \qquad \cdots\cdots (8)$$

$$D = \begin{vmatrix} \mu & \mu\Pi_1^* & \mu\Pi_2^* & \Pi_3^* & \cdots\cdots & \Pi_{n-1}^* & \Pi_n^* \\ X^* & -1 & 0 & 0 & \cdots\cdots & 0 & 0 \\ 0 & X^* & -1 & 0 & \cdots\cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots\cdots & X^* & -1 \end{vmatrix} \qquad \cdots\cdots (9)$$

It is to be understood that in the right-hand side of (7) the factor 1/D means its power-series in ascending powers of X^*. The factor X^* in denominator and mumerator may be omitted. It is seen that

$$\begin{aligned} N = \ & q^*[X^*]^n + \mu\{A_{n-1}[X^*]^{n-1} + \cdots\cdots + A_0\} \\ & + \mu\Pi_1^*\{A_{n-2}[X^*]^{n-2} + \cdots\cdots + A_0X^*\} \\ & + \mu\Pi_2^*\{A_{n-3}[X^*]^{n-3} + \cdots\cdots + A_0[X^*]^2\} \\ & + \Pi_3^*\{A_{n-4}[X^*]^{n-4} + \cdots\cdots + A_0[X^*]^3\} \\ & + \cdots\cdots \cdots\cdots \\ & + p_{n-1}A_0[X^*]^{n-1} \end{aligned} \qquad \cdots\cdots (10)$$

$$D = \{1 + \Pi_1^*X^* + \Pi_2^*[X^*]^2\}\mu + \Pi_3^*[X^*]^3 + \cdots\cdots + \Pi_n[X^*]^n \quad (11)$$

By the analogy with algebraic equation of previous section, the operator D can be factorized as follows;

$$\begin{aligned} D \equiv \ & [\nu - \lambda_1^*(\xi)X^*][\nu - \lambda_2^*(\xi)X^*][\nu - \lambda_3^*(\xi)X^*] \cdot \\ & [1 - \lambda_4^*(\xi)X^*]\cdots\cdots\cdots[1 - \lambda_n^*(\xi)X^*] \end{aligned} \qquad \cdots\cdots (12)$$

Where we put $\nu = \sqrt[3]{\mu}$. We may also write

$$\begin{aligned} \lambda_1(\xi) &= 1 + \nu L_{11}(\xi) + \nu^2 L_{12}(\xi) + \cdots\cdots \\ & \text{etc., etc.} \\ \lambda_4(\xi) &= \Lambda_4(\xi) + \mu L_{41}(\xi) + \mu^2 L_{42}(\xi) + \cdots\cdots \\ & \text{etc., etc.} \end{aligned} \qquad \cdots\cdots (13)$$

IV Separation of the Solution into Different Parts

Since the operational solution (7) is expressed as quotient of two algebraic expresessions, each of degrees n, with respect to the unit operator X^*, we can, by analogy with partial fraction of algebraic fraction, put U_o into the form;-

$$U_o = Q(\xi) + V_1(\xi) + \cdots + V_n(\xi) \qquad \cdots\cdots (14)$$

where

$$Q(\xi) = [\Pi_n^*(\xi)]^{-1} q^*(\xi)$$

$$V_i(\xi) = \frac{W_i^*(\xi)}{\nu - \lambda_i^*(\xi) X^*} \qquad (i = 1,2,3,)$$

$$V_j(\xi) = \frac{W_j^*(\xi)}{1 - \lambda_j^*(\xi) X^*} \qquad (j = 4,\cdot\cdot n)$$

Each expressions for $V(\xi)$ is to be understood to represent the value, when the right hand side expression is replaced by power series in ascending powers of X^*. But some light on nature of functions $V(\xi)$ can be given by inversion as follows: -

$$W_i(\xi) = [\nu - \lambda_i^*(\xi) X^*] V_i^*$$

$$W_j(\xi) = [1 - \lambda_j^*(\xi) X^*] V_j^*$$

These equations are operational form of the integral equations;

$$W_i(\xi) = \nu V_i(\xi) - \int_\tau^t V_i(\xi)\lambda_i(\xi)\,dt$$

$$W_j(\xi) = V_j(\xi) - \int_\tau^t V_j(\xi)\lambda_j(\xi)\,dt$$

These integeral equtions can easily be solved, by returning to differential equations, and we have

$$V_i(\xi) = \frac{1}{\nu G_i(\xi)} \left[\int_0^\xi W_i'(\xi) G_i(\xi)\,d\xi + W_i(o)\right] \cdots\cdots (15)$$

$$V_j(\xi) = \frac{1}{G_j(\xi)} \left[\int_0^\xi W_j'(\xi) G_j(\xi)\,d\xi + W_j(o)\right] \cdots\cdots (16)$$

where we put

$$G_i(\xi) = \exp\left[-\frac{1}{\nu}\int_0^\xi \lambda_i(\xi)\,d\xi\right] \qquad \cdots\cdots\cdots (17)$$

$$G_j(\xi) = \exp\left[-\int_0^\xi \lambda_j(\xi)\,d\xi\right] \qquad \cdots\cdots\cdots (18)$$

As we assumed, the numerical constant μ (and consequently $\nu = \sqrt[3]{\mu}$) has very small valve. In such a case, we see from (15), (16) that the partial function $V_i(\xi)$ ($i = 1,2,3,$) contained in the solution has the property of varying very rapidly when the independent variable ξ varies. If $\mu =$ o the terms $V_i(\xi)$ will disappear from the solution. Thus the so-called "End Effect" is seen to be represented by the partial functions $V_i(\xi)$ thus obtained.

It will also be observed that when all the coefficients $p_0(t) \cdots p_n(t)$ of the given linear differential equation are constants, the functions $V_i(\xi)$, $V_j(\xi)$ reduce themselves to exponential functions of the form

$$V_i(\xi) = \frac{w_i}{\nu} \exp\left\{\frac{\alpha_i}{\nu}\xi\right\} \qquad (i = 1, 2, 3)$$

$$V_j(\xi) = w_j \exp\{\alpha_j \xi\} \qquad (j = 4, \cdot\cdot\cdot n)$$

where ν/α_i, $1/\alpha_j$ are roots of algebraic equation

$$\mu\,[\,p_0 + p_1 x + p_2 x^2\,] + p_3 x^3 + \cdots + p_n x^n = 0.$$

The above statement has been made about the differential equation of the form (3), but similar statement can also be made, regarding the differential equation of the form;-

$$\mu\left[p_0(t)\frac{d^n u}{dt^n} + p_1(t)\frac{d^{n-1}u}{dt^{n-1}} + \cdots + p_s(t)\frac{d^{n-s}u}{dt^{n-s}}\right]$$

$$+ p_{s+1}(t)\frac{d^{n-s-1}u}{dt^{n-s-1}} + \cdots + p_{n-1}(t)\frac{du}{dt} + p_n(t)u = q(t).$$