Title	Bending of a circular plate of non-uniform thickness
Sub Title	
Author	鬼頭, 史城(Kito, Fumiki)
Publisher	慶應義塾大学藤原記念工学部
Publication year	1949
Jtitle	慶應義塾大学藤原記念工学部研究報告 (Proceedings of Faculty of Engineering, Keiogijuku University). Vol.2, No.5 (1949. 7) ,p.76(32)- 80(36)
JaLC DOI	
Abstract	A circular plate whose thickness varies linearly with the radius γ is subjected to non-uniform load which is a specified function of γ. This problem of bending of plate has been studied theoretically, and the result obtained is shown as charts, so that the designer could make estimates of stresses etc., in such a plate with easiness.
Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00020005-0032

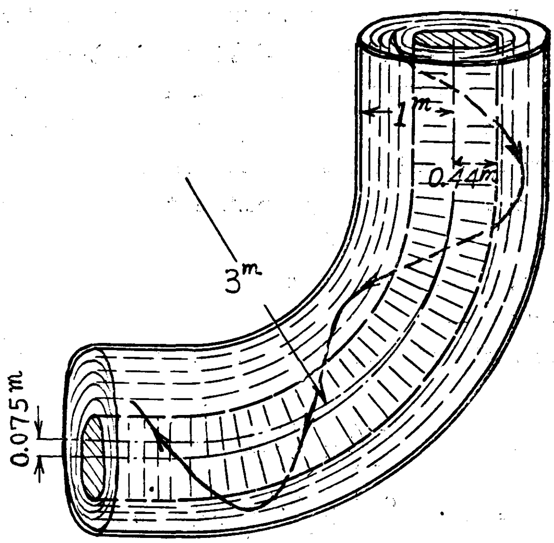

Fig. 6.
Sketch of flow of Whirling Water
through an Elbow.

Note:—Some copies giving the main feature of this paper has already been distributed by the Author in Oct. 1941.

Bending of a Circular Plate of Non-Uniform Thickness.

Recieved Jan. 15, 1949.
Fumiki Kito*

A circular plate whose thickness varies linearly with the radius r is subjected to non-uniform load which is a specified function of r. This problem of bending of plate has been studied theoretically, and the result obtained is shown as charts, so that the designer could make estimates of stresses etc., in such a plate with easiness.

Section I. Preliminary Remarks.

Let us consider a circular plate of radius R, as sqown in Fig. 1. Its thickness h is not uniform, but it is a function of radial distance r. On this plate lateral load of strength p/unit area is applied. This distributed load p is also a function of radial distance r. Thus, all things being symmetrical about the center O of the plate, the stresses occurring in the plate must also be functions of radius r only. The Author has made some calculation on such a state of stress, and it is reported here. A case of stress in blades of a marine propeller (especially wide-bladed one) under the action

* Dr. of Eng. Prof. of Keiogijuku University.

of hydraulic pressures upon their surface, has some resemblance with the case of the present paper.

In order to obtain simple solution as far as possible, we confine ourselves to the case in which the thickness varies linearly with the radius r, and we use the following notation: E=Young's modulus of the material of the plate, ν=its Poisson's ratio, h=thickness at r, N=Flexural rigidity$=Eh^3/12(1-\nu^2)$, σ_r=radial stress, σ_t=tangential stress, p=load per unit area of plate surface, p_r=shearing force at circular section r, w=lateral deflection, φ= inclination of middle surface$=dw/dr$, R=Outside radius of the plate, R_1=virtual radits at which the thickness h vanishes.

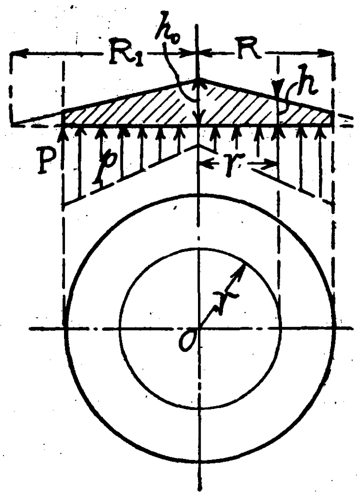

Fig. 1.
Circular Plate of non-uniform thickness.

The thickness h of the plate is given by $h=h_0(R_1-r)/R_1$. Further, we put $x=r/R_1$, and regard x as new independent variable. At the outer edge of platc we have $x=x_1=R/R_1$. h_0 being the thickness at center, we put

$$N_0=Eh_0^3/12(1-\nu^2) \qquad \cdots(1)$$

As to the dependent variable, we take instead of φ, the quantity y defined by the formnla

$$\varphi=\frac{x}{(1-x)^2}y$$

Using these notations, it is found out that the equation of equilibrium of plate as given in usual text-books can be transformed into:—

$$x(1-x)\frac{d^2y}{dx^2}+(3-2x)\frac{dy}{dx}+3(1-\nu)y=\frac{R_1^2}{N_0}p_r \qquad \cdots(2)$$

The homogeneous equation to be obtained by putting the r.h.s. of this eq. to zero is of hypergeometric type and could be solved by means of the hypergeometric functions. But if we assume tentatively that $\nu=1/3$, then they reduce into elementary functions, and the complementary solution of eq. (2) can be written in the form;

$$y=A\left(1-\frac{2}{3}x\right)+B\frac{1}{x^2}$$

Knowing the complementary solution, we can build up the general solution of (2) by usual method.

Section II. Circular Plate under the action of Lateral Loads along the Edge. Thus the general solution of eq. (2) becomes

$$y=A\left(1-\frac{2}{3}x\right)+\frac{B}{x^2}.$$

$$+\frac{R_1^2}{2N_0}\int_0^x\left[\left(1-\frac{2}{3}x\right)-\frac{\xi^2}{x^2}\left(1-\frac{2}{3}\xi\right)\right]\frac{p_r(\xi)}{(1-\xi)^2}d\xi \qquad \cdots(3)$$

and if we know the value p_r of the shearing force at radius r, the integration in the formula (3) could be performed. When the plate is acted upon by edge load of strength P per unit peripheral length, and there exist no distributed load, we must have $rp_r=RP$ (a const.) so that

$$p_r(\xi)=\frac{PR}{R_1\xi}$$

and we can obtain value of y by putting this expression for $p_r(\xi)$ into the equation (3).

Section III. Circular Plate under the action of non-uniform lateral loads.

Here we take, k being a numerical parameter,

$$p=p_0[f(x)+kg(x)]$$

where

$$f(x)=\frac{1}{x}\left[\left(\frac{1}{2}-x\right)x_1^2-\frac{2}{3}x^2+2x^3\right]$$

$$g(x)=\frac{1}{x}\left[\frac{1}{3}x_1^3(1-2x)-\frac{4}{3}x^2+\frac{5}{3}x^4\right]$$

so that

$$p_r=p_0\frac{1-x}{x}\left[\left(\frac{1}{2}x^2+\frac{1}{3}kx^3\right)-x_1^2\left(\frac{1}{2}+\frac{1}{3}kx_1\right)\right]R_1$$

Putting this expression for p_r into the eq. (3) we obtain:—

$$y=A\left(1-\frac{2}{3}x\right)+\frac{B}{x^2}-\frac{p_0R_1^3}{2N_0}[F_1(x)+kx_1F_2(x)] \qquad \cdots(4)$$

where $F_1(x)$ and $F_3(x)$ are functions of x which is shown later on. The arbitrary constants A and B are to be determined by **Boundary Condirions** which we take to be as follows;

(a) owing to symmetry we must have $\varphi=0$, or $xy=0$ at $x=0$ $(r=0)$

(b) the outer radius $r=R$, being free edge, we must have at $x=x_1$,

$$M_r=N\left[\frac{d\varphi}{dr}+\nu\frac{\varphi}{r}\right]=0$$

which is the same things as

$$x(1-x)\frac{dy}{dx}+\frac{2}{3}(2+x)y=0$$

In this way, arbitrary constants A, B are determined. The stresses σ_r and σ_t in the plates are then given by,

$$\left.\begin{aligned}\sigma_r&=-\frac{12N}{h^2}\left[\frac{d\varphi}{dr}+\nu\frac{\varphi}{r}\right]\\ \sigma_t&=-\frac{12N}{h^2}\left[\nu\frac{d\varphi}{dr}+\frac{\varphi}{r}\right]\end{aligned}\right\} \qquad \cdots(5)$$

The Author has carried out the calculation for σ_t and σ_r and obtained the following result, the detail of which, being lengthy, is omitted here.

$$\sigma_r=\frac{2p_0R_1^3}{h_0^2(1-x)^2}\left[\frac{2}{3}(6-4x+x^2)C+G_1(x)+kx_1G_2(x)\right] \qquad \cdots(6)$$

$$\sigma_t=\frac{2p_0R_1^3}{h_0^2(1-x)^2}\left[2\left(2-\frac{8}{3}x+x^2\right)+H_1(x)+kx_4H_2(x)\right] \qquad \cdots(7)$$

where we put

$$C=\left[\frac{1}{3}G_1(x_:)+kx_1\times\frac{1}{3}G_2(x_1)\right]\div\left[-\frac{2}{9}(6-4x_1+x_1^2)\right]$$

and the functions G_1, G_2, H_1 and H_2 are defined as follows:—

$$F_1(x)=\frac{1}{2}\left(1-\frac{2}{3}x\right)I_1(x)-\frac{1}{2x^2}I_2(x)$$

$$F_2(x)=\frac{1}{3}\left(1-\frac{2}{3}x\right)I_2(x)-\frac{1}{3x^2}I_4(x)$$

$$F_1'(x)=-\frac{1}{3}I_1(x)+\frac{1}{x^3}I_2(x)$$

$$F_2'(x)=-\frac{2}{9}I_3(x)+\frac{2}{3x^3}I_4(x)$$

$$I_1(x)=\left(x+\frac{1}{2}x^2\right)+(1-x_1^2)\log(1-x)$$

$$I_2(x)=\frac{1}{3}x_1^2\left(\frac{2}{3}x^2-\frac{1}{2}x-1\right)x$$
$$-\frac{1}{3}x^3\left(\frac{2}{5}x^2-\frac{1}{4}x-\frac{1}{3}\right)+\frac{1}{3}x\left(\frac{1}{2}x+1\right)$$
$$+\frac{1}{3}(1-x_1^2)\log(1-x)$$

$$I_3(x)=(1-x_1^3)\log(1-x)+x\left(1+\frac{1}{2}x+\frac{1}{3}x^2\right)$$

$$I_4(x)=\frac{1}{3}x_1^3x\left(\frac{2}{3}x^2-\frac{1}{2}x-1\right)$$
$$-\frac{1}{3}x^4\left(\frac{1}{3}x^2-\frac{1}{5}x-\frac{1}{4}\right)+\frac{1}{3}x\left(1+\frac{1}{2}x+\frac{1}{3}x^2\right)$$
$$+\frac{1}{3}(1-x_1^3)\log(1-x)$$

$$G_1(x)=3x(1-x)F_1'(x)+2(2+x)F_1(x)$$
$$G_2(x)=3x(1-x)F_2'(x)+2(2+x)F_2(x)$$
$$H_1(x)=x(1-x)F_1'(x)+2(2-x)F_1(x)$$
$$H_2(x)=x(1-x)F_2'(x)+2(2-x)F_2(x)$$

Practically, the calculation of stresses as given by the Formulae (6) and (7) can easily be carried out if we know values of the functions $G_1(x)$, $G_2(x)$, $H_1(x)$, and $H_2(x)$. The numerical values of these functions has been obtained by the Author, and they are shown as curves in Figs 2 to 5.

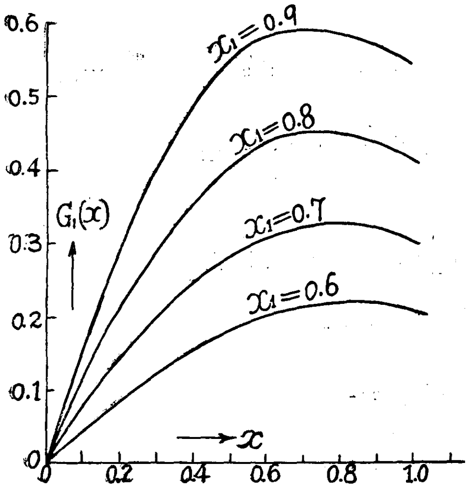

Fig. 2. Chart for $G_1(x)$.

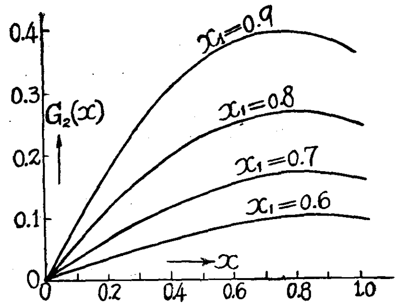

Fig. 3. Chart for $G_2(x)$.

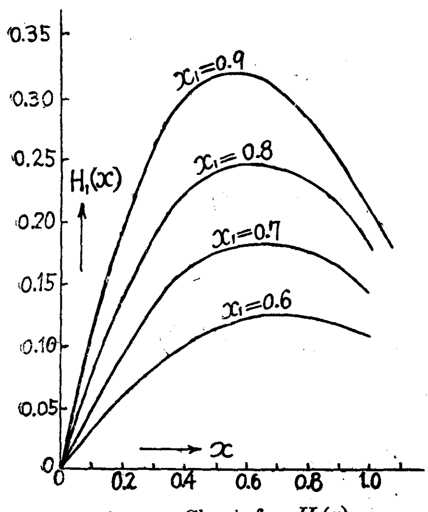

Fig. 4. Chart for $H_1(x)$.

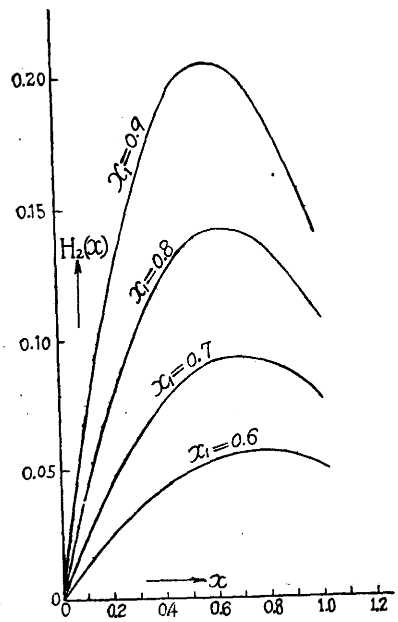

Fig. 5. Chart for $H_2(x)$.