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The Research

Featured in this issue is Associate Professor Kenichi Bannai, a researcher in arithmetic geometry – a field of mathematics – tackling unsolved problems in number theory via an approach based on geometry.

Bringing new insight to mathematical conjectures

Balance between logic and intuition

The combination of logic and intuition is a powerful tool in solving many problems. Arithmetic geometry is a field which uses geometric intuition to solve problems in number theory – intuition which may come to you, for example, when you look at geometric figures comprising of points, lines, etc. We paid a visit to Associate Professor Bannai, who is researching problems in number theory – related to properties of integers – using methods of arithmetic geometric.

A delicate balance between logic and intuition

"When trying to solve a problem in geometry, one may have had the experience of being able to solve it easily by drawing an additional line in the original figure – an approach which stimulates intuition. Arithmetic geometry approaches problems in number theory using concepts of geometry as represented by such figures," Dr. Bannai outlined his field of study.

What do we mean by geometry? Let's take for the example the problem of finding all the rational number solutions of the equation: $x^2 + y^2 = 1$. You may of course solve this problem by manipulating the equation. On the other hand, the solution becomes much easier to understand if you think of the equation as describing a unit circle with a radius 1 on a plane (see Fig. 1). (x, y) = (-1, 0) is a rational number solution of this equation. Consider a straight line passing through the point (-1, 0); this line meets the unit circle only at one more point other than (-1, 0). If the slope of this line is a rational number, then the point of intersection provides a rational number solution of the equation. Conversely, if a rational number solution of the equation is given, then if we connect this point to (-1,0), then we obtain a line through (-1, 0) with rational slope. This implies that the rational number solution to the original equation corresponds one to one to the lines through (-1,0) whose slope is a rational number.

Arithmetic geometry, being useful in solving number theoretical problems, has developed alongside number theory. Number theory has a very long history. Even a simple problem concerning integers in many cases has behind it highly sophisticated mathematical theory. Number theory also has many real-word applications; it has been applied to cryptography and authentication technology. As an approach to solving problems in number theory, arithmetic geometry is expected to contribute to solving many problems. In fact, the methods were used to prove Fermat's last theorem and Mordell's conjecture(*1).

"What I've pursued so far in my research are arithmetic geometric functions known as the polylogarithm. The polylogarithm is related to many important invariants in number theory. There are certain conjectures in number theory that predicts the relationship between two number theoretical invariants which a priori do not appear to be related to each other. If we can present a higher mathematical concept that can explain both invariants in a unified manner, it will help us get one step closer to shedding light on their relationship. The polylogarithm can serve as a useful bridge between them," remarks Dr. Bannai.

Finding relationships between important invariants by capturing the essence of the problem

The polylogarithm, on which Dr. Bannai focuses, is a kind of which we call a "geometric object." To better explain its concept, we interpret a "geometric object" as some sort of "figure." If one has a figure, it is possible to derive various invariants, such as the "area" and "the number of vertex." Therefore, suppose there are several invariants

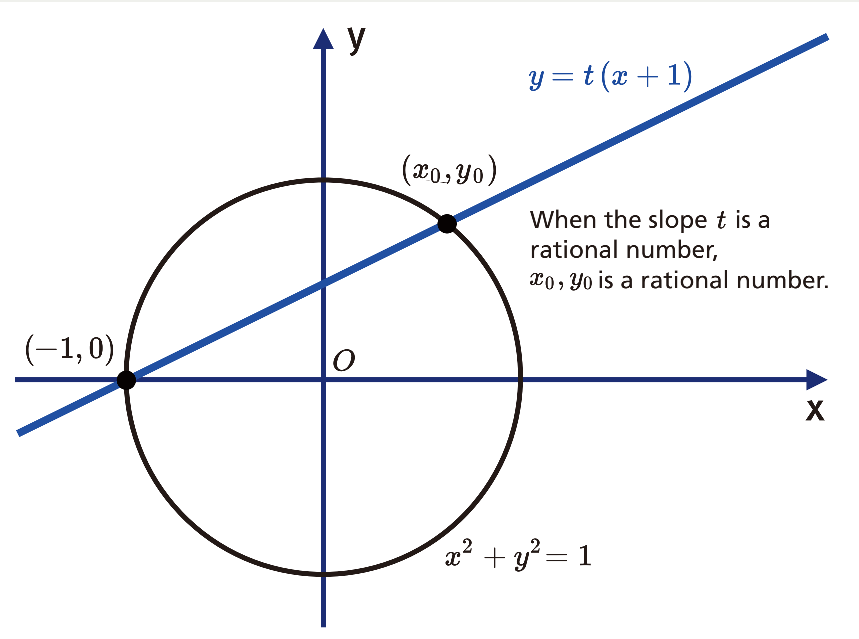

Fig.1 Solving problems in number theory using geometry
A rational number solution for the equation $x^2 + y^2 = 1$ corresponds one to one to a line through the point (-1, 0) with rational slope that. The solution is easily visualized if one uses geometry and views the equation as defining a unit circle.

which you wish to compare. If one could find a figure that intrinsically contains these invariants, then this figure would be the starting point for finding some relationship between the various invariants. In a sense, the polylogarithm is a figure that intrinsically contains important invariants in number theory.

"Suppose you can represent an invariant 'A' as a simple figure by abstracting its properties. Using the same technique, one next studies another invariant 'B' which looks completely different from 'A.' If one can express the invariant 'B' using the same figure as that of 'A,' then we can hope to find a formula or rule that governs both A and B." Alexander Grothendieck(*2), who contributed greatly to algebraic geometry, referred to such geometric objects obtained through abstraction as "motives." The polylogarithm is an important example of a "motive," which plays a vital role in arithmetic geometry (Fig. 2).

Dr. Bannai adds, "The term 'motive' used by Grothendieck comes from a commentary of the work of Paul Cézanne(*3). The post-impressionist painter Cézanne distanced himself from impressionists who depicted objects using the ever changing quality of light and shadow. Cézanne instead establish his own style emphasizing the "vital intensity that the motif possessed in its actual existence." The 'motive' in arithmetic geometry also sees through the surface and captures the true essence."

Solving mathematical problems through cooperation

In stark contrast to the depth of his research, the reason Dr. Bannai chose the polylogarithm as his research topic seems very simple.

"I chose this theme because I was attracted by its simplicity," says Dr. Bannai smilingly. One may imagine mathematicians as working alone when doing research. In reality, this is not the case.

"Remembering struggling through mathematics in high school, one may have the impression that mathematical research is a solitary endeavor. However, in reality, mathematical research typically advances through discussions among mathematicians around the world. Mathematics is a highly internationalized field of study. For example, my recent series of work is a joint project with Prof. G. Kings of Regensburg University in Germany. The situation is the same in my lab. While everyone has their own theme, members advance their studies through active discussions. They are taking advantage of being in a group and are enjoying its benefits."

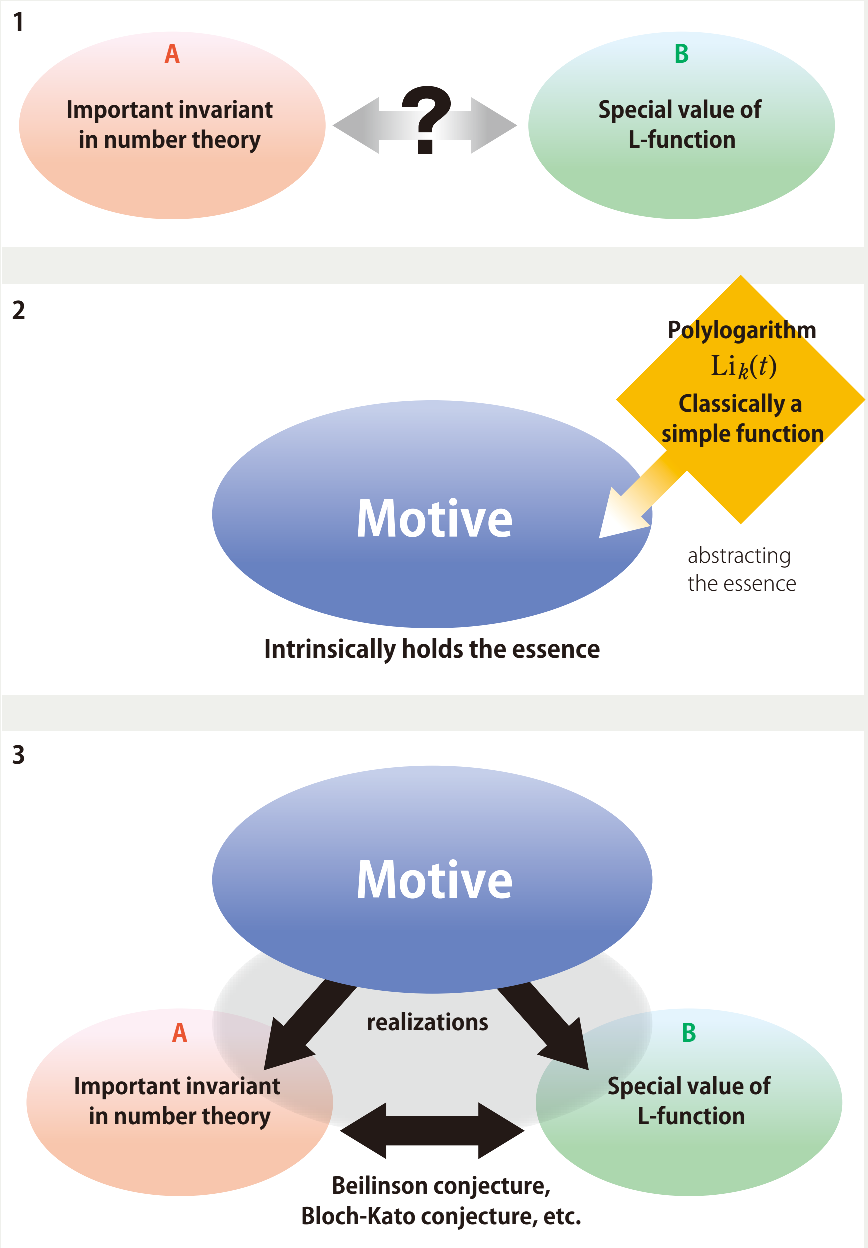

Fig. 2 Polylogarithm at work

(1) We consider "An important invariant in number theory" and "The special value of an L-function" – two invariants which appears unrelated to each other. (2)By starting from and abstracting the polylogarithm function, we construct a "motive" which gives a higher conceptual interpretation of the two invariants. (3) By going through the "motive," we are able to find the relationship between two invariants which at first seemed unrelated. Such relationships may derive solutions to problems, such as the Beilinson conjecture and the Bloch-Kato conjecture.

Dr. Bannai believes that polylogarithm will play a key role in understanding important problems in number theory. Above all, he is fascinated by the idea of finding unified rules through abstraction of various properties and quantities. He dreams that such methods would give clues to solving important conjectures.

(Reporter & text writer: Kaoru Watanabe)

＊1 **Fermat's last theorem and Mordell's conjecture:** Both had long been unsolved mathematical problems, but Mordell's conjecture was solved in 1983 by Gerd Faltings, and Fermat's last theorem was solved in 1995 by Andrew Wiles.

＊2 **Grothendieck:** Alexander Grothendieck is a founding member of the Institute of Advanced Scientific Studies (IHÉS) of France. Working on the foundation of algebraic geometry, he expanded and enriched its horizon by bringing in commutative ring theory and various other mathematical methods –revolutionizing the world of mathematics.

＊3 **Cézanne:** Paul Cézanne is a post-impressionist painter. Originally an impressionist, Cézanne gradually came to realize the importance of capturing the "motif." He had great impact on 20th-century art.