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Essays on Mechanism Design

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Abstract

This thesis is a collection of three essays on mechanism design. I study three economic problems on auction theory, matching theory, and social choice theory. In each problem, I design a mechanism which satisfies "desirable properties", such as *efficiency, fairness*, and *incentive compatibility.*

In Chapter 1, I study multi-unit ascending-bid auctions. Lawrence M. Ausubel (2004) introduces a new ascending-bid auction rule for multiple homogeneous objects, called *the Ausubel auction*, which is a dynamic counterpart of the Vickrey auction. He claims that in the Ausubel auction with private values, sincere bidding by all bidders is an ex post perfect equilibrium, which is a tuple of strategies constituting ex post equilibria at all nodes of the dynamic auction game. However, I show that this claim does not hold in general. In my counterexample, there exists a node at which sincere bidding by all bidders is not an ex post equilibrium. I then examine properties of the sincere bidding equilibrium. Finally, I provide two modifications of the Ausubel auction in which sincere bidding by all bidders is an ex post perfect equilibrium.

In Chapter 2, I study a rescheduling problem in the Ground Delay Program. The Ground Delay Program is an air traffic control program in the Uniter States. When inclement weather strikes an airport, the airport needs to reduce arrival slots and reassigns flights to available slots. I first show that FAA's current mechanism may not maximize the number of flights assigned to available

slots. To resolve this inefficiency, I introduce a new efficiency criterion, *universal non-wastefulness.* Then, I design a new mechanism that satisfies *universal non-wastefulness* and a fairness requirement. Furthermore, I show that no airline has an incentive to misreport flight delay under our mechanism.

In Chapter 3, I study voting rules. Jean-Charles de Borda (1774) provided an example in which the plurality rule selects an alternative, so-called *pairwise-majority-loser*, which is defeated by any other alternative in pairwise comparison. To avoid selecting such an alternative, he introduced a new social choice rule, called *the Borda rule.* A social choice rule satisfies *Borda's criterion* if it never selects a pairwise-majority-loser. I show that the Borda rule is the only social choice rule which satisfies *anonymity, neutrality, consistency, continuity*, and *Borda's criterion.*

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Chapter 1

Perfect Incentive Compatibility on Multi-Unit Ascending-Bid Auctions

[*1]

1.1 Introduction

In his seminal work, Lawrence M. Ausubel (2004) designs a new ascending-bid auction rule for multiple homogeneous objects, called *the Ausubel auction*. This auction yields the Vickrey outcome at the sincere bidding equilibrium with private values.[*2] His main result claims that sincere bidding by all bidders is an *ex post perfect equilibrium* that constitutes ex post equilibria at *all nodes* of the dynamic auction game.

In this chapter, we show that this claim does not hold by providing a counterexample. That is, sincere bidding by all bidders is *not* always an ex post perfect equilibrium. In our counterexample of a dynamic auction game, there exists a subgame such that some bidder has an incentive not to sincerely bid (Examples 1 and 3).

[*1] This chapter is based on my paper "An Efficient Ascending-Bid Auction for Multiple Objects: Comment", which is forthcoming in the *American Economic Review*.

[*2] For example, see Milgrom (2004), for a detail of the Vickrey auction.

We then show that for any subgame, if a bidder dose *not* bid in excess of her demand just before the subgame, then she has an incentive to bid sincerely. Therefore, for any subgame, if each bidder does not bid more quantity than her demand just before the subgame, then sincere bidding by all bidders is an ex post equilibrium (Proposition 1).

Next, we provide two modifications of the Ausubel auction in which sincere bidding by all bidders is an ex post perfect equilibrium. In the first modification, we introduce a new rationing rule for tie-breaking. In the second modification, we introduce a new tie-breaking system such that each bidder can select whether she accepts an excess supply or not. In these modified auction, sincere bidding by all bidders is an ex post perfect equilibrium (Theorems 1 and 2).

This chapter is organized as follows. In Section 2, we introduce definitions. In Section 3, we give a counterexample to Ausubel's claim. In Section 4, we examine equilibrium properties of the Ausubel auction and give two modifications. In Section 5, we conclude the paper. All proofs are relegated to Appendix.

1.2 Definitions

Our definitions and notation almost follow Section Ⅱ and Ⅲ of Ausubel (2004), but we generalize some definitions so as to investigate details of the dynamic auction games.

1.2.1 Bidders

We construct the model of auction for multiple objects with *private values*. A seller puts M *homogeneous goods* for an auction. A finite set of *bidders* is $N =$

$\{1, 2, \ldots, n\}$ with $n \geq 2$. Each bidder $i \in N$ has a *consumption set* $X_i = [0, \lambda_i]$ with $0 < \lambda_i \leq M$ and a *valuation function* $U_i : X_i \to \mathbb{R}_+$. When a bidder $i \in N$ is assigned $x_i \in X_i$ and pays $y_i \in \mathbb{R}$, bidder i's utility is $U_i(x_i) - y_i$. For each $x_i \in X_i$, the value $U_i(x_i)$ can be calculated by the integral of a corresponding *marginal value function* $u_i : X_i \to \{0, 1, \ldots, \overline{u}\}$, so that

$$U_i(x_i) = \int_0^{x_i} u_i(q)dq \quad \forall x_i \in X_i.$$

We also assume that each u_i is a *weakly decreasing* function in X_i, and $u_i(x_i)$ is an integer in $\{0, 1, \ldots, \overline{u}\}$ for all $x_i \in X_i$. We assume that $\overline{u} < \infty$.

1.2.2 The auction rule

We revisit the rule of the Ausubel auction with discrete times $\{0, 1, \ldots, T\}$ where $T < \overline{u}$. For each time $t \in \{0, 1, \ldots, T\}$, we define the price by $p^t = t$. All bidders know the price at each time. An auction starts at $t = 0$, and it proceeds as follows.

$\boldsymbol{t = 0}$**:** Each bidder $i \in N$ simultaneously bids a quantity $x_i^0 \in X_i$. If $\sum_{i \in N} x_i^0 \leq M$, then the auction ends at $t = 0$ with the assignment $(x_i^*)_{i \in N}$ such that

$$x_i^* = x_i^0 \quad \forall i \in N.$$

Otherwise, for each bidder $i \in N$, let

$$C_i^0 = \max\{0, M - \sum_{j \neq i} x_j^0\}$$

be bidder i's *cumulative clinches* at $t = 0$, and the auction continues to $t = 1$.

$\boldsymbol{t = s < T}$**:** The auctioneer announces the prior bids by all bidders to each bidder Each bidder $i \in N$ simultaneously bids a quantity $x_i^s \in X_i$ satisfying the constraint

$$C_i^{s-1} \leq x_i^s \leq x_i^{s-1}.$$

If $\sum_{i \in N} x_i^s \leq M$, the auction ends at $t = s$ with an assignment $(x_i^*)_{i \in N}$ such that

$$\sum_{i \in N} x_i^* = M$$
$$x_i^s \leq x_i^* \leq x_i^{s-1} \quad \forall i \in N.$$

Otherwise, let $C_i^s = \max \left\{0, M - \sum_{j \neq i} x_j^s\right\}$ be bidder i's *cumulative clinches* at $t = s$, and the auction continues to $s + 1$.

$\boldsymbol{t = T}$**:** The auctioneer announces the prior bids by all bidders to each bidder Each bidder $i \in N$ simultaneously bids a quantity $x_i^T \in X_i$ with $C_i^{T-1} \leq x_i^T \leq x_i^{T-1}$. In any case, the auction ends. If $\sum_{i \in N} x_i^T > M$, an assignment $(x_i^*)_{i \in N}$ is such that $\sum_{i \in N} x_i^* = M$ and

$$x_i^* \leq x_i^T \quad \forall i \in N.$$

Otherwise, similarly to the case which ends at $t = s < T$, an assignment $(x_i^*)_{i \in N}$ is such that $\sum_{i \in N} x_i^* = M$ and $x_i^T \leq x_i^* \leq x_i^{T-1}$ for each $i \in N$.

This auction process finishes in at most $T+1$ steps. Let L be a *last time* of the auction game, that is, $\sum_{i \in N} x_i^L \leq M$ or $L = T$. For each bidder $i \in N$, define cumulative clinches of the last time by i's assignment, $C_i^L = x_i^*$. Then, by this process, we obtain a vector of cumulative clinches $\left\{(C_i^t)_{i \in N}\right\}_{t=0}^{L}$. We define the vector of *current clinches* $\left\{(c_i^t)_{i \in N}\right\}_{t=0}^{L}$ as follows: For each $i \in N$ and $t \geq 1$,

$$c_i^t = C_i^t - C_i^{t-1},$$

and $c_i^0 = C_i^0$.

Each bidder's payment is calculated as follows: For each $i \in N$, the *payment* is given by

$$y_i = \sum_{t=0}^{L} p^t c_i^t.$$

When an auction ends with the last bid $(x_i^L)_{i \in N}$ such that $\sum_{i \in N} x_i^L < M$ $(L \geq 1)$,[*3] a *rationing rule for tie-breaking* assigns the excess supply quantity $M - \sum_{i \in N} x_i^L$ to bidders such that $\sum_{i \in N} x_i^* = M$ and

$$x_j^L \leq x_j^* \leq x_j^{L-1} \quad \text{for all } j \in N.$$

Some bidder may be assigned a quantity in excess of the bidder's final bid at the final price. However, the following lemma ensures that such "over assignment" does not reduce the bidder's utility if the bidder bids sincerely.

[*3] In the case that $\sum_{i \in N} x_i^L > M$ with $L = T$, we need to consider another type of rationing rules for tie-breaking. However, our results hold under all rationing rules for this case.

Lemma 1. *For each $i \in N$ and each $p \in \{1, 2, \ldots, \overline{u}\}$,*

$$Q_i(p-1) = \max\{\arg\max_{x_i \in X_i}(U_i(x_i) - px_i)\}.$$

For sincere bidding $x_i^t = Q_i(p^t)$, this lemma implies that

$$\min\{\arg\max_{x_i \in X_i}(U_i(x_i) - p^L x_i)\} \leq x_i^* \leq \max\{\arg\max_{x_i \in X_i}(U_i(x_i) - p^L x_i)\}. \qquad (1.1)$$

Ausubel employs rationing rules that satisfy the following monotonicity property, which is introduced in Footnotes 17 and 18 of Ausubel (2004).

Monotonicity property: If i's final bid x_i^L strictly increases and the final bids x_{-i}^L of the other bidders do not change, then i's assignment x_i^* strictly increases.[*4]

An example of a rationing rule that satisfies the monotonicity property is *proportional rationing*, which is introduced by Ausubel (1997):[*5]

$$x_j^* = x_j^L + \left(M - \sum_{i \in N} x_i^L\right) \frac{x_j^{L-1} - x_j^L}{\left(\sum_{i \in N} x_i^{L-1} - x_i^L\right)}.$$

1.2.3 Histories

At each $t \in \{1, \ldots, T+1\}$, a *history* h^t is a vector of prior bids to t

$$h^t = (x_1^s, x_2^s, \ldots, x_n^s)_{s \leq t-1} \in \left(\times_{i \in N} X_i\right)^{\{0,1,\ldots,t-1\}}$$

[*4]This is the formulation for a divisible-good. By changing the assigned quantity x_i^* to the expected assigned quantity $E(x_i^*)$, we can also apply this property to the indivisible-good model.

[*5]Ausubel (1997) is a working paper version of Ausubel (2004).

such that for each $i \in N$ and each $s \leq t-1$,

$$C_i^{s-1} \leq x_i^s \leq x_i^{s-1}, \tag{1.2}$$

$$\sum_{j \in N} x_j^{t-2} > M. \tag{1.3}$$

Define the history of starting point $t = 0$ by *empty sequence*, $h^0 = \emptyset$. Let H^t be the set of histories at t. Then, the set of all histories is given by $H \equiv \bigcup_{t=0}^{T+1} H^t$.

We call a history $z^{t+1} = (x_1^s, \ldots, x_n^s)_{s \leq t} \in H^{t+1}$ *terminal* if $\sum_{i \in N} x_i^t \leq M$ or $t = T$; i.e., $t = L$. Let Z^t be the set of terminal histories at t, and $Z \equiv \bigcup_{t=1}^{T+1} Z^t$ be the set of all terminal histories. We can see a terminal history as a *result of the auction game*, because this represents all bids from beginning to end. Then, for each $z \in Z$, an assignment vector $(x_i^*)_{i \in N}$ and a payment vector $(y_i)_{i \in N}$ are determined by the auction rule.

1.2.4 Strategies

At each time $t \in \{1, 2, \ldots, T\}$, the auctioneer observes a history $h^t \in H^t \setminus Z^t$. Then, the auctioneer announces some information h_i^t to each bidder $i \in N$. Ausubel introduces three important *informational rules*: "full bid information", "aggregate bid information", and "no bid information". We study auctions with *full bid information* so that each bidder $i \in N$ can observe all prior bids $h_i^t = h^t$ at each time t.

With full bid information, the set of observable histories of each bidder is $H \setminus Z$. Then, a strategy of bidder i is a function $\sigma_i : H \setminus Z \to X_i$ such that for any $t \in \{0, 1, \dots, T\}$ and $h^t = (x_1^s, x_2^s, \dots, x_n^s)_{s \leq t-1} \in H^t \setminus Z^t$,

$$C_i^{t-1} \leq \sigma_i(h^t) \leq x_i^{t-1}$$

where $C_i^{t-1} = \max\{0, M - \sum_{j \neq i} x_j^{t-1}\}$.

For each $i \in N$, let Σ_i be the set of bidder i's strategies. For any n-tuple of strategies $(\sigma_i)_{i \in N} \in \times_{i \in N} \Sigma_i$, we attain a terminal history z^{L+1} which represents a result of an auction game. We denote $\pi_i((\sigma_j)_{j \in N})$ the utility of some bidder i at an n-tuple of strategies $(\sigma_j)_{j \in N}$.

We define *sincere bidding*, which is the strategy reporting truthfully the minimum demand at any history unless the bidder breaks the bidding constraint.

Definition 1. Bidder i's *sincere demand* at price $p \in \mathbb{Z}_+$ is defined by

$$Q_i(p) = \min\{\arg\max_{x_i \in X_i}(U_i(x_i) - px_i)\}.$$

Bidder i's *sincere bidding* is the strategy σ_i^* such that for any $t \geq 1$ and $h^t \in H^t \setminus Z^t$,

$$\sigma_i^*(h^t) = \min\{x_i^{t-1}, \max\{Q_i(p^t), C_i^{t-1}\}\},$$

and $\sigma_i^*(h^0) = Q_i(p^0)$.

We note that for each $p \in \{0, 1, 2, \dots, \overline{u}\}$, the existence of $Q_i(p)$ is guaranteed in this model. By Lemma 1, for each $i \in N$, if the bidder plays the auction game sincerely, then an assignment x_i^* satisfies

$$\min\{\arg\max_{x_i \in X_i}(U_i(x_i) - px_i)\} \leq x_i^* \leq \max\{\arg\max_{x_i \in X_i}(U_i(x_i) - px_i)\}.$$

That is, although some bidder may be assigned more quantity than her final bid, the quantity maximizes her utility at the final price.

1.2.5 Subgames

The Ausubel auction provides an extensive form game with simultaneous actions. In the case with full bid information, for each nonterminal history $h \in H \setminus Z$, a subgame that follows h is well-defined.[*6]

Consider any $h \in H \setminus Z$. The set of histories in the subgame that follows h is given by

$$H|_h = \{h' \in H : h' = (h, h'') \text{ for some sequence } h''\}.$$

Then, the set of terminal histories in the subgame is given by

$$Z|_h = Z \cap H|_h.$$

For each $z \in Z$ and each $z' \in Z|_h$, if $z = z'$, then let the result of z' in the subgame be identical to the result of z in the original game.

A strategy of bidder i in the subgame is a function $\sigma_i : H|_h \setminus Z|_h \to X_i$ such that for any $h' = (x_1^s, x_2^s, \ldots, x_n^s)_{s \leq t-1} \in H|_h \setminus Z|_h$,

$$C_i^{t-1} \leq \sigma_i(h') \leq x_i^{t-1}$$

[*6] An auction game with full bid information is an extensive form game with perfect information. Thus, we can define subgames for all histories. On the other hand, the auction with aggregate bid information or no bid information is an extensive form game with imperfect information. Therefore, a subgame for some history is not always well-defined. Although Ausubel (2004) states that sincere bidding by all bidders is an ex post equilibrium or a weakly dominant strategy equilibrium after *any* history, his statement is not entirely well-defined. We cannot use the usual definitions of such equilibrium notions after some history. See, for example, Osborne and Rubinstein (1994) for details on extensive form games.

where $C_i^{t-1} = \max\{0, M - \sum_{j \neq i} x_j^{t-1}\}$. For each strategy $\sigma_i \in \Sigma_i$ in the original game, we denote $\sigma_i|_h \in \Sigma_i|_h$ the strategy which is induced in the subgame, that is, for each $h' \in H|_h \setminus Z|_h$, $\sigma_i|_h(h') = \sigma_i(h')$.

Let $\Sigma_i|_h$ be the set of bidder i's strategies in the subgame. Similarly to the original game, we denote $\pi_i((\sigma_j)_{j \in N})$ bidder i's utility at an n-tuple of strategies of the subgame $(\sigma_j)_{j \in N} \in \times_{j \in N} \Sigma_j|_h$.

1.3 Counterexample

Ausubel (2004) extends Selten's (1975) perfectness concept of the extensive form game to the dynamic auction game. In extensive form games, a famous equilibrium notion is *subgame perfect equilibrium*, which is a tuple of strategies constituting Nash equilibria in all subgames. On the other hand, we sometimes investigate an *ex post equilibrium* in auction games. Ausubel (2004) combines these two concepts and introduces the notion of *ex post perfect equilibrium.* Then, he claims that sincere bidding by all bidder is an ex post perfect equilibrium in the Ausubel auction.

Definition 2. An n-tuple of strategies $(\sigma_i)_{i \in N} \in \times_{i \in N} \Sigma_i$ is an *ex post perfect equilibrium* if for each $h \in H \setminus Z$, $(\sigma_i|_h)_{i \in N}$ is an ex post equilibrium in the subgame that follows h.[*7]

Claim 1 (Ausubel 2004, Theorem 1). *In the Ausubel auction with private values, sincere bidding by all bidders is an ex post perfect equilibrium, yielding the efficient outcome of the Vickrey auction.*

[*7] An n-tuple of strategies is an *ex post equilibrium* if it is a Nash equilibrium for any realization of all bidders' valuation functions. See, for example, Crémer and Richard (1985) and Krishna (2009).

Here, we provide a simple counterexample showing that sincere bidding by all bidders is *not* an ex post perfect equilibrium under proportional rationing. In the Appendix, we give a general counterexample showing that Claim 1 does not hold under all rationing rules that satisfy the monotonicity property.

Example 1: Consider the case with two bidders, A and B, and two quantities of an object. The marginal value function of bidder A is given by

$$u_A(q) = \begin{cases} 3 & \text{if } q \in [0,1), \\ 1 & \text{if } q \in [1,2]. \end{cases}$$

Consider the history $h^2 = (x_A^t, x_B^t)_{t=0,1} = \big((2,2),(2,2)\big)$. Assume that B's sincere bidding is $x_B^2 = 0$ after h^2. At $t = 2$, A's sincere demand is 1. Let us consider the results with sincere bidding $x_A^2 = 1$ and misreporting $\hat{x}_A^2 = 0$.

The result with sincere bidding

If bidder A bids $x_A^2 = 1$ and bidder B bids $x_B^2 = 0$ after h^2, then the auction ends at $z_3 = \big((2,2),(2,2),(1,0)\big)$. First, bidder A is assigned $x_A^2 = 1$ which is A's final bid. In addition, since there is an excess supply quantity 1, proportional rationing assigns $\frac{1}{3}$ to bidder A. Then, bidder A is assigned $\frac{4}{3}$ at price 2. Therefore, A's utility is

$$\int_0^{\frac{4}{3}} u_A(q)dq - 2 \cdot \frac{4}{3} = 3 + 1 \cdot \frac{1}{3} - \frac{8}{3} = \frac{2}{3}.$$

The result with misreporting

If bidder A bids $\hat{x}_A^2 = 0$ and bidder B bids $x_B^2 = 0$ after h^2, then the auction ends at $\hat{z}_3 = \big((2,2),(2,2),(0,0)\big)$. Since there is an excess supply quantity 2, proportional rationing assigns 1 to bidder A. Then, bidder A is assigned 1 at price 2. Therefore, A's utility is

$$\int_0^1 u_A(q)dq - 2 \cdot 1 = 3 - 2 = 1.$$

Hence, bidder A has an incentive to misreport $\hat{x}_A^2 = 0$ after h^2.

1.4 Amending the result

In Section 2, we give a counterexample to a result of Ausbel. In this section, we amend this result. First, we examine equilibrium properties. Next, we modify the rule of the Ausubel auction.

1.4.1 Examining equilibrium properties

In our counterexample, we considered a history where some bidder bids a quantity in excess of the bidder's sincere demand just before the history. An assignment of the bidder may be more than $Q_i(p^{L-1})$, even if the bidder's final bid is $Q_i(p^L)$. Therefore, inequality (1.1) does not hold, and Ausubel's proof is invalid after such a history.

On the other hand, for any history, if a bidder does not bid a quantity in excess of the bidder's sincere demand just before the history, the bidder has an incentive to bid sincerely in the subgame that follows the history. Before the starting point $h^0 = \emptyset$, since all bidders do not bid a quantity in excess of their sincere demands, Ausubel's proof is valid at h^0; that is, sincere bidding by all bidders is an ex post equilibrium in the Ausubel auction. The following lemma summarized what Ausubel has essentially shown.

Lemma 2. *Consider any bidder and history under which the bidder did not overbid just before the history. After the history, the bidder's sincere bidding is an "ex post best response" to the other bidders' sincere bidding; that is, the bidder's sincere bidding is a best response to the other bidders' sincere bidding for any realization of all bidders' valuation functions. The same argument holds under all rationing rules.*

Proof. See Appendix. □

Applying this Lemma to all bidders, we have the following result.

Proposition 1. *In the Ausubel auction, sincere bidding by all bidders is an ex post equilibrium.*

Proof. Immediately follows from Lemma 2. □

1.4.2 The sequential rationing rule

In this section, we shall design a rationing rule for tie-breaking such that each bidder has an incentive to bid sincerely after any history. Our "sequential" rationing rule assigns the excess supply quantity $M - \sum_{i \in N} x_i^L$ as follows: Bidder 1 is assigned the excess supply up to x_1^{L-1}, bidder 2 is assigned the remaining excess supply up to x_2^{L-1}, and so on. Formally, the assignment $(x_i^*)_{i \in N}$ is

$$
\begin{aligned}
x_1^* &= \min\{x_1^{L-1}, x_1^L + M - \sum_{i=1}^{n} x_i^L\}, \\
x_2^* &= \min\{x_2^{L-1}, x_2^L + M - \sum_{i=2}^{n} x_i^L - x_1^*\}, \\
&\vdots
\end{aligned}
$$

$$x_j^* = \min\{x_j^{L-1}, x_j^L + M - \sum_{i=j}^{n} x_i^L - \sum_{i=1}^{j-1} x_i^*\},$$

$$\vdots$$

$$x_n^* = \min\{x_n^{L-1}, x_n^L + M - x_n^L - \sum_{i=1}^{n-1} x_i^*\}.$$

Note that because

$$\begin{aligned} x_j^* &= \min\{x_j^{L-1}, x_j^L + M - \sum_{i=j}^{n} x_i^L - \sum_{i=1}^{j-1} x_i^*\}, \\ &= \min\{x_j^{L-1}, M - \sum_{i=j+1}^{n} x_i^L - \sum_{i=1}^{j-1} x_i^*\}, \end{aligned}$$

each j's final bid x_j^L does not explicitly appear in x_j^*.

In the next example, we show that bidder A of Example 1 has no incentive to misreport under our rationing rule.

Example 2: Consider the case with two bidders, A and B, and two quantities of an object. The marginal value function of bidder A is given by

$$u_A(q) = \begin{cases} 3 & \text{if } q \in [0,1), \\ 1 & \text{if } q \in [1,2]. \end{cases}$$

Consider the history $h^2 = (x_A^t, x_B^t)_{t=0,1} = \big((2,2),(2,2)\big)$. Assume that bidder B bids $x_B^2 = 0$ after h^2. There are two possible orderings on the bidders: (i) $A = 1$ and $B = 2$ and (ii) $A = 2$ and $B = 1$.

(i) $\boldsymbol{A = 1}$ and $\boldsymbol{B = 2}$: For any bidding x_A^2, the excess supply $2 - x_A^2$ is assigned to bidder A by our rationing rule. Bidder A is always assigned 2. Therefore, bidder A has no incentive to misreport.

(ii) $\boldsymbol{A = 2}$ and $\boldsymbol{B = 1}$: For any bidding x_A^2, the excess supply $2 - x_A^2$ is assigned to bidder B by our rationing rule. Then, bidder A is assigned only x_A^2, which is A's final bid. Therefore, bidder A has no incentive to misreport.

Theorem 1. *In the Ausubel auction with sequential rationing, sincere bidding by all bidders is an ex post perfect equilibrium.*

1.4.3 Selecting option of the excess supply

In the original Ausubel auction, bidders may be assigned in excess of their last bid without they are not asked whether they want or not. In *the Ausubel auction with selecting option of the excess supply*, at each time, each bidder reports quantity and selects whether she accepts an excess supply or not. If at the last time of the auction, a bidder reports that she does not want to be assigned more quantity than her last bid, then she is assigned only her last bid.

We use the same model in Section 2. The process of the Ausubel auction with selecting option of the excess supply is as follows.

$\boldsymbol{t = 0}$**:** Each bidder $i \in N$ simultaneously reports quantity $x_i^0 \in X_i$ and a *signal* $a_i^0 \in \{0, 1\}$ for *tie-breaking*. If $\sum_{i \in N} x_i^0 \leq M$, the auction ends at $t = 0$ with the assignment $(x_i^*)_{i \in N}$ which is

$$x_i^* = x_i^0 \quad \forall i \in N.$$

Otherwise, for each bidder $i \in N$, let

$$C_i^0 = \max\left\{0, M - \sum_{j \neq i} x_j^0\right\}$$

be bidder i's cumulative clinches at $t = 0$, and the auction continues to $t = 1$.

$\boldsymbol{t = s < T}$**:** The auctioneer announces information of prior bids to each bidder. Each bidder $i \in N$ simultaneously reports quantity $x_i^s \in X_i$ satisfying the constraint

$$C_i^{s-1} \leq x_i^s \leq x_i^{s-1},$$

and a signal $a_i^s \in \{0, 1\}$. If $\sum_{i \in N} x_i^s \leq M$, the auction ends at $t = s$ with an assignment $(x_i^*)_{i \in N}$ which is decided by the following way: Let $N_0 = \{i \in N : a_i = 0\}$ and $N_1 = \{i \in N : a_i = 1\}$. Then

$$\begin{aligned} x_i^* &= x_i^s & \forall i \in N_0 \\ x_j^* &= x_j^s + \min\{x_j^{s-1}, \frac{x_j^{s-1} - x_j^s}{\sum_{k \in N_1} x_k^{s-1} - x_k^s} \cdot \big(M - \sum_{i \in N} x_i^s\big)\} & \forall i \in N_1 \end{aligned}$$

Otherwise, let $C_i^s = \max\left\{0, M - \sum_{j \neq i} x_j^s\right\}$ be bidder i's cumulative clinches, and the auction continues to $s + 1$.

$\boldsymbol{t = T}$**:** The auctioneer announces information of prior bids to each bidder. Each bidder $i \in N$ simultaneously bids quantity $x_i^T \in X_i$ with $C_i^{T-1} \leq x_i^T \leq x_i^{T-1}$ and signal $a_i^T \in \{0, 1\}$. In any case, the auction ends, even when there is excess demand. If $\sum_{i \in N} x_i^T > M$, an assignment $(x_i^*)_{i \in N}$ is such that $\sum_{i \in N} x_i^* = M$ and

$$x_i^* \le x_i^T \quad \forall i \in N.$$

Otherwise, similarly to the case ends at $t = s < T$ an assignment $(x_i^*)_{i \in N}$ is decided.

Note that M homogenous good may *not* be assigned *entirely* by the modified Ausubel auction. However, we can assign entirely and achieve an efficient outcome at sincere bidding equilibrium by defining sincere bidding as follows.

To define the strategies, we introduce the histories of the Ausubel auction with selecting option of excess supply which are very similar to Section 3. At each time $t \in \{1, 2, \ldots, T+1\}$, a *history* h^t is a summary of prior reports to t

$$h^t = \big((x_1^s, a_1^s), (x_2^s, a_2^s) \ldots, (x_n^s, a_n^s)\big)_{s \le t-1} \in \Bigg(\times_{i \in N} \big(X_i \times \{0, 1\}\big)\Bigg)^{\{0,1,\ldots,t-1\}}$$

such that for each $i \in N$ and each $s \le t-1$,

$$C_i^{s-1} \le x_i^s \le x_i^{s-1},$$
$$\sum_{j \in N} x_j^{t-2} > M.$$

Let $h^0 = \emptyset$ be the history of starting point $t = 0$. Let H^t be the set of histories at t, and $H \equiv \bigcup_{t=0}^{T+1} H^t$. A history $z^{t+1} = \big((x_1^s, a_1^s), \ldots, (x_n^s, a_n^s)\big)_{s \le t} \in H^{t+1}$ is *terminal* if $t = L$. Let Z^t be the set of terminal histories at t, and $Z \equiv \bigcup_{t=1}^{T+1} Z^t$ be all terminal histories. In this modified auction, we use the similar informational rule to full bid information, so $h^t = h_i^t$ for all $i \in N$.

A strategy of bidder i is a function $\sigma_i : H \setminus Z \to X_i \times \{0, 1\}$ satisfying the bidding constraint. In the modified Ausubel auction, sincere bidding is defined as follows.

Definition 3. Bidder i's *sincere bidding in the Ausubel auction with selecting option excess supply* is the strategy σ_i^* such that for any $t \geq 1$ and $h^t \in H^t \setminus Z^t$,

$$\sigma_i^*(h^t) = (\min\{x_i^{t-1}, \max\{Q_i(p^t), C_i^{t-1}\}\}, 1_{(x_i^{t-1}) \leq Q_i(p^{t-1})}),$$

and $\sigma_i^*(h^0) = (Q_i(p^0), 1)$.

We define subgames and ex post perfect equilibrium same as Section 2. Then, we have the following result.

Theorem 2. *In the Ausubel auction with selecting option of the excess suppy, sincere bidding by all bidders is an ex post perfect equilibrium.*

Proof. See Appendix. $\square$

1.5 Conclusion

We have investigated the sincere bidding equilibrium in the Ausubel auction. We first gave a counterexample to one of main results of Ausubel (2004). That is, we showed that sincere bidding by all bidders is not always an ex post perfect equilibrium. We then amended this result. We showed if a bidder does not bid her demand just before a node, then she has an incentive to sincerely bid at the node. Furthermore, sincere bidding by all bidders is an ex post equilibrium in the Ausubel aution. We also introduced two modifications of the Ausubel auction in which sincere bidding by all bidders is an ex post perfect equilibrium.

1.6 Appendix

The following counterexample shows that sincere bidding by all bidders is not always an ex post perfect equilibrium under all rationing rules that satisfy the monotonicity property.

Example 3

Consider a case where there are two bidders, A and B, and three quantities of an object. Let u_A, u_B be the marginal value functions of the two bidders such that

$$u_A(q) = u_B(q) = \begin{cases} 5 & \text{if } q \in [0, 1) \\ 1 & \text{if } q \in [1, 3]. \end{cases}$$

Consider the history $h^4 = (x_A^t, x_B^t)_{t=0,1,2,3} = \big((3,3),(3,3),(3,3),(3,3)\big)$. After h^4, sincere bidding of each bidder is 1.

The result under sincere bidding $\boldsymbol{x_A^4 = 1}$

If the bidders report $x_A^4 = x_B^4 = 1$ after h^4, then the auction ends at $z^5 = (h^4, (1,1))$, yielding an assignment (x_A^*, x_B^*) such that

$$1 \leq x_A^* \leq 3,$$
$$1 \leq x_B^* \leq 3,$$
$$x_A^* + x_B^* = 3.$$

Without loss of generality, suppose that $x_A^* \geq \frac{3}{2}$. Since bidder A did not clinch at $t \leq 3$, A's payment is $y_A^* = 4x_A^*$. Therefore, A's utility is $U_A(x_A^*) - 4x_A^*$ at z^5.

The result under misreporting $\hat{x}_A^4 = 0$

If bidder A reports $\hat{x}_A^4 = 0$, and bidder B reports $x_B^4 = 1$ after h^4, then the auction ends at $\hat{z}^5 = (h^4, (0,1))$, yielding an assignment $(\hat{x}_A, \hat{x}_B)$. Since $0 = \hat{x}_A^4 < x_A^4 = 1$ and any other condition of $\hat{z}^5$ is the same as z^5, by the monotonicity property, $\hat{x}_A$ must be strictly less than x_A^*. Similarly to case with sincere bidding, A's utility at $\hat{z}^5$ is $U_A(\hat{x}_A) - 4\hat{x}_A$.

We calculate the difference between A's utilities at z^5 and $\hat{z}^5$,

$$\begin{aligned}
&\Big(U_A(x_A^*) - 4x_A^*\Big) - \Big(U_A(\hat{x}_A) - 4\hat{x}_A\Big) \\
&= \Big(\int_0^{x_A^*} u_A(q)dq - 4x_A^*\Big) - \Big(\int_0^{\hat{x}_A} u_A(q)dq - 4\hat{x}_A\Big) \\
&= \Big(\int_0^{x_A^*} u_A(q)dq - \int_0^{\hat{x}_A} u_A(q)dq\Big) - 4\cdot\Big(x_A^* - \hat{x}_A\Big) \\
&= \int_{\hat{x}_A}^{x_A^*} u_A(q)dq - 4\cdot\Big(x_A^* - \hat{x}_A\Big). \qquad (1.4)
\end{aligned}$$

Case 1: $\hat{x}_A \geq 1$. We calculate (1.4) such that

$$x_A^* - \hat{x}_A - 4\cdot\Big(x_A^* - \hat{x}_A\Big) = -3\cdot\Big(x_A^* - \hat{x}_A\Big) < 0.$$

Case 2: $\hat{x}_A < 1$. We calculate (1.4) such that

$$\begin{aligned}
&\Big(x_A^* - 1\Big) + 5\cdot\Big(1 - \hat{x}_A\Big) - 4\cdot\Big(x_A^* - \hat{x}_A\Big) \\
&= -3x_A^* - \hat{x}_A + 4 < 0 \qquad (\because x_A^* \geq \frac{3}{2}).
\end{aligned}$$

Thus, A's utility at $\hat{z}^5$ is strictly greater than that at z^5, and bidder A has an incentive to misreport after h^4. Therefore, sincere bidding by all bidder is not an ex post perfect equilibrium.

Proof of Lemma 1

Since u_i is a weakly decreasing integer-valued function, there is a partition $\{a_0, \ldots, a_m\} \subset X_i$ with $0 = a_0 < \cdots < a_m = \lambda_i$ and values $\{b_1, \ldots, b_m\} \subset \{0, 1, \ldots, \overline{u}\}$ with $b_1 > b_2 > \cdots > b_m$ such that for each k with $1 \leq k \leq m$,

$$u_i(x_i) = b_k \text{ if } a_{k-1} < x_i < a_k.$$

Note that $m \leq T$. Consider any $x_i' \in X_i$. Let

$$k = \arg\min_{\ell}\{a_\ell : x_i' \leq a_\ell\}.$$

By the definition of Riemann Integral,

$$U_i(x_i') = \int_0^{x_i'} u_i(q)dq = \sum_{\ell=1}^{k-1} b_\ell(a_\ell - a_{\ell-1}) + b_k(x_i' - a_{k-1}). \tag{1.5}$$

Take any $p \in \{1, \ldots, T\}$. Define $b_0 = T + 1$. Let

$$r = \arg\min_{\ell}\{b_\ell : p - 1 < b_\ell\},$$
$$r' = \arg\min_{\ell}\{b_\ell : p \leq b_\ell\}.$$

By equation (1.5), we can verify that

$$a_r = \min\{\arg\max_{x_i \in X_i} U_i(x_i) - (p-1)x_i\},$$
$$a_{r'} = \max\{\arg\max_{x_i \in X_i} U_i(x_i) - p x_i\}.$$

Because $b_\ell \in \mathbb{Z}$ for each ℓ,

$$\{b_\ell : p - 1 < b_\ell\} = \{b_\ell : p \leq b_\ell\}.$$

Therefore $a_r = a'_r$, that is,

$$\min\{\arg\max_{x_i \in X_i} U_i(x_i) - (p-1)x_i\} = \max\{\arg\max_{x_i \in X_i} U_i(x_i) - px_i\}.$$

To prove Lemma 2 and Theorem 1, we explain a property of the Ausubel auction.

Property 1

For each $t \geq 1$, if there exists a bidder $i \in N$ such that $x_i^t = C_i^{t-1}$ and $C_i^{t-1} > 0$, then the auction ends at t, i.e., $t = L$. Therefore, if the auction does not end at t, then for each bidder $i \in N$, $x_i^t \neq C_i^{t-1}$ or $C_i^{t-1} = 0$.

Proof. Suppose that $x_i^t = C_i^{t-1}$ and $C_i^{t-1} > 0$. Then, $x_i^t = M - \sum_{j \neq i} x_j^{t-1}$. By bidding constraint for each $j \in N$, $x_j^t \leq x_j^{t-1}$. Therefore $\sum_{j \in N} x_j^t \leq M$. □

Note that this property holds under all rationing rules. We use the property in proofs of Lemma 2 and Proposition 1.

Proof of Lemma 2

Consider any $t \in \{0, 1, \ldots, T\}$,

$$h^t = (x_1^s, x_2^s, \ldots, x_n^s)_{s \leq t-1} \in H^t \setminus Z^t,$$

and $(u_j)_{j \in N}$. For each $j \in N$, let σ_j^* be sincere bidding which is corresponding to u_j, and $\sigma_j^*|_{h^t}$ be induced sincere bidding in the subgame that follows h^t.

Take any $i \in N$ and $\sigma_i \in \Sigma_i|_{h^t}$. Suppose that $x_i^{t-1} \leq Q_i(p^{t-1})$. We shall show that

$$\pi_i((\sigma_j^*|_{h^t})_{j \in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j \neq i}).$$

Let

$$z^{L+1} = (x_1^s, x_2^s, \ldots, x_n^s)_{s \leq L}$$

be the terminal history which is reached by $(\sigma_j^*|_{h^t})_{j \in N}$, and

$$w^{L'+1} = (\hat{x}_1^s, \hat{x}_2^s, \ldots, \hat{x}_n^s)_{s \leq L'}$$

be the terminal history which is reached by $(\sigma_i, (\sigma_j^*|_{h^t})_{j \neq i})$. Denote $\{(C_j^t)_{j \in N}\}_{t=0}^{L}$ the cumulative clinches of z^{L+1}, and $\{(\hat{C}_j^t)_{j \in N}\}_{t=0}^{L'}$ the cumulative clinches of $w^{L'+1}$.

Step 1. $x_i^{L-1} \leq Q_i(p^{L-1})$.

If $L - 1 = t - 1$, $x_i^{L-1} = x_i^{t-1} \leq Q_i(p^{t-1}) = Q_i(p^{L-1})$. Then, let $L - 1 \geq t$. By the definition of sincere bidding,

$$x_i^{L-1} = \sigma_i^*|_{h^t}\big((x_1^\ell, \ldots, x_n^\ell)_{\ell \le L-2}\big) = \min\{x_i^{L-2}, \max\{C_i^{L-2}, Q_i(p^{L-1})\}\}.$$

By Property 1, $x_i^{L-1} \neq C_i^{L-2}$ or $C_i^{L-2} = 0$. Then, $x_i^{L-1} = \min\{x_i^{L-2}, Q_i(p^{L-1})\}$. Therefore, $x_i^{L-1} \le Q_i(p^{L-1})$.

Step 2. For each $j \neq i$ and $s \le \min\{L-1, L'-1\}$, $x_j^s = \hat{x}_j^s$. This implies that for each $s \le \min\{L-1, L'-1\}$,

$$C_i^s = M - \sum_{j \neq i} x_j^s = M - \sum_{j \neq i} \hat{x}_j^s = \hat{C}_i^s.$$

For each $s \le t-1$, obviously $x_j^s = \hat{x}_j^s$.

For the cases with $t \le s \le \min\{L-1, L'-1\}$, we shall show by induction.
Let $s = t$. Because $x_j^t = \sigma_j^*|_{h^t}(h^t)$ and $\hat{x}_j^t = \sigma_j^*|_{h^t}(h^t)$, $x_j^t = \hat{x}_j^t$.
Let $s = k$ with $t+1 \le k \le \min\{L-1, L'-1\}$). Suppose that $x_j^\ell = \hat{x}_j^\ell$ for all ℓ with $t+1 \le \ell \le k-1$. By the definition of sincere bidding,

$$x_j^k = \sigma_j^*|_{h^t}\big((x_1^\ell, \ldots, x_n^\ell)_{\ell \le k-1}\big) = \min\{x_j^{k-1}, \max\{C_j^{k-1}, Q_j(p^k)\}\},$$
$$\hat{x}_j^k = \sigma_j^*|_{h^t}\big((\hat{x}_1^\ell, \ldots, \hat{x}_n^\ell)_{\ell \le k-1}\big) = \min\{\hat{x}_j^{k-1}, \max\{\hat{C}_j^{k-1}, Q_j(p^k)\}\}.$$

Since $k \le \min\{L-1, L'-1\}$, by Property 1, $x_j^k \neq C_j^{k-1}$ or $C_j^{k-1} = 0$. Thus, $x_j^k = \min\{x_j^{k-1}, Q_j(k)\}$. Similarly, we have $\hat{x}_j^k = \min\{\hat{x}_j^{k-1}, Q_j(k)\}$. Since $x_j^{k-1} = \hat{x}_j^{k-1}$, $x_j^k = \hat{x}_j^k$.

Step 3. $\pi_i((\sigma_j^*|_{h^t})_{j \in N}) \ge \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j \neq i})$.

We consider three cases; $L = L'$, $L > L'$ and $L < L'$.

Case 1. $L = L'$.

By step 2, for all $s \leq L-1 = L'-1$, $C_i^s = \hat{C}_i^s$. We calculate C_i^L and $\hat{C}_i^L$ for two cases with $x_i^L \geq Q_i(p^L)$ and $x_i^L < Q_i(p^L)$.

Case 1-1. $x_i^L \geq Q_i(p^L)$.

By step 1, $x_i^{L-1} \leq Q_i^{L-1}$. Thus,

$$Q_i(p^L) \leq x_i^L \leq C_i^L \leq x_i^{L-1} \leq Q_i(p^{L-1}).$$

Therefore, by lemma 1,

$$\min\{\arg\max_{x_i \in X_i}(U_i(x_i) - p^L x_i)\} \leq C_i^L \leq \max\{\arg\max_{x_i \in X_i}(U_i(x_i) - p^L x_i)\}.$$

Hence,

$$\pi_i((\sigma_j^*|_{h^t})_{j \in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j \neq i}).$$

Case 1-2. Let $x_i^L < Q_i(p^L)$.

We shall show that $x_i^L = x_i^{t-1}$. By the definition of sincere bidding,

$$x_i^L = \sigma_i^*|_{h^t}((x_1^s, \ldots, x_n^s)_{s \leq L-1}) = \min\{x_i^{L-1}, \max\{C_i^{L-1}, Q_i(p^L)\}\}.$$

Since $x_i^L < Q_i(p^L)$, $x_i^L = x_i^{L-1}$. If $L-1 = t-1$, $x_i^{L-1} = x_i^{t-1}$. Then, we assume $t-1 \neq L-1$. By the definition of sincere bidding,

$$x_i^{L-1} = \sigma_i^*|_{h^t}((x_1^s, \ldots, x_n^s)_{s \leq L-2}) = \min\{x_i^{L-2}, \max\{C_i^{L-2}, Q_i(p^{L-1})\}\}.$$

Since $Q_i(p^L) \leq Q_i(p^{L-1})$, $x_i^{L-1} = x_i^L < Q_i(p^{L-1})$. Hence, we have $x_i^{L-1} = x_i^{L-2}$. By repeating this procedure, $x_i^L = x_i^{L-1} = \cdots = x_i^{t-1}$. Thus, $C_i^L = x_i^{t-1}$.

Since bidder i cannot bid more quantity than x_i^{t-1} after h^t, $\hat{C}_i^L \leq x_i^{t-1}$. Then,

$$\hat{C}_i^L \leq C_i^L < Q_i(p^L).$$

Hence,

$$\pi_i((\sigma_j^*|_{h^t})_{j\in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j\neq i}).$$

Case 2. $L > L'$.

By step 2, for each $s \leq L'-1$, $C_i^s = \hat{C}_i^s$. Then, we calculate $\{C_i^s\}_{s=L'}^L$ and $\hat{C}_i^{L'}$. Since the auction does not end at L' in the history z^{L+1}, by Property 1 for each $j \neq i$, $x_j^{L'} \neq C_j^{L'-1}$ or $C_j^{L'-1} = 0$. Then, by the definition of sincere bidding, for each $j \neq i$,

$$x_j^{L'} = \min\{x_j^{L'-1}, Q_j(p^{L'})\}.$$

On the other hand,

$$\hat{x}_j^{L'} = \min\{\hat{x}_j^{L'-1}, \max\{\hat{C}_j^{L'-1}, Q_j(p^{L'})\}\}.$$

Since $x_j^{L'-1} = \hat{x}_j^{L'-1}$, $x_j^{L'} \leq \hat{x}_j^{L'}$. Thus,

$$\hat{C}_i^{L'} \leq M - \sum_{j\neq i} \hat{x}_j^{L'} \leq M - \sum_{j\neq i} x_j^{L'} = C_i^{L'}.$$

By the definition of cumulative clinches, for each $s \in \{L', \ldots, L-1\}$, $C_i^s \leq x_i^s$ and $x_i^L \leq C_i^L \leq x_i^{L-1}$. For each $s \in \{L', \ldots, L-1\}$, because $s \geq t$, x_i^s is sincere bidding. That is,

$$x_i^s = \min\{x_i^{s-1}, \max\{C_i^{s-1}, Q_i(p^s)\}\}.$$

Since the auction does not end at $s \leq L-1$ in the history z^{L+1}, by Property 1,

$$x_i^s = \min\{x_i^{s-1}, Q_i(p^s)\}.$$

Therefore, for each $s \in \{L', \ldots, L-1\}$, $x_i^s \leq Q_i(p^s)$. Thus,

$$\begin{aligned}
&C_i^s \leq Q_i(p^s) \qquad \forall s \in \{L', \ldots, L-1\},\\
&\hat{C}_i^{L'} \leq C_i^{L'} \leq Q_i(p^{L'}),\\
&C_i^L \leq x_i^{L-1} \leq Q_i(p^{L-1}) = \max\{\arg\max_{x_i \in X_i}(U_i(x_i) - p^L x_i)\}.
\end{aligned}$$

Hence,

$$\pi_i((\sigma_j^*|_{h^t})_{j\in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j\neq i}).$$

Case 3. $L < L'$.

We first show that $Q_i(p^L) \leq x_i^L$. Suppose that $Q_i(p^L) > x_i^L$. Similarly to case 1-2, we have $x_i^L = x_i^{t-1}$. By bidding constraint, $\hat{x}_i^L \leq x_i^{t-1}$. Then, $\hat{x}_i^L \leq x_i^L$. Since $L < L'$, the auction does not end at L in the history $w^{L'+1}$. Therefore, by Property 1, for each $j \neq i$, $\hat{x}_i^L \neq \hat{C}_i^L$ or $\hat{C}_i^L = 0$. By the definition of sincere bidding

$$\begin{aligned}
&x_j^L = \min\{x_j^{L-1}, \max\{C_j^{L-1}, Q_j(p^L)\}\},\\
&\hat{x}_j^L = \min\{\hat{x}_j^{L-1}, \max\{\hat{C}_j^{L-1}, Q_j(p^L)\}\} = \min\{\hat{x}_j^{L-1}, Q_j(p^L)\}.
\end{aligned}$$

For each $j \neq i$, since by step 2, $x_j^{L-1} = \hat{x}_j^{L-1}$, we have $x_j^L \geq \hat{x}_j^L$. Hence for each $j \in N$, $x_j^L \geq \hat{x}_j^L$. Since the auction ends at L in the history z^{L+1}, $\sum_{j \in N} x_j^L \leq M$. Therefore, $\sum_{j \in N} \hat{x}_j^L \leq M$. This implies the auction ends at L in the history $w^{L'+1}$. This contradicts to $L < L'$. Thus, $Q_i(p^L) \leq x_i^L$.

By step 2, for each $s \leq L-1$, $C_i^s = \hat{C}_i^s$. Similarly to case 2, we have $C_i^L \leq \hat{C}_i^L$. Because $x_i^L \leq C_i^L$, $Q_i(p^L) \leq C_i^L \leq \hat{C}_i^L$. Moreover, for each $s \geq L$, $\hat{C}_i^L \leq \hat{C}_i^s$ and $Q_i(p^s) \leq Q_i(p^L)$. Thus, for each $s \geq L$, $Q_i(p^s) \leq \hat{C}_i^s$. Hence,

$$\pi_i((\sigma_j^*|_{h^t})_{j \in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j \neq i}).$$

Proof of Theorem 1

Consider any $t \in \{0, 1, \ldots, T\}$,

$$h^t = (x_1^s, x_2^s, \ldots, x_n^s)_{s \leq t-1} \in H^t \setminus Z^t,$$

and $(u_j)_{j \in N}$. For each $j \in N$, let σ_j^* be sincere bidding which is corresponding to u_j, and $\sigma_j^*|_{h^t}$ be induced sincere bidding in the subgame that follows h^t.

Take any $i \in N$ and $\sigma_i \in \Sigma_i|_{h^t}$. We shall show that

$$\pi_i((\sigma_j^*|_{h^t})_{j \in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j \neq i}).$$

If $x_i^{t-1} \leq Q_i(p^{t-1})$, we can show by Lemma 2. Suppose that $x_i^{t-1} > Q_i(p^{t-1})$.

Let

$$z^{L+1} = (x_1^s, x_2^s, \ldots, x_n^s)_{s \leq L}$$

be the terminal history which is reached by $(\sigma_j^*|_{h^t})_{j\in N}$, and

$$w^{L'+1} = (\hat{x}_1^s, \hat{x}_2^s, \dots, \hat{x}_n^s)_{s\leq L'}$$

be the terminal history which is reached by $(\sigma_i, (\sigma_j^*|_{h^t})_{j\neq i})$. Denote $\{(C_j^t)_{j\in N}\}_{t=0}^{L}$ the cumulative clinches of z^{L+1}, and $\{(\hat{C}_j^t)_{j\in N}\}_{t=0}^{L'}$ the cumulative clinches of $w^{L'+1}$.

We consider three cases; $L > t$, $L' > L = t$ and $L' = L = t$.

Case 1. $L > t$.

Since $L - 1 \geq t$, by the definition of sincere bidding,

$$x_i^{L-1} = \sigma_i^*|_{h^t}\big((x_1^\ell, \dots, x_n^\ell)_{\ell\leq L-2}\big) = \min\{x_i^{L-2}, \max\{C_i^{L-2}, Q_i(p^{L-1})\}\}.$$

By Property 1, $x_i^{L-1} \neq C_i^{L-2}$ or $C_i^{L-2} = 0$. Then, $x_i^{L-1} = \min\{x_i^{L-2}, Q_i(p^{L-1})\}$. Therefore, $x_i^{L-1} \leq Q_i(p^{L-1})$, which is the same argument as step 1 of Lemma 2. Note that we only use the assumption $x_i^{t-1} \leq Q_i(p^{t-1})$ in step 1 of Lemma 2. Thus, we can prove this case similarly to Lemma 2.

Case 2. $L' > L = t$.

For each $j \in N$ and each $s \leq t-1$, obviously $x_j^s = \hat{x}_j^s$. Therefore, for each $s \leq t - 1 = L - 1$, $C_i^s = \hat{C}_i^s$. We will calculate C_i^L and $\{\hat{C}_i^s\}_{s=L}^{L'}$.

We first show that $Q_i(p^L) \leq C_i^L$. By the definition of sincere bidding,

$$x_i^L = \min\{x_i^{L-1}, \max\{C_i^{L-1}, Q_i(p^L)\}\}.$$

Since $x_i^{L-1} \geq C_i^{L-1}$ and $x_i^{L-1} > Q_i(p^{L-1}) \geq Q_i(p^L)$,

$$x_i^L = \max\{C_i^{L-1}, Q_i(p^L)\}.$$

Therefore, $Q_i(p^L) \leq x_i^L$. Because $x_i^L \leq C_i^L \leq x_i^{L-1}$, $Q_i(p^L) \leq C_i^L$.

Next we show that $C_i^L \leq \hat{C}_i^L$. For each $j \neq i$, because $t = L$, $x_j^L = \sigma_j^*|_{h^t}(h^t)$ and $\hat{x}_j^L = \sigma_j^*|_{h^t}(h^t)$. Therefore, for each $j \neq i$, $x_j^L = \hat{x}_j^L$. Since the auction does not end at L in the history $w^{L'+1}$,

$$\hat{C}_i^L = M - \sum_{j \neq i} \hat{x}_j^L = M - \sum_{j \neq i} x_j^L.$$

On the other hand, since the auction ends at L in the history z^{L+1},

$$C_i^L \leq M - \sum_{j \neq i} x_j^L.$$

Therefore, $C_i^L \leq \hat{C}_i^L$.

Hence, $Q_i(p^L) \leq C_i^L \leq \hat{C}_i^L$. Furthermore, for all $s \geq L+1$, $Q_i(p^s) \leq \hat{C}_i^s$, because $Q_i(p^s) \leq Q_i(p^L)$ and $\hat{C}_i^L \leq \hat{C}_i^s$. Thus,

$$\pi_i((\sigma_j^*|_{h^t})_{j \in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j \neq i}).$$

Case 3. $L' = L = t$.

For each $j \in N$ and each $s \leq t-1$, obviously $x_j^s = \hat{x}_j^s$. Furthermore, for each $j \neq i$, $x_j^L = \sigma_j^*|_{h^t}(h^t) = \hat{x}_j^L$. Since for each $s \leq L-1$, $C_i^s = \hat{C}_i^s$, we calculate C_i^L and $\hat{C}_i^L$.

Case 3-1. $C_i^L = x_i^L$.

By the definition of sincere bidding,

$$x_i^L = \min\{x_i^{L-1}, \max\{C_i^{L-1}, Q_i(p^L)\}\}.$$

Since $x_i^{L-1} \geq C_i^{L-1}$ and $x_i^{L-1} > Q_i(p^{L-1}) \geq Q_i(p^L)$,

$$x_i^L = \max\{C_i^{L-1}, Q_i(p^L)\}.$$

If $x_i^L = Q_i(p^L)$, then $C_i^L = Q_i(p^L)$ and we have

$$\pi_i((\sigma_j^*|_{h^t})_{j\in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j\neq i}).$$

Suppose that $x_i^L = C_i^{L-1}$. Then, $C_i^{L-1} \geq Q_i(p^L)$ and $C_i^L = C_i^{L-1}$. Since $\hat{C}_i^L \geq \hat{C}_i^{L-1}$ and $C_i^{L-1} = \hat{C}_i^{L-1}$, $\hat{C}_i^L \geq C_i^L$. Therefore, $\hat{C}_i^L \geq C_i^L \geq Q_i(p^L)$. Hence

$$\pi_i((\sigma_j^*|_{h^t})_{j\in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j\neq i}).$$

Case 3-2. $C_i^L > x_i^L$.

First, we show that for each $j \in \{1, \ldots, i-1\}$, $C_j^L = x_j^{L-1}$. Suppose that there exists $j \in \{1, \ldots, i-1\}$ such that $C_j^L \neq x_j^{L-1}$. By the definition of our rationing rule,

$$C_j^L = \min\{x_j^{L-1}, x_j^L + M - \sum_{k=j}^{n} x_k^L - \sum_{k=1}^{j-1} C_k^L\}$$

$$= x_j^L + M - \sum_{k=j}^{n} x_k^L - \sum_{k=1}^{j-1} C_k^L.$$

Therefore,

$$M = \sum_{k=j+1}^{n} x_k^L - \sum_{k=1}^{j} C_k^L.$$

Since for each $k \in N$, $x_k^L \leq C_k^L$, and $\sum_{k\in N} C_k^L = M$,

$$M = \sum_{k=j+1}^{n} x_k^L - \sum_{k=1}^{j} C_k^L \leq \sum_{k\in N} C_k^L = M.$$

Therefore, for each $k \geq j+1$, $x_k^L = C_k^L$. Because $i \geq j+1$, this contradicts to $C_i^L > x_i^L$.

Next, we show that $C_i^L \leq \hat{C}_i^L$. By the definition of our rationing rule,

$$C_i^L = \min\{x_i^{L-1}, x_i^L + M - \sum_{j=i}^{n} x_j^L - \sum_{j=1}^{i-1} C_j^L\} \qquad = \min\{x_i^{L-1}, M - \sum_{j=i+1}^{n} x_j^L - \sum_{j=1}^{i-1} C_j^L\},$$

$$\hat{C}_i^L = \min\{x_i^{L-1}, \hat{x}_i^L + M - \hat{x}_j^L - \sum_{j=i+1}^{n} x_j^L - \sum_{j=1}^{i-1} \hat{C}_j^L\} \quad = \min\{x_i^{L-1}, M - \sum_{j=i+1}^{n} x_j^L - \sum_{j=1}^{i-1} \hat{C}_j^L\}.$$

For each $j \leq i-1$, since $x_j^{L-1} = C_j^L$ and $x_j^{L-1} \geq \hat{C}_j^L$,

$$C_j^L \geq \hat{C}_j^L.$$

Therefore,

$$\min\{x_i^{L-1}, M - \sum_{j=i+1}^{n} x_j^L - \sum_{j=1}^{i-1} C_j^L\} \leq \min\{x_i^{L-1}, M - \sum_{j=i+1}^{n} x_j^L - \sum_{j=1}^{i-1} \hat{C}_j^L\}.$$

Hence, $C_i^L \leq \hat{C}_i^L$.

Similarly to case 2, we can show that $Q_i(p^L) \leq C_i^L$. Therefore, $Q_i(p^L) \leq C_i^L \leq \hat{C}_i^L$. Thus,

$$\pi_i((\sigma_j^*|_{h^t})_{j\in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j\neq i}).$$

Proof of Theorem 2

Consider any $t \in \{0, 1, \ldots, T\}$,

$$h^t = \big((x_1^s, a_1^s), (x_2^s, a_2^s) \ldots, (x_n^s, a_n^s)\big)_{s\leq t-1} \in \Bigg(\times_{i\in N} \big(X_i, \times\{0,1\}\big)\Bigg)^{\{0,1,\ldots,t-1\}}$$

and $(u_j)_{j\in N}$. For each $j \in N$, let σ_j^* be sincere bidding which is corresponding to u_j, and $\sigma_j^*|_{h^t}$ be induced sincere bidding in the subgame that follows h^t.

Take any $i \in N$ and $\sigma_i \in \Sigma_i|_{h^t}$. If $x_i^{t-1} \leq Q_i(p^{t-1})$, then we can show similarly to Proposition 1. Suppose that $x_i^{t-1} > Q_i(p^{t-1})$.

Let

$$z^{L+1} = \Big((x_1^s, a_1^s)(x_2^s, a_2^s), \ldots, (x_n^s, a_n^s)\Big)_{s\leq L}$$

be the terminal history which is reached by $(\sigma_j^*|_{h^t})_{j\in N}$, and

$$w^{L'+1} = \Big((\hat{x}_1^s, \hat{a}_1^s)(\hat{x}_2^s, \hat{a}_2^s), \ldots, (\hat{x}_n^s, \hat{a}_n^s)\Big)_{s\leq L'}$$

be the terminal history which is reached by $(\sigma_i, (\sigma_j^*|_{h^t})_{j\neq i})$. Denote $\{(C_j^t)_{j\in N}\}_{t=0}^{L}$ the cumulative clinches of z^{L+1}, and $\{(\hat{C}_j^t)_{j\in N}\}_{t=0}^{L'}$ the cumulative clinches of $w^{L'+1}$.

We first consider the case $L > t$. Since the auction does not end at t in the history z^{L+1},

$$x_i^t \neq C_i^{t-1}.$$

Because $Q_i(p^t) \leq Q_i(p^{t-1}) < x_i^{t-1}$,

$$x_i^t \neq x_i^{t-1}.$$

That is, $x_i^t = Q_i(p^t)$. If the auction ends at t in the history $w^{L'+1}$,

$$\hat{C}_i^{L'+1} \leq C_i^t \leq Q_i(p^t) = \min\{\arg\max_{x_i \in X_i}(U_i(x_i) - px_i)\}.$$

On the other hand, the equation $\hat{C}_i^{t+1} = C_i^t$ holds. Then, since $x_i^t = Q_i(p^t)$, we can prove the case $L < t$ similarly to Proposition 1.

We next consider the case $L = t$. Then, $\sigma_i^*|_{h^t}(h^t) = (\max(C_i^{t-1}, Q_i(p^t)), 0)$, because $x_i^{t-1} > Q_i(p^{t-1}) \geq Q_i(p^t)$. Thus,

$$C_i^L = \max\{C_i^{t-1}, Q_i(p^t)\}.$$

Then

$$\hat{C}_i^L \geq C_i^L \geq Q_i(p^t) = \min\{\arg\max_{x_i \in X_i}(U_i(x_i) - px_i)\}.$$

Then for all $s \leq t$, the clinches $\hat{C}_i^s \geq \min\{\arg\max_{x_i \in X_i}(U_i(x_i) - px_i)\}$ $s \leq t$ reduce the utility of bidder i. Thus,

$$\pi_i((\sigma_j^*|_{h^t})_{j \in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j \neq i}).$$

Chapter 2

Non-Wasteful Rescheduling on the Ground Delay Program

2.1 Introduction

The Ground Delay Program is an air traffic control program in the United States. When inclement weather strikes an airport, the airport needs to reconstruct the landing schedule. There are mainly two reasons to reschedule: airport's condition and flights' condition. First reason is that airport's acceptable rate of flights declines, and the number of available slots decreases. Second reason is that some flights may delay its arrival or be canceled.

The purpose of this paper is to design a mechanism which always obtains a non-wasteful schedule. In this paper, we introduce a new efficiency condition, namely *universal non-wastefulness*. To construct a non-waste schedule, we need to aggregate correct information. However, flights' conditions are private information of airlines. Thus, we also investigate incentive conditions of rescheduling rules.

In the FAA's current mechanism on the Ground Delay Program, for airport's condition, FAA's decides which arrival slots remain active at the first step.[*8] The center aggregates earliest feasible arrival times of flights from airlines, and assigns available slots to flights by a rescheduling rule at the second step. We explain

[*8] See Vossen and Ball (2006a, 2006b), for example, for a survey on FAA's current mechanism.

the mechanism by an example.

Example 1: There are ten slots and ten flights in the original schedule. Because bad weather strikes the airport, airport's acceptable rate declines one flight per a minute to one flight per two minutes. The airport selects five slots as available slots, 10:01, 10:03, 10:05, 10:07, and 10:09. Suppose flights' conditions are as in Table 1, that is, f_1 can arrive at earliest 10:01, f_2 has to be canceled, f_3 can arrive at earliest 10:04, and so on. The FAA's current mechanism assigns the schedule such that slot 10:01 is assigned to f_1, slot 10:03 is vacant, slot 10:05 is assigned to f_3, slot 10:07 is assigned to f_6, and slot 10:09 is assigned to f_5.

time	flight	feasible arrival time	rescheduling
10:01	f_1	10:01	f_1
10:02	f_2	canceled	–
10:03	f_3	10:04	vacant
10:04	f_4	canceled	–
10:05	f_5	10:08	f_3
10:06	f_6	10:06	–
10:07	f_7	canceled	f_6
10:08	f_8	canceled	–
10:09	f_9	10:10	f_5
10:10	f_{10}	canceled	–

Table 1: The existing procedure

However, we can construct a more efficient schedule. In the schedule of Table 2, slot 10:04 is assigned to f_3, slot 10:06 is assigned to f_6, slot 10:08 is assigned to f_5, and slot 10:10 is assigned to f_9. Therefore, three flights f_3, f_6 and f_5 are assigned earlier slots than the above schedule. Furthermore, in this schedule, one more flight f_9 is assigned an available slot. Note that this schedule keeps the airline's acceptable rate of one flight per two minutes.

time	flight	feasible arrival time	rescheduling
10:01	f_1	10:01	f_1
10:02	f_2	canceled	–
10:03	f_3	10:04	–
10:04	f_4	canceled	f_3
10:05	f_5	10:08	–
10:06	f_6	10:06	f_6
10:07	f_7	canceled	–
10:08	f_8	canceled	f_5
10:09	f_9	10:10	–
10:10	f_{10}	canceled	f_9

Table 2: Non-waste schedule

Since FAA's current mechanism fixes available slots, FAA may have to select a schedule that is wasteful in some situation. Hence, we develop a new procedure such that available slots are flexible. Then, we investigate properties of non-wastefulness and fairness on the new procedure. Finally, we analyze airlines' incentives to report earliest feasible arrival times.

This chapter is organized as follows. In Section 2, we introduce definitions and notation. In Section 3, we investigate problems of the existing model. In Section 4, we introduce a new model and design a new mechanism. In Section 5, we conclude this chapter. Some proofs are relegated to Appendix.

2.2 Notation and Definitions

*9

There is a set of finite *original arrival slots* $\overline{S} = \{1, 2, \ldots, |\overline{S}|\}$. For any $s, s' \in \overline{S}$, $s < s'$ means that the time of slot s is earlier than the time of slot s'. The *the null slot*, denoted by $|\overline{S}| + 1$, means *cancellation*.

There is a finite set of *airlines* $\mathscr{A}$. Each airline $A \in \mathscr{A}$ has a finite set of *flights* F_A. Let $F \equiv \bigcup_{A \in \mathscr{A}} F_A$ be the set of all flights. Each flight is *initially* assigned to a slot, that is, we assume that $|F| = |\overline{S}|$.

Each flight $f \in F$ has an *earliest feasible arrival time* $e_f \in \overline{S} \cup \{|\overline{S}| + 1\}$; that is, for any $s \in \overline{S}$, flight f can arrive at the time of slot s if $s \geq e_f$. Note that $f \in F$ cannot arrive at the time of any slot if $e_f = |\overline{S}| + 1$. An *earliest arrival*

*9 In this section, we follow definitions and notation of Schummer and Vohra (2013) and Schmmer and Abizada (2017).

time profile, or simply a *time profile*, is a list of earliest feasible arrival times

$$e = (e_f)_{f \in F}.$$

For each $A \in \mathscr{A}$, we denote

$$e_A = (e_f)_{f \in F_A},$$

$$e_{-A} = (e_f)_{f \in F \setminus F_A}.$$

An *original schedule* is a bijection $g : F \to \overline{S}$ and its *inverse function* is denoted by $g^{-1} : \overline{S} \to F$. A *schedule* is a function $\pi : F \to \overline{S} \cup \{|\overline{S}| + 1\}$ such that for each $s \in \overline{S}$, $|\{f \in F : s = \pi(f)\}| \leq 1$. This condition means that at most one flight is assigned to a slot except for the null slot.

Let Π be the set of schedules. A schedule $\pi \in \Pi$ is *feasible* for e_A if for any $f \in F_A$, $e_f \leq \pi(f)$. Let $\Pi(e_A) \subset \Pi$ be the set of schedules that are feasible for e_A. Similarly, $\pi \in \Pi$ is *feasible* for e if for any $f \in F$, $e_f \leq \pi(f)$. Let $\Pi(e) = \bigcap_{A \in \mathscr{A}} \Pi(e_A)$.

2.3 Fixed slots problems

In this section, we analyze schedules which are constructed on fixed available slots. The center decides which slots remain active given any $S \subset \overline{S}$.

A schedule $\pi \in \Pi$ is *feasible* for $S \subset \overline{S}$ if for any $s \in \overline{S} \setminus S$,

$$|\{f \in F : \pi(f) = s\}| = 0.$$

Let $\Pi(S) \subset \Pi$ be the set of schedules which are feasible for S.

A *fixed slots problem* is $x = (S, (e_A)_{A \in \mathscr{A}})$. Let $X \equiv 2^{\overline{S}} \times \left(\overline{S} \cup \{|\overline{S}| + 1\}\right)^F$ be the set of fixed slots problems. An *assignment rule on fixed slots problem* is a function $\varphi : X \to \Pi$ which satisfies feasibility for the domain X. Thus, for any $x = (S, (e_A)_{A \in \mathscr{A}}) \in X$, $\varphi(x) \in \Pi(x) \equiv \Pi(S) \cap \Pi(e)$.

We first introduce an efficiency axiom on the fixed slots problem. If a schedule is partially non-wasteful, then there is no available slot that any unassigned flight can use.

Axiom 1. Given any $x = (S, (e_A)_{A \in \mathscr{A}}) \in X$, a schedule $\pi \in \Pi(x)$ is *partially non-wasteful* for x if there exists no slot $s \in S$ such that

$$\begin{aligned} \pi(f) \neq s \qquad & \forall f \in F \\ e_{f'} \leq s < \pi(f') \qquad & \exists f' \in F. \end{aligned}$$

An assignment rule φ satisfies *partial non-wastefulness* if for any problem $x \in X$, $\varphi(x)$ is partially non-wasteful for x.

Next, we introduce an order preservation axiom. If a schedule is order preserving, any flight is not overtaken by other flights which are later than the flight in the original schedule.

Axiom 2. Given any problem $x = (S, (e_A)_{A \in \mathscr{A}})$, a schedule $\pi \in \Pi$ is *order preserving* if there exist no flights $f, f' \in F$ such that $g(f) < g(f')$ and $\pi(f) > \pi(f') \geq e_f$.

Order preserving is a certain fairness axiom to flights in rescheduling. If this condition is satisfied, flights do not face with a situation such that thought a flight can use a slot, another flight which is later in the original schedule is assigned to the slot.

In each problem, we want to find a schedule that is partially non-wasteful and order preserved. Here, we describe the rule of the flight proposing deferred acceptance algorithm with the original schedule priority.[*10]

[*10]Gale and Shapley (1962) introduces this algorithm. See Roth (2004), for example, for a suvery.

Step 1-a.

Each flight proposes to the earlies available slot in its earliest feasible arrival time. If a flight cannot arrive at any slot, then the flight is assigned to the null slot.

Step 1-b.

Each slot except for the null slots rejects all but the earliest in the original schedule of flights which have proposed to the slot, and keeps the earliest flight.

Step 1-c.

If all flights are kept by slots or assigned to the null slot, the algorithm ends and each flight is assigned to the that keeps it. If not, go to Step 2-a.

$$\vdots$$

Step k-a.

Each flight that has been rejected in the previous step proposes to the earliest of those available slots that have not yet rejected the flight. If a flight cannot arrive at any slot which has not yet rejected the flight, then the flight is assigned to the null slot.

Step k-b.

Each slot except for the null slots rejects all but the earliest in the original schedule of flights which have proposed to the slot, and keeps the earliest flight.

Step k-c.

If all flights are kept by slots or assigned to the null slot, the algorithm ends and each flight is assigned to the slot that keeps it. If not, go to Step k+1-a.

Figure 1: The flight proposing deferred acceptance algorithm

In Proposition 1, we prove that the flight proposing algorithm satisfies partial non-wastefulness and original schedule monotonicity. Moreover, we show the uniqueness of the schedule that is partially non-wasteful and order preserving.[*11]

Proposition 1. *For each problem, there exists a unique schedule that is partial non-wasteful and order preserving. The flight proposing deferred acceptance algorithm with the original schedule priority selects this schedule.*

Proof. See Appendix. □

Next, we investigate incentives in the flight proposing deferred acceptance algorithm with the original schedule priority. It is well-known that under Gale and Shapley's DA algorithm, no one has an incentive to misreport his private information.[*12] In the following results, we show that under the flight proposing deferred acceptance algorithm with the original schedule priority, no airline has an incentive to misreport.

Proposition 2. *Let φ be the flight proposing Deferred Acceptance algorithm with the original schedule priority. Consider any $x = (S, (e_A)_{A \in \mathscr{A}}) \in X$, any $A \in \mathscr{A}$, and e'_A. Let $x' = (S, (e'_A, e_{-A}))$, $\varphi(x) = \pi$ and $\varphi(x') = \pi'$. Then, if there exists $f \in F_A$ such that $e_f \leq \pi'(f) < \pi(f)$, there must exist $f' \in F_A$ such that $e_{f'} \leq \pi(f') < \pi'(f')$ or $\pi'(f') < e_{f'} \leq \pi(f') < |\overline{S}| + 1$.*

Proof. See Appendix. □

[*11]We note that slots proposing deferred acceptance algorithm with same orderings obtains the same results of flight proposing in these problems. Crawford (1991) showed that if priorities satisfy certain monotonicity, then the both of man and woman proposing deferred acceptance algorithm select the same outcome.

[*12]See Dubins and Freedman (1981) and Roth (1982).

Proposition 2 states that if an airline which has a flight assigned to a earlier feasible slot by misreporting, then the airline must have a flight which is assigned to a later or infeasible slot. That is, any airline cannot improve the schedule by misreporting on the flight proposing deferred acceptance algorithm.

2.4 Minimum interval problem

So far, we have discussed rescheduling problems on the fixed slots domain. In this section, we extend the domain so as to accommodate more flights.

The center decides a *minimum interval* $d \in \overline{S}$ of the intervals between flights. A schedule $\pi \in \Pi$ is *feasible* for d if for any $s \in \overline{S}$,

$$\left|\left\{f \in F : \pi(f) \in [s-d, s+d] \text{ and } \pi(f) \neq |\overline{S}|+1\right\}\right| \leq 1.$$

Let $\Pi(d)$ be the set of schedules which are feasible for d.

A *minimum interval problem* is $y = (d, (e_A)_{A \in \mathscr{A}})$. Let $Y \equiv \overline{S} \times (\overline{S} \cup \{|\overline{S}|+1\})^F$ be the set of problems. An *assignment rule on minimum interval problem* is a function $\phi : Y \to \Pi$ which satisfies feasibility for the domain Y: For any $y \in Y$, $\phi(y) \in \Pi(y) \equiv \Pi(d) \cap \Pi(e)$.

Remark 1. Consider any $y = (d, (e_A)_{A \in \mathscr{A}}) \in Y$, any $S \subset \overline{S}$ such that for any $s, s' \in S$ with $s < s'$, $s + d < s'$, and let $x = (S, (e_A)_{A \in \mathscr{A}})$. Then, $\Pi(x) \subset \Pi(y)$.

In fact, a minimum interval problem extends the domain of feasible schedules. Thus, we also extend the efficiency axiom in this problem. If a schedule of the minimum interval problem is universally non-wasteful, then the schedule does not have a waste interval of times. Thus, it maximizes the number of flights that are assigned available slots.[*13]

Axiom 3. Given $y = (d, (e_A)_{A\in\mathscr{A}}) \in Y$, a schedule $\pi \in \Pi(y)$ is *universally non-wasteful* for y if there exists no $s \in \overline{S}$ such that

$$\begin{aligned} &\pi(f) \notin [s-d, s] && \forall f \in F \\ &e_{f'} \leq s < \pi(f') && \exists f' \in F. \end{aligned}$$

An assignment rule ϕ satisfies *universal non-wastefulness* if for any $y \in Y$, $\phi(y)$ is universally non-wasteful for y.

In this model, we also introduce the same axiom of order preserving.

Axiom 4. Given any problem $y = (d, (e_A)_{A\in\mathscr{A}}) \in Y$, a schedule $\pi \in \Pi$ is *order preserving* if there exist no flights $f, f' \in F$ such that $g(f) < g(f')$ and $\pi(f) > \pi(f') \geq e_f$.

An assignment rule φ satisfies *order preserving* if for any problem $x \in X$, $\varphi(x)$ is order preserving.

For each minimum interval problem, we also search a schedule that is universally non-wasteful and order preserving. Here, we describe a new rule, *the sequential assignment algorithm*, in Figure 2.

[*13]Here, we assume that each airline has a preference order as follows: For any two schedule π, π', (i) an airline A strictly prefer π to π' if for any $f \in F_A$, $e_f \leq \pi(f) \leq \pi'(f)$ and for some $f' \in F_A$, $e_f \leq \pi(f') < \pi(f')$, (ii) π and π' is incomparable for A if there exist $f, f' \in F_A$ such that $\pi(f) < \pi'(f)$ and $\pi(f') > \pi'(f')$, and (iii) if there exists $f \in F_A$, $\pi(f) < e_f$, then π is preferred to all the other schedules by A. A partial non-wasteful schedule is Pareto efficient in $\Pi(x)$, but it may not be Pareto efficient in $\Pi(y)$. On the other hand, a universal non-wasteful schedule is always Pareto efficient in $\pi(y)$.

Step 0.

Given any $y = (d, (e_A)_{A \in \mathscr{A}}) \in Y$.

Step 1.

Start at slot 1. Let $F_1 = \{f \in F : e_f \leq 1\}$. If $F_1 = \emptyset$, go to slot 2. If $F_1 \neq \emptyset$, assign $\arg\min_{f \in F_1} g(f)$ to slot 1, and go to slot $\min\{2+d, |\overline{S}|+1\}$.

$$\vdots$$

Step k.

At slot $s \in \overline{S}$, let

$$F_s = \{f \in F : e_f \leq s \text{ and } f \text{ has not been assigned to a slot}\}.$$

If $F_s = \emptyset$, go to $s+1$. If $F_s \neq O$, assign $\arg\min_{f \in F_s} g(f)$ to s, and go to slot $\min\{s+1+d, |\overline{S}|+1\}$.

Step n.

At $|\overline{S}|+1$, assign the flights that have not been assigned to slots to the null slot, and the algorithm ends.

Figure 2: The sequential assignment algorithm

Theorem 1. *For each problem, there exists a unique schedule that is universal non-wasteful and order preserving. The sequential assignment algorithm selects this schedule.*

Proof. See Appendix. □

Next, we investigate airlines' incentives to misreport earliest feasible arrival times.

Theorem 2. *Let ϕ be the sequential assignment algorithm. Consider any $y = (d, (e_A)_{A \in \mathscr{A}}) \in Y$, any $A \subset \mathscr{A}$ and any e'_A. Let $y' = (d, (e'_A, e_{-A}))$, $\phi(x) = \pi$ and $\phi(x') = \pi'$. If there exists $f \in F_A$ such that $e_f \leq \pi'(f) < \pi(f)$, then there exist $f' \in F_A$ such that $e_{f'} \leq \pi(f') < \pi'(f')$ or $\pi'(f') < e_{f'} \leq \pi(f')$.*

Proof. See Appendix. □

Theorem 2 states that if an airline which has a flight assigned to an earlier feasible slot by misreporting, then the airline must have a flight which is assigned to a later or infeasible slot. We note that there may exists an airline which can improve the schedule by making dummy flight in the sequential acceptance algorithm. That is, by misreporting some airline may make a flight such that although a flight is assigned to an infeasible slot, the flight is assigned to the null slot in the schedule of truth-reporting.[*14] However, in real situations, it is difficult for airlines to make a dummy flight, and such airlines may be punished for making a dummy flight.

[*14]The difference point between Proposition 2 and Proposition 4 is $\pi'(f') < e_{f'} \leq \pi(f') < |\overline{S}| + 1$ and $\pi'(f') < e_{f'} \leq \pi(f')$. Thus, in Proposition 4, $\pi'(f') < e_{f'} \leq \pi(f') = |\overline{S}| + 1$ may occur.

2.5 Conclusion

In this study, we first showed that FAA's current mechanism may not maximize the number of flights assigned to available slots for all possible cases. Then, we extended the domain of schedules, and introduce an efficiency axiom, *universal non-wastefulness.* We designed a new mechanism, namely *the sequential assignment algorithm*, that satisfies universal non-wastefulness and order preserving. Furthermore, we showed that under this mechanism, all airlines have no incentive to misreport. To find a universal non-wasteful mechanism under which there exists no incentives to make a dummy flight is left to the future research.

2.6 Appendix

Proof of Proposition 1. Take any $x = (S, (e_A)_{A \in \mathscr{A}})$. Let φ be the flight proposing deferred acceptance algorithm with the original schedule priority, and $\varphi(x) = \pi$.

We first show that π is partial non-wasteful. Note that φ satisfies feasibility, that is, for any $f \in F$, $e_f \leq \pi(f)$. Suppose that there exists $s \in S$ such that

$$\begin{aligned} \pi(f) &\neq s && \forall f \in F \\ e_{f'} \leq s &< \pi(f') && \exists f' \in F. \end{aligned}$$

However, since $e_{f'} \leq s < \pi(f')$, f' has proposed to s at some step, and s accept f'. This is a contradiction.

Next, we shall show the schedule satisfies order preserving. Suppose that there exists flights $f, f' \in F$ with $g(f) < g(f')$ such that

$$\pi(f) > \pi(f') \geq e_f$$

Then, f has proposed $\pi(f')$ at some step. Since $g(f) < g(f')$, f' was rejected by $\pi(f')$ at some step, or does not proposed to $\pi(f')$. This is a contradiction.

Finally, we shall show the uniqueness of the schedule that is partially non-wasteful and monotone to the original schedule. Suppose that there exists two schedule π, π' which is partially non-wasteful and monotone to the original schedule, and $\pi \neq \pi'$. Then, there exists $f_1 \in F$ such that $\pi(f_1) \neq \pi'(f_1)$. Without loss of generality, $\pi(f_1) < \pi'(f_1)$.

By partial non-wastefulness, we have $e_{f_1} \leq \pi(f_1) < \pi'(f_1)$, and there exists $f_2 \in F$ such that $e_{f_2} \leq \pi'(f_2) = \pi(f_1)$. By $\pi'(f_2) < \pi'(f_1)$ and order preserving,

$g(f_2) < g(f_1)$. Then, $\pi(f_2) < \pi(f_1) = \pi'(f_2)$. By this way, we have

$$\cdots < \pi(f_k) < \cdots < \pi(f_2) < \pi(f_1).$$

However, since the numbers of slots and flights are finite, this is a contradiction.

□

Proof of Proposition 2. Let φ be the flight proposing deferred acceptance algorithm with the original schedule priority. Take any $x = (S, (e_A)_{A \in \mathscr{A}})$, $A \in \mathscr{A}$, and e'_A. Let $x' = (S, (e'_A, e_{-A}))$, $\varphi(x) = \pi$ and $\varphi(x') = \pi'$. Suppose that there exists $f \in F_A$ such that $e_f \leq \pi'(f) < \pi(f)$.

Since the deferred acceptance algorithm satisfies group strategy-proofness. There must exists $f' \in F_A$ such that $e_{f'} \leq \pi(f') < \pi'(f')$ or $\pi'(f') < e_{f'} \leq \pi(f')$. Then, we consider the case that there exists only the flights $f' \in F_A$ such that $e_{f'} \leq \pi(f') < \pi'(f')$ or $\pi'(f') < e_{f'} \leq \pi(f')$. We consider the case that there exists no flight $f' \in F_A$ such that $e_{f'} \leq \pi(f') < \pi'(f') = |\overline{S}| + 1$.

Suppose that for any flight $f' \in F_A$ with $\pi'(f') < e_{f'} \leq \pi(f')$, $\pi(f') = |\overline{S}| + 1$. Since $\pi(f) < \pi(f')$, $g(f) < g(f')$. Then, $\pi(f) < \pi(f')$. Thus, there exists no flight $f' \in F_A$ such that $\pi'(f') < e_{f'} \leq \pi(f')$ and $e_f \leq \pi(f') < \pi(f')$. This contradicts to $e_f \leq \pi'(f) < \pi(f)$. Therefore, there exist $f' \in F_{\mathscr{B}}$ such that $e_{f'} \leq \pi(f') < \pi'(f')$ or $\pi'(f') < e_{f'} \leq \pi(f') < |\overline{S}| + 1$. □

Proof of Proposition 3. Take any $y = (d, (e_A)_{A \in \mathscr{A}})$. Let ϕ be the generalized deferred acceptance algorithm, and $\phi(y) = \pi$.

We first show universal non-wastefulness. Note that ϕ satisfies feasibility, that is, for any $f \in F$, $e_f \leq \pi(f)$. Suppose that there exists exists no $s \in \overline{S}$ such that

$$\pi(f) \notin [s-d, s] \qquad \forall f \in F \tag{2.6}$$

$$e_{f'} \leq s < \pi(f') \qquad \exists f' \in F. \tag{2.7}$$

By (2.6), in some step, algorithm reached to slot s. By (2.7), $F_s \neq \emptyset$. Therefore, slot s must be assigned to some flight. This is a contradiction.

Next, we shall show the schedule satisfies order preserving. Suppose that there exists flights $f, f' \in F$ with $g(f) < g(f')$ such that

$$\pi(f) > \pi(f') \geq e_f$$

Then, at some step, algorithm reached to slot $\pi(f')$. Since $\pi(f) > \pi(f') \geq e_f$, $f \in F_{\pi(f')}$. By $g(f) < g(f')$, slot $\pi(f')$ must not be assigned to f'. This is a contradiction.

By the same way of Proposition 1, we can prove the uniqueness. $\square$

Proof of Proposition 4. Let ϕ be generalized deferred acceptance algorithm. Take any $y = (d, (e_A)_{A \in \mathscr{A}})$, $A \in \mathscr{A}$, and e'_A. Let $y' = (S, (e'_A, e_{-A}))$, $\phi(y) = \pi$ and $\phi(y') = \pi'$. Then, by Proposition 3, π, π' satisfies universal non-wastefulness and order preserving for reported earliest feasible arrival times. Suppose that there exists $f \in F_A$ such that $e_f \leq \pi'(f) < \pi(f)$.

By feasibility of π', for any $f' \neq f$, $\pi'(f') \notin [\pi'(f) - d, \pi'(f)]$. However, by universal non-wastefulness, there exists $f_1 \in F$ such that $\pi(f_1) \in [\pi'(f) - d, \pi'(f)]$. We consider two case: $\pi'(f_1) < \pi(f_1)$ and $\pi(f_1) < \pi'(f_1)$.

Case 1: $\boldsymbol{\pi'(f_1) < \pi(f_1)}$.

If $f_1 \notin F_A$, the earliest feasible arrival time is identical in π, π'. Then, by feasibility of π', for any $f' \neq f_1$, $\pi'(f') \notin [\pi'(f_1) - d, \pi'(f_1)]$. By universal non-wastefulness and order preserving, there exists $f_2 \in F$ such that $\pi(f_2) \in [\pi'(f) - d, \pi'(f)]$. Then, we also consider two case: $\pi'(f_2) < \pi(f_2)$ and $\pi(f_2) < \pi'(f_2)$.

Then, $f_1 \in F_A$. Suppose that $\pi'(f_1) \geq e_A$. Then, by universal non-wastefulness and order preserving, for any $f' \neq f_1$, $\pi'(f') \notin [\pi'(f_1) - d, \pi'(f_1)]$ and there exists $f_2 \in F$ such that $\pi(f_2) \in [\pi'(f) - d, \pi'(f)]$. Therefore, $\pi'(f_1) < e_A$.

Case 2: $\boldsymbol{\pi(f_1) < \pi'(f_1)}$.

If $f_1 \notin F_A$, the earliest feasible arrival time is identical in π, π'. Then, by universal non-wastefulness and order preserving, there exists $f_2 \in F$, $\pi'(f_2) \in [\pi(f_1) - d, \pi(f_2)]$ and $\pi'(f') < e_{f_2}$. Since π' satisfies feasibility $e'_{f_2} \leq \pi'(f_2) < e_{f_2}$. Since earliest feasible arrival time of f_2 is different between e_{f_2} and e'_{f_2}, $f_2 \in F_A$.

Otherwise, $f_1 \in F_A$ and $e_{f_1} \leq \pi(f_1) < \pi'(f_1)$. $\square$

Chapter 3

A Characterization of the Borda Rule

3.1 Introdcution

In his classical work, Jean-Charles de Borda (1784) pointed out that the plurality rule may select an alternative which is defeated by any other alternatives in pairwise comparison. We call such an alternative a *pairwise-majority loser.* He then introduced a new social choice rule, called *the Borda rule*, to avoid selecting a pairwise-majority-loser. We say that a social choice rule satisfies *Borda's criterion* if it never selects a pairwise-majority-loser.

Young (1974, 1975) developed the model of social choice rules with variable set of voters.[*15] In the model, He characterized the Borda rule (a social choice rule based on the Borda score) and the scoring rules using variants of the Borda score by a set of natural conditions.[*16] However, Young's characterization did not use Borda's criterion.

The purpose of this study is to characterize the Borda rule by Borda's criterion and other standard axioms.[*17] In fact, we show that the Borda rule is the only social choice rule that satisfies anonymity, neutrality, consistency, continuity and

[*15]Smith (1973) studied social *ranking* rules in a similar model.

[*16]See Hansson and Sahlquist (1976) for a survey.

[*17]Fishburn and Gehrlein (1976) and Okamoto and Sakai (2013) consider the same topic. Their results show that the Borda rule is the only *scoring rule* that satisfies Borda's criterion. That is, they focus on scoring rules. We study *all* social choice rules.

Borda's criterion.

This chapter is organized as follows. In Section 3.2 we introduce definitions. In Section 3.3, we characterize the Borda rule. In Section 3.4, we show the tightness of axioms.

3.2 Definitions

Let $X = \{x_1, x_2, \ldots, x_m\}$ be the finite set of alternatives with $m \geq 3$. Let $\mathbb{N}$ be set of of the potential voters and $\mathcal{N}$ the collection of all nonempty finite subsets of $\mathbb{N}$. A *preference relation* is a linear ordering $\succsim_i$ on X, where the symmetric, asymmetric parts of $\succsim_i$ are denoted by $\sim_i, \succ_i$, respectively.*18 Let $\mathcal{P}$ be the set of preference relations, and $\bigcup_{N\in\mathcal{N}} \mathcal{P}^N$ the set of all preference profiles. For any $N \in \mathcal{N}$, any $\succsim^N \in \mathcal{P}^N$, and any $x, y \in X$, let

$$\pi_{xy}(\succsim^N) = |\{i \in N | x \succsim_i y\}|.$$

A *social choice rule* is a function $f : \bigcup_{N\in\mathcal{N}} \mathcal{P}^N \to 2^X \setminus \{\emptyset\}$ that maps each preference profile to a nonempty set of alternatives. *The Borda rule* is the social choice rule f_B such that for each $x \in X$, each $N \in \mathcal{N}$ and each $\succsim^N \in \mathcal{P}^N$,

$$x \in f_B(\succsim^N) \iff \sum_{y\neq x} \big(\pi_{xy}(\succsim^N) - \pi_{yx}(\succsim^N)\big) \geq \sum_{y\neq z} \big(\pi_{zy}(\succsim^N) - \pi_{yz}(\succsim^N)\big) \quad \forall z \in X.$$

We introduce axioms of social choice rules. The first two axioms an fairly standard in the literature.

*18 A linear order is a binary relation which is *complete, transitive* and *anti-symmetry.*

Anonymity requires that a social choice function treat all voters equally.

Anonymity: For any $N, N' \in \mathcal{N}$, any $\succsim^N \in \mathcal{P}^N$, and any $\succsim^{N'} \in \mathcal{P}^N$, if for each preference relation, the number of voters who have the preference relation in $\succsim^N$ is same as that in $\succsim^{N'}$, then $f(\succsim^N) = f(\succsim^{N'})$.

Let Σ be the set of permutations on X. For each $\sigma \in \Sigma$ and $\succsim \in \mathcal{P}$, let $\hat{\sigma}(\succsim)$ be such that for all $x, y \in X$

$$x \succsim y \implies \sigma(x)\ \hat{\sigma}(\succsim)\ \sigma(y).$$

Neutrality requires that a social choice rule is independent of labeling of alternatives.

Neutrality: For any $N \in \mathcal{N}$, any $\succsim^N \in \mathcal{P}^N$, and any $\sigma \in \Sigma$, $f(\hat{\sigma}(\succsim^N)) = \sigma(f(\succsim^N))$.

The next two axioms are due to Young(1974, 1975). Consider two disjoint subset of voters and two preference profiles. *Consistency* requires that if a social choice rule selects the same alternative for the two preference profile, the social choice rule selects the alternative for the union of preference profile.

Consistency: For any $N, N' \in \mathcal{N}$, $\succsim^N \in \mathcal{P}^N$ and $\succsim^{N'} \in \mathcal{P}^{N'}$, if $N \cap N' = \emptyset$ and $f(\succsim^N) \cap f(\succsim^{N'}) \neq \emptyset$, $f(\succsim^N, \succsim^{N'}) = f(\succsim^N) \cap f(\succsim^{N'})$.

Continuity requires that if an anonymous social choice rule selects a single alternative for a particular preference profile, then for any preference profile which contains sufficiently large numbers of replications of the particular preference profile, it selects the same alternative.

Continuity: Assume that f satisfies anonymity. For any $N, N' \in \mathcal{N}$, any $\succsim^N \in$

$\mathcal{P}^N$ and $\succsim^{N'} \in \mathcal{P}^{N'}$ if $f(\succsim^N)$ is a singleton, then there exists $\bar{n} \in \mathbb{N}$ such that for any $n \geq \bar{n}$,

$$f(\succsim^{N'}, n(\succsim^N)) = f(\succsim^N)$$

where

$$n(\succsim^N) = (\underbrace{\succsim^N, \ldots, \succsim^N}_{n \text{ times}}).$$

Borda's criterion requires that a social choice rule never select a pairwise-majority-loser. Borda's criterion is often called "Condorcet loser criterion" in the literature. However, this condition is essentially introduced by Borda in 1770. For the history of social choice in 18th century, see McLean and Hewitt (1994) and McLean and Orken (1995).

Borda's criterion: For any $N \in \mathcal{N}$ and $\succsim^N \in \mathcal{P}^N$, there exists no $x \in f(\succsim^N)$ such that

$$\pi_{xy}(\succsim^N) < \pi_{yx}(\succsim^N), \quad \text{for all } y \neq x.$$

3.3 Characterization

In this section, we characterize the Borda rule by the five axioms introduced so far.

Theorem 1. *A social choice rule satisfies anonymity, neutrality, consistency, continuity and Borda's criterion if and only if it is the Borda rule.*

First, we show that the Borda rule satisfies the five axioms.

Lemma 1. *The Borda rule satisfies anonymity, neutrality, consistency, continuity and Borda's criteiron.*

Proof. One can easily check that the Borda rule satisfies anonymity, neutrality, consistency and continuity. We here only show that the Borda rule satisfies Borda's criterion.

Consider any $N \in \mathcal{N}$ and $\succsim^N$. Let $x \in f_B(\succsim^N)$. Then, by the definition of the Borda rule, for any $x' \in X$,

$$\sum_{y \neq x} (\pi_{xy}(\succsim^N) - \pi_{yx}(\succsim^N)) \geq \sum_{y \neq x'} (\pi_{x'y}(\succsim^N) - \pi_{yx'}(\succsim^N)) \tag{3.8}$$

Since

$$\sum_{x' \in X} \sum_{y \neq x'} (\pi_{x'y}(\succsim^N) - \pi_{yx'}(\succsim^N)) = 0,$$

by (3.8),

$$\sum_{y \neq x} (\pi_{xy}(\succsim^N) - \pi_{yx}(\succsim^N)) \geq 0. \tag{3.9}$$

Suppose, by contradiction, that for each $y \neq x$, $\pi_{xy}(\succsim^N) < \pi_{yx}(\succsim^N)$. Therefore,

$$\sum_{y \neq x} (\pi_{xy}(\succsim^N) - \pi_{yx}(\succsim^N)) < 0.$$

This contradicts to (3.9). Therefore, f_B satisfies Borda's criterion. $\square$

To show the only if part, we need some lemmas.

Lemma 2. *Suppose that a social choice rule f satisfies anonymity, neutrality consistency, and Borda's criterion. For any $N \in \mathcal{N}$, any $\succsim^N \in \mathcal{P}^N$, and any*

$x \in f(\succsim^N)$,

$$\sum_{y \neq x} (\pi_{xy}(\succsim^N) - \pi_{yx}(\succsim^N)) \geq 0.$$

Proof. Consider any $N \in \mathcal{N}$, any $\succsim^N \in \mathcal{P}^N$, and any $x \in f(\succsim^N)$. Suppose, by contradiction, that

$$\sum_{y \neq x} (\pi_{xy}(\succsim^N) - \pi_{yx}(\succsim^N)) < 0. \tag{3.10}$$

Let Σ_x be the set of permutations such that for any $\sigma \in \Sigma_x$, $\sigma(x) = x$. By neutrality, since for each $\sigma \in \Sigma_x$, $f(\hat{\sigma}(\succsim^N)) = \sigma(f(\succsim^N))$, $x \in f(\hat{\sigma}(\succsim^N))$. By anonymity, for each $N' \in \mathcal{N}$ with $|N'| = |N|$ if $\succsim^{N'}$ consists of the same preference relations as $\succsim^N$, then $f(\succsim^N) = f(\succsim^{N'})$. We note that $|\Sigma_x| = (m-1)!$. Let $\succsim^{N^x}$ the preference profile by an arbitrary voters N_x with $N_x = (m-1)!|N|$ such that

$$\succsim^{N_x} = \big(\sigma(\succsim^N)\big)_{\sigma \in \Sigma_x} = (\sigma_1(\succsim^N), \sigma_2(\succsim^N), \ldots, \sigma_{(m-1)!}(\succsim^N)).$$

By consistency, $x \in f(\succsim^{N_x})$. By construction of $\succsim^{N_x}$,

$$(m-1)! \sum_{y \neq x} (\pi_{xy}(\succsim^N) - \pi_{yx}(\succsim^N)) = \sum_{y \neq x} (\pi_{xy}(\succsim^{N_x}) - \pi_{yx}(\succsim^{N_x})), \tag{3.11}$$

and for each $y, y' \in X \setminus \{x\}$,

$$\pi_{xy}(\succsim^{N_x}) = \pi_{xy'}(\succsim^{N_x}).$$

Therefore for an arbitrary $y \neq x$,

$$\sum_{y \neq x} (\pi_{xy}(\succsim^{N_x}) - \pi_{yx}(\succsim^{N_x})) = (m-1)(\pi_{xy}(\succsim^{N_x}) - \pi_{yx}(\succsim^{N_x})).$$

By (3.10) and (3.11), $(m-1)(\pi_{xy}(\succsim^{N_x}) - \pi_{yx}(\succsim^{N_x})) < 0$. Hence, $\pi_{xy}(\succsim^{N_x}) < \pi_{yx}(\succsim^{N_x})$ for all $y \neq x$. However, $x \in f(\succsim^{N_x})$, a contradiction to Borda's criterion.

□

Lemma 3. *Suppose that a social choice rule f satisfies anonymity, neutrality, continuity, consistency, and Borda's criterion. Consider any $N \in \mathcal{N}$ and any $\succsim^N \in \mathcal{P}^N$. If for any $x \in X$,*

$$\sum_{y \neq x} (\pi_{xy}(\succsim^{N}) - \pi_{yx}(\succsim^{N})) = 0$$

then $f(\succsim^N) = X$. Furthermore, for any $x \in X$, if

$$\sum_{y \neq x} (\pi_{xy}(\succsim^{N}) - \pi_{yx}(\succsim^{N})) = 0$$

$$x \in f(\succsim^N) \iff f(\succsim^N) = X.$$

Proof. Consider any $N \in \mathcal{N}$ and any $\succsim^N \in \mathcal{P}^N$ such that for any $x \in X$,

$$\sum_{z \neq x} (\pi_{xz}(\succsim^{N}) - \pi_{zx}(\succsim^{N})) = 0 \tag{3.12}$$

Suppose that $f(\succsim^N) \neq X$.

Take any $y \in f(\succsim^N)$. Similarly to the proof of Lemma 2, we construct a preference profile

$$\succsim^{N_y} = \left(\sigma(\succsim^N)\right)_{\sigma \in \Sigma_y} = (\sigma_1(\succsim^N), \sigma_2(\succsim^N), \ldots, \sigma_{(m-1)!}(\succsim^N)).$$

Because $f(\succsim^N) \neq X$, for each $x \neq y$, there exists $\sigma \in \Sigma_y$ such that

$$x \notin f(\sigma(\succsim^N)).$$

Therefore, by consistency,

$$f(\succsim^{N_y}) = \{y\}.$$

By construction of $\succsim^{N_y}$, for all $x \in X$,

$$\begin{aligned}
\pi_{xy}(\succsim^{N_y}) &= (m-1)!\pi_{xy}(\succsim^N), \\
\pi_{yx}(\succsim^{N_y}) &= (m-1)!\pi_{yx}(\succsim^N)
\end{aligned}$$

and for each $x, x' \in X \setminus \{y\}$,

$$\begin{aligned}
\pi_{xy}(\succsim^{N_y}) &= \pi_{x'y}(\succsim^{N_y}), \\
\pi_{yx}(\succsim^{N_y}) &= \pi_{yx'}(\succsim^{N_y}).
\end{aligned}$$

Therefore, by (3.12), for all $x \neq y$,

$$\pi_{xy}(\succsim^{N_y}) = \pi_{yx}(\succsim^{N_y}). \tag{3.13}$$

Let $\succsim_j$ be such that for all $x \in X$, $x \succsim_j y$. By continuity, there exists $n \in \mathbb{N}$

such that

$$f(\succsim_j, n(\succsim^{N_y})) = \{y\}.$$

However, by (3.13), for all $x \neq y$,

$$\pi_{xy}(\succsim_j, n(\succsim^{N_y})) > \pi_{yx}(\succsim_j, n(\succsim^{N_y}).$$

This contradicts to Borda's criterion. □

Lemma 4. *Suppose that a social choice rule f satisfies anonymity, neutrality, continuity, consistency, and Borda's criterion. Consider any $N, N' \in \mathcal{N}$, any $\succsim^N \in \mathcal{P}^N$ and any $\succsim^{N'} \in \mathcal{P}^{N'}$. If for any $x \in X$,*

$$\sum_{y \neq x}(\pi_{xy}(\succsim^N) - \pi_{yx}(\succsim^N)) = \sum_{y \neq x}(\pi_{xy}(\succsim^{N'}) - \pi_{yx}(\succsim^{N'})),$$

then $f(\succsim^N) = f(\succsim^{N'})$.

Proof. Consider any $N, N' \in \mathcal{N}$, any $\succsim^N \in \mathcal{P}^N$ and any $\succsim^{N'} \in \mathcal{P}^{N'}$ such that for any $x \in X$,

$$\sum_{y \neq x}(\pi_{xy}(\succsim^N) - \pi_{yx}(\succsim^N)) = \sum_{y \neq x}(\pi_{xy}(\succsim^{N'}) - \pi_{yx}(\succsim^{N'})). \tag{3.14}$$

Let $\bar{N} \in \mathcal{N}$ be such that $|N| = |\bar{N}|$, $\bar{N} \cap N = \emptyset$ and $\bar{N} \cap N' = \emptyset$. Let $\bar{\succsim}^{\bar{N}}$ be the inverting preference profile of $\succsim^N$; that is, for each $i \in N$, there uniquely exists $j \in \bar{N}$ such that for any $x, x' \in X$

$$x \succsim_i x' \iff x' \bar{\succsim}_j x.$$

By construction of $\succsim^{\bar{N}}$, for each $x, x' \in X$

$$\pi_{xx'}(\succsim^{N}, \succsim^{\bar{N}}) = \pi_{x'x}(\succsim^{N}, \succsim^{\bar{N}}).$$

Hence, for all $x, x' \in X$,

$$\sum_{y \neq x}(\pi_{xy}(\succsim^{N}, \succsim^{\bar{N}}) - \pi_{yx}(\succsim^{N}, \succsim^{\bar{N}}) = \sum_{y \neq x'}(\pi_{x'y}(\succsim^{N}, \succsim^{\bar{N}}) - \pi_{yx}(\succsim^{N}, \succsim^{\bar{N}})).$$

By (3.14), we also obtain

$$\sum_{y \neq x}(\pi_{xy}(\succsim^{N'}, \succsim^{\bar{N}}) - \pi_{yx}(\succsim^{N'}, \succsim^{\bar{N}}) = \sum_{y \neq x'}(\pi_{x'y}(\succsim^{N'}, \succsim^{\bar{N}}) - \pi_{yx}(\succsim^{N'}, \succsim^{\bar{N}})).$$

Then, by Lemma 3,

$$f(\succsim^{N}, \succsim^{\bar{N}}) = f(\succsim^{N'}, \succsim^{\bar{N}}) = X.$$

Hence, by consistency,

$$\begin{aligned}
f(\succsim^{N}) =& f(\succsim^{N}) \cap X \\
=& f(\succsim^{N}) \cap f(\succsim^{N'}, \succsim^{\bar{N}}) \\
=& f(\succsim^{N}, \succsim^{N'}, \succsim^{\bar{N}}) \\
=& f(\succsim^{N}, \succsim^{\bar{N}}) \cap f(\succsim^{N'}) \\
=& X \cap f(\succsim^{N'}) \\
=& f(\succsim^{N'}).
\end{aligned}$$

$\square$

Lemma 5. *Suppose that a social choice rule f satisfies anonymity, neutrality, continuity, consistency, and Borda's criterion. Consider any $N \in \mathcal{N}$ and any $\succsim^N \in \mathcal{P}^N$. For any $x, y \in X$, if*

$$\sum_{z \neq x} (\pi_{xz}(\succsim^N) - \pi_{zx}(\succsim^N)) > \sum_{z \neq y} (\pi_{yz}(\succsim^{N'}) - \pi_{zy}(\succsim^{N'})),$$

then $y \notin f(\succsim^N)$.

Proof. Consider any $N \in \mathcal{N}$ and any $\succsim^N \in \mathcal{P}^N$ such that

$$\sum_{z \neq x} (\pi_{xz}(\succsim^N) - \pi_{zx}(\succsim^N)) > \sum_{z \neq y} (\pi_{yz}(\succsim^{N'}) - \pi_{zy}(\succsim^{N'})).$$

Suppose, by contradiction, $y \in f(\succsim^N)$. Note that the Borda score is even for all alternatives. Let

$$\begin{aligned}
2k &= \sum_{z \neq x} (\pi_{xz}(\succsim^N) - \pi_{zx}(\succsim^N)) - \sum_{z \neq y} (\pi_{yz}(\succsim^{N'}) - \pi_{zy}(\succsim^{N'})), \\
2\ell &= \sum_{z \neq y} (\pi_{yz}(\succsim^{N'}) - \pi_{zy}(\succsim^{N'})).
\end{aligned}$$

Assume that the number of alternatives m is odd.

Define two preference relations $\succsim_1$ and $\succsim_2$ such that

$$\begin{aligned}
|\{z \in X : z \succsim_1 x\}| &= \frac{m+1}{2}, \\
|\{z \in X : z \succsim_1 y\}| &= \frac{m+3}{2}, \\
|\{z \in X : z \succsim_2 y\}| &= \frac{m+3}{2}, \\
|\{z \in X : z \succsim_2 x\}| &= \frac{m+1}{2}.
\end{aligned}$$

Take any $N' \in \mathcal{N}$ with $|N'| = k + \ell$. Let $\hat{\succsim}^{N'}$ such that

$$\hat{\succsim}^{N'} = \big(k(\succsim_1), \ell(\succsim_1, \succsim_2)\big) = (\succsim^K, \succsim^L).$$

Then, by construction of $\hat{\succsim}^{N'}$,

$$\sum_{z \neq x} (\pi_{xz}(\succsim^N) - \pi_{zx}(\succsim^N)) = \sum_{z \neq x} (\pi_{xz}(\hat{\succsim}^{N'}) - \pi_{zx}(\hat{\succsim}^{N'})),$$
$$\sum_{z \neq y} (\pi_{yz}(\succsim^N) - \pi_{zy}(\succsim^N)) = \sum_{z \neq y} (\pi_{xy}(\hat{\succsim}^{N'}) - \pi_{zy}(\hat{\succsim}^{N'})).$$

Let Σ_{xy} be the set of all permutations such that for any $\sigma \in \Sigma_{xy}$, $\sigma(x) = x$ and $\sigma(y) = y$. By a similar way to Lemma 2, define $\succsim^{N_{xy}}$ and $\hat{\succsim}^{N'_{xy}} = (\succsim^{K_{xy}}, \succsim^{L_{xy}})$. Therefore, for all $w \in X$,

$$\sum_{z \neq w} (\pi_{wz}(\succsim^{N_{xy}}) - \pi_{zw}(\succsim^{N_{xy}})) = \sum_{z \neq w} (\pi_{xy}(\hat{\succsim}^{N'_{xy}}) - \pi_{zw}(\hat{\succsim}^{N'_{xy}})).$$

Hence, by lemma 4, $f(\succsim^{N_x y}) = f(\hat{\succsim}^{N'_{xy}})$. By lemmas 2 and 3, $f(\succsim^{K'_{xy}}) = \{x\}$ and $f(\succsim^{L'_{xy}}) = \{x, y\}$. Thus, $f(\hat{\succsim}^{N'_{xy}})) = f(\succsim^{K'_{xy}}, \succsim^{L'_{xy}}) = \{x\}$. This contradicts to $y \in f(\succsim^{N_{xy}})$. □

Finally, we show Theorem 1.

Proof of Theorem 1. Consider any $N \in \mathcal{N}$ and any $\succsim^N \in \mathcal{P}^N$. Consider any $x \in X$ such that for any $y \in X$,

$$\sum_{z \neq x} (\pi_{xz}(\succsim^N) - \pi_{zx}(\succsim^N)) \geq \sum_{z \neq y} (\pi_{yz}(\succsim^N) - \pi_{zy}(\succsim^N)).$$

We use Lemma 5. If for all $y \in X \setminus \{x\}$,

$$\sum_{z \neq x} (\pi_{xz}(\succsim^N) - \pi_{zx}(\succsim^N)) > \sum_{z \neq y} (\pi_{yz}(\succsim^N) - \pi_{zy}(\succsim^N)),$$

then $\{x\} = f(\succsim^N)$. Otherwise, there exists $y \in f(\succsim^N)$ such that

$$\sum_{z \neq x} (\pi_{xz}(\succsim^N) - \pi_{zx}(\succsim^N)) = \sum_{z \neq y} (\pi_{yz}(\succsim^N) - \pi_{zy}(\succsim^N)).$$

Suppose that $x \neq y$. In this case, let $\hat{\succsim}^{N'} = \ell(\succsim_2)$. Then, by a similar way to Lemma 5, we can show $x \in f(\succsim^N)$. $\square$

3.4 Tightness of the characterization result

Finally, we discuss tightness of the characterization result. That is, if we drop one of the axioms, there are other social choice rules that satisfy the rest of the axioms.

- Let f_1 be the social choice rule such that for any $N \in \mathcal{N}$ and any $\succsim^N \in \mathcal{P}^N$,

$$f_1(\succsim^N) = \{x \in X_1 | x \succsim_i y \text{ for all } y \in X_1\},$$

  where $i \equiv \min_{j \in N} j$ and $f_B(\succsim^N) \equiv X_1$. One can easily check that, f_1 satisfies neutrality and consistency, but it violates anonymity. By a similar way to Lemma 1, we obtain that f_1 satisfies Borda's criterion, because for any preference profile, f_1 winner is one of the Borda winners.

- Let f_2 be the social choice rule such that for any $N \in \mathcal{N}$ and any $\succsim^N \in \mathcal{P}^N$,

$$f_2(\succsim^N) = \begin{cases} f_B(\succsim^N) \setminus \arg\min_{x_k \in X} k & \text{if } f_B(\succsim^N) = X, \\ f_B(\succsim^N) & \text{otherwise} \end{cases}$$

  Obviously, f_2 violates neutrality. One can easily check that f_2 satisfies anonymity, consistency and continuity. By a similar way to Lemma 1, we obtain that f_1 satisfies Borda's criterion.

- Let f_3 be the social choice function such that for any $N \in \mathcal{N}$ and any $\succsim^N \in \mathcal{P}^N$,

$$f_3(\succsim^N) = \arg\max_{x \in X_3} |\{i \in N | x \succsim_i y \text{ for all } y \in X_3\}|.$$

  where $X_3 = f_B(\succsim^N)$. One can easily check that f_3 satisfies anonymity, neutrality and consistency. By a similar way to Lemma 1, we obtain that f_3 satisfies Borda's criterion. For example, let

$$\succsim_1 : x \ y \ z \ w$$

$$\succsim_2 : w \ y \ x \ z$$

$$\succsim_2 : y \ x \ z \ w \ .$$

  Then, $f(\succsim_1, \succsim_2) = \{x\}$. However, for all $n \in \mathbb{N}$, $f(\succsim_3, n(\succsim_1, \succsim_2)) = \{y\}$. Therefore, f_3 violates continuity.

- Let f_4 be *the Kemeny-Young rule.*[*19] One can easily check that the Kemeny-Young rule satisfies anonymity, neutrality and continuity, but violates consistency. Sarri and Merlin (2000) show that the Kemmeny Young rule satisfies Borda's criterion.

- Let f_5 be *the plurality rule*. Obviously, f_5 satisfies anonymity, neutrality, continuity, and consistency, but violate Bora's criterion.

The satisfaction and the violation of axioms by these functions are summarized by Table 3. It shows the independence of the axioms in our Theorem 1.

	Anonymity	Neutrality	Anonymity+Continuity	Consistency	Borda's criteiron
f_1	$-$	$+$	$-$	$+$	$+$
f_2	$+$	$-$	$+$	$+$	$+$
f_3	$+$	$+$	$-$	$+$	$+$
f_4	$+$	$+$	$+$	$-$	$+$
f_5	$+$	$+$	$+$	$+$	$-$

Table 3: Tightness of the characterization result

[*19]For example, see, Young and Levenglick (1978) for the definition of the Kemeny-Young rule.

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