Nonelementary Computational Complexity of Substructural Logic

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Contents

Chapter 1

Introduction

> After the theory explaining which problems can and cannot be solved by computer was well developed, it was natural to ask about the relative computational difficulty of computable functions. This is the subject matter of computational complexity.
>
> Stephen A. Cook
> *An overview of computational complexity* [28], 1983

In this thesis, we explore the nonelementary computational complexity of various substructural logics. This chapter aims to: (i) give an overview of general background on substructural logic and computational complexity theory, (ii) describe the central motivation behind the thesis, and (iii) make our contributions clear.

Since substructural logics are closely related to sequent calculus, we start by discussing the sequent calculus LJ of Gentzen [46, 47] (Section 1.1). We next carefully describe how to obtain *full Lambek calculus* [40, 39], often viewed as the most rudimentary substructural logic, from Gentzen's LJ (Section 1.2). We furthermore discuss: (i) the significance of investigating the (un)decidability and complexity of substructural logics (Section 1.3) and (ii) some caveats to keep in mind when tackling such problems (Section 1.4). In Section 1.5, we summarize the complexity status of various substructural logics and discuss the effect of structural rules on the complexity of a logical system. In Section 1.6, we outline our contributions. In Section 1.7, we present a brief overview of the organization of the thesis.

1.1 Sequent calculus and structural rules

Sequent calculus, introduced by Gerhard Gentzen [46, 47], is an epoch-making formalism for logical systems. Even though some typical substructural logics (e.g., $\mathsf{FL_{ew}}$) can be well-captured via other formalisms such as natural deduction (see, e.g., [126]), substructural logicians place great importance on sequent calculus. This, in a nutshell, is because the notion of *structural rules* [46, 47] is unique to sequent calculus. Considering, for instance, the success of *algebraic proof theory* [22, 23, 24, 25] in recent years, one may understand that the current development of substructural logic would not have been possible without the discovery of sequent calculus.[1]

For our purposes here, we first recall the sequent calculus LJ [46, 47] for intuitionistic propositional logic; the reader unfamiliar with sequent calculus may refer to Ono's introductory guide [97]. Roughly speaking, the system LJ consists of (i) the initial sequent, (ii) the rule of cut, and (iii) several inference rules. Moreover, we classify the inference rules of LJ into two kinds of groups. The inference rules from the first group are called *logical inference rules*, which stipulate how logical connectives behave in LJ. For instance, the role of the binary logical connective $\to$, called *logical implication*, is operationally determined by the following logical inference rules:

$$\frac{\Gamma \Rightarrow s \qquad \Sigma, t, \Delta \Rightarrow \sigma}{\Sigma, \Gamma, s \to t, \Delta \Rightarrow \sigma}\ (\to\Rightarrow) \qquad\qquad \frac{s, \Gamma \Rightarrow t}{\Gamma \Rightarrow s \to t}\ (\Rightarrow\to)$$

where Γ, Σ stand for a finite sequence of terms[2], s, t stand for terms, and σ is a term or empty.

On the other hand, of interest to us here are the second group of inference rules, i.e., *structural rules*. The sequent calculus LJ is equipped with the structural rules of exchange, contraction, left-weakening and right-weakening, denoted by (e), (c), (i), and (o), respectively:[3]

[1] Algebraic proof theory, which Ciabattoni, Galatos, and Terui developed in a series of papers [22, 23, 24, 25], established the solid superiority of sequent calculus in the framework of substructural logic. They demonstrated that (hyper)sequent calculus serves as a powerful tool for a uniform understanding of a wide range of logical axioms and (hyper)structural rules. To date, this advantage does not seem to be found in other formalisms, such as natural deduction and Hilbert-style calculus.

[2] The mathematical objects called "terms" in the thesis are usually referred to as "formulas" in the context of proof theory.

[3] We can also take the rule of associativity into consideration; however, logics without this rule are beyond the scope of this thesis. All the logics falling within our interests are implicitly associative.

$$\frac{\Gamma, s, t, \Delta \Rightarrow \sigma}{\Gamma, t, s, \Delta \Rightarrow \sigma}\ (\mathsf{e}) \qquad \frac{\Gamma, s, s, \Delta \Rightarrow \sigma}{\Gamma, s, \Delta \Rightarrow \sigma}\ (\mathsf{c})$$

$$\frac{\Gamma, \Delta \Rightarrow \sigma}{\Gamma, s, \Delta \Rightarrow \sigma}\ (\mathsf{i}) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow s}\ (\mathsf{o})$$

In contrast to the logical inference rules, the above rules are not directly related to any logical connectives. Instead, these rules stipulate the "structural" properties of sequents. Since terms occurring on the left-hand side of a sequent are intuitively considered to be assumptions in logical reasoning, the role of each of the structural rules may be understood as follows.

- The exchange rule says that we can use assumptions in any order we desire.
- The left-weakening rule says that we can freely add assumptions that may not be used.
- The contraction rule says that we can make use of the same assumptions more than once.

The right-weakening rule corresponds to the so-called *law of explosion*; that is to say, any term can be deduced from the constant 0.

1.2 Embodiment of substructural logics

From intuitionistic logic to full Lambek calculus

We now remove all the structural rules from LJ and tentatively write LJ^- for the resulting logic. There are some points to keep in mind. In LJ, each comma (i.e., ",") appearing on the left-hand side of a sequent can be seen as an implicit form of logical conjunction (i.e., "$\wedge$"). Namely, the following holds for arbitrary terms s, t, and u:

$s, t \Rightarrow u$ is provable in LJ if and only if $s \wedge t \Rightarrow u$ is provable in LJ.

As pointed out in, e.g., [39, Section 2.1.2], the above equivalence fails to hold in LJ^-; the provability of $s, t \Rightarrow u$ is not equivalent to that of $s \wedge t \Rightarrow u$ in LJ^-. In order to interpret commas properly, we require a new binary logical connective "$\circ$," called *fusion* (also known as *product*). It is characterized by the following inference rules:

$$\frac{\Gamma, s, t, \Delta \Rightarrow \sigma}{\Gamma, s \circ t, \Delta \Rightarrow \sigma}\ (\circ \Rightarrow) \qquad \frac{\Gamma \Rightarrow s \qquad \Delta \Rightarrow t}{\Gamma, \Delta \Rightarrow s \circ t}\ (\Rightarrow \circ)$$

By virtue of these rules, we can safely replace any comma that appears in the left-hand side of a sequent with the fusion connective, i.e., we have:

$$s, t \Rightarrow u \text{ is provable in } \mathsf{LJ}^- \text{ if and only if } s \circ t \Rightarrow u \text{ is provable in } \mathsf{LJ}^-.$$

The logical connective of implication also needs to be reconsidered carefully. The sequent calculus LJ enjoys the *law of residuation* (see, e.g., [39, Section 1.1.4]), which is a fundamental property of the intuitionistic implication:

$$\begin{aligned} s, t \Rightarrow u \text{ is provable in } \mathsf{LJ} \quad &\text{iff} \quad s \Rightarrow t \to u \text{ is provable in } \mathsf{LJ} \\ &\text{iff} \quad t \Rightarrow s \to u \text{ is provable in } \mathsf{LJ} \end{aligned}$$

The absence of exchange however destroys these equivalences. In order to properly maintain the law of residuation in LJ^-, substructural logicians usually divide the intuitionistic implication $\to$ into two types of implication, called *right division* (/) and *left division* (\). These two connectives are operationally characterized as follows:

$$\frac{\Gamma \Rightarrow s \qquad \Sigma, t, \Delta \Rightarrow \sigma}{\Sigma, \Gamma, s \backslash t, \Delta \Rightarrow \sigma}\ (\backslash \Rightarrow) \qquad\qquad \frac{s, \Gamma \Rightarrow t}{\Gamma \Rightarrow s \backslash t}\ (\Rightarrow \backslash)$$

$$\frac{\Gamma \Rightarrow s \qquad \Sigma, t, \Delta \Rightarrow \sigma}{\Sigma, t \,/\, s, \Gamma, \Delta \Rightarrow \sigma}\ (/ \Rightarrow) \qquad\qquad \frac{\Gamma, s \Rightarrow t}{\Gamma \Rightarrow t \,/\, s}\ (\Rightarrow /)$$

As a result, the residuation property is restored in the following sense:

$$\begin{aligned} s, t \Rightarrow u \text{ is provable in } \mathsf{LJ}^- \quad &\text{iff} \quad s \Rightarrow u \,/\, t \text{ is provable in } \mathsf{LJ}^- \\ &\text{iff} \quad t \Rightarrow s \backslash u \text{ is provable in } \mathsf{LJ}^- \end{aligned}$$

When we add the exchange rule to LJ^-, $s \backslash t$ and $t \,/\, s$ are *provably equivalent*; that is to say, $s \backslash t \Rightarrow t \,/\, s$ and $t \,/\, s \Rightarrow s \backslash t$ are both provable in LJ^- with (e).

Furthermore, we usually include the two nullary connective symbols (i.e., constant symbols) 1 and 0 in the language of LJ^-.[4] These symbols are captured by the following rules:

[4] One can also naturally define the constants $\top$ and $\bot$, which behave as the units for the binary connectives $\wedge$ and $\vee$, respectively. Since these are often omitted in many papers concerning substructural logics, we follow this practice and exclude them from our framework; however, all the main results in this thesis should hold even when these constants are considered.

$$\frac{\Gamma, \Delta \Rightarrow \sigma}{\Gamma, 1, \Delta \Rightarrow \sigma}\ (1 \Rightarrow) \qquad \frac{}{\Rightarrow 1}\ (\Rightarrow 1)$$

$$\frac{}{0 \Rightarrow}\ (0 \Rightarrow) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (\Rightarrow 0)$$

It is clear that the terms $s \circ 1$, $1 \circ s$, and s are provably equivalent in LJ^- for every term s; that is, the constant 1 plays the role of the neutral element for fusion. In contrast to this algebraic property of 1, the constant 0 seems to have no noteworthy properties. In fact, the rules of $(0 \Rightarrow)$ and $(\Rightarrow 0)$ merely guarantee that the empty succedent of a sequent is always interpreted as 0; i.e., $\Gamma \Rightarrow 0$ is provable in LJ^- if and only if $\Gamma \Rightarrow$ is provable in LJ^-. Nevertheless, we usually require this constant in order to define negation operations (i.e., ${\sim}s = s \backslash 0$ and $-s = 0 / s$). This seems to be the main reason why the seemingly meaningless constant 0 is often employed in the literature.[5]

Following the usual practice in substructural logic, we henceforth refer to LJ^- as *full Lambek calculus* (FL, for short) [40, 39]; see the next chapter for the precise definition of FL. The term "full Lambek calculus" comes from the fact that the system is a natural extension of the logical calculus introduced by Lambek [75] in 1958. Although his original motivation was to provide a logical account for natural language syntax, FL is nowadays regarded, independently of such linguistic motivation, as one of the most rudimentary logical systems in substructural logic.[6]

Substructural logics

Substructural logics [101, 40, 39] are sometimes described as the logics obtained from a standard logical system (typically, classical logic) by restricting some of structural rules; see, e.g., [37, p. 6]. However, it is often not clear what the words "restricting" and "structural rules" exactly mean.[7] As we will describe in detail in Chapter 2, we thus employ the following clear-cut

[5] Some prominent substructural logics (e.g., the exponential free fragment of linear logic [48, 49], Łukasiewicz logic [55], and relevance logic R [2]) are however defined by imposing some additional properties on the constant 0.

[6] *Full nonassociative Lambek calculus* [41], which is the system obtained from FL by removing the rule of associativity, is another candidate for the most basic substructural logic.

[7] Of course, one can plausibly define the notion of structural rules; see [22, 24, 25]. For the argument below, it however suffices to understand that structural rules are inference rules that are involved with no logical connectives.

definition:

$$\text{Substructural logics} \quad = \quad \text{extensions of } \mathsf{FL} \text{ by axiom schemes.}$$

This definition allows us to get from a single perspective an overview of an extremely wide range of nonclassical logics, such as superintuitionistic logics (see, e.g., [39, Section 2.3.7]) and many-valued logics (see, e.g., [39, Section 2.3.5]). In practice, this definition may well be the most standard one in the literature; see, e.g., [39, p. 88].

Since the name "substructural logic"[8] was coined in the early 1990s, a number of structural rules and substructural logics have been discovered and investigated. Nevertheless, substructural logicians give a special status to the structural rules (e), (c), (i), and (o). This may be primarily because these rules reflect fundamental—yet sometimes troublesome—properties underlying usual (i.e., classical-logical) reasoning.[9] Following the terminology of [39, p. 86], we therefore refer to the 14 extensions of FL by some (possibly none or all) of (e), (c), (i), and (o) as *basic substructural logics*.[10]

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[8] According to Ono [98], the term is suggested by Kosta Došen in a conference at Tübingen.

[9] For instance, the rules of weakening and contraction are often bothersome for logicians and philosophers.

The contraction rule has a catastrophic consequence in naïve set theory. Roughly speaking, as we have mentioned earlier, the contraction rule allows us to use the same assumptions more than once in a logical inference. Unfortunately, this feature of contraction, in conjunction with the principle of naïve comprehension, causes the inconsistency of naïve set theory. Grišin [52, 53] observed that the naïve comprehension axiom does not necessarily entail the falsum in the absence of contraction. With this in mind, he proposed a naïve set theory (without the principle of extensionality) that is based on a logic without contraction and showed that his set theory is consistent. See [119] for a nice exposition of Grišin's set theory.

On the other hand, some philosophers have traditionally considered the rule(s) of weakening to be problematic; see, e.g., [37, Section 3] for historical background. The left-weakening rule allows us to add unlimitedly new premises—even if they are completely unnecessary for deducing the conclusion! Consequently, the rightful relevance between the conclusion and the premises in inference is destroyed in classical-logical reasoning. *Relevance logics* were born from the motivation to circumvent the problem; see, e.g., [2, 87] for more on this topic.

[10] In some other papers, basic substructural logics are defined in a slightly different way; see, e.g., [98, 40].

1.3 Decision problems and computational complexity

Why study the computational complexity of a logic?[11]

The ultimate mission of substructural logic is to capture various proof-theoretic properties (e.g., cut elimination [22, 24, 25]) in a uniform (typically, algebraic) manner. Among such properties, making it clear whether a logical system has a decision procedure is a well-established topic in the literature on substructural logic. However, even if such a decision procedure were to be fortunately discovered, modern logicians would not be satisfied. Their next concern is to clarify the computational complexity of the problem under consideration, i.e., how much time and/or space resources are required to decide it. Indeed, computational complexity serves not only as a touchstone for testing the applicability of logics for practical use but also as a means to uncover quantitative information, often overlooked by the age-old dichotomy of decidability and undecidability. This is the reason why computational analysis is of importance to many logicians and especially to those who engage in *logic in computer science*.

Extremely intractable problems

A critical insight in the theory of computational complexity is that the amount of time and/or space resources is measured by a function of the size of the input. This insight leads to the idea of formulating various classes of decision problems using various functions with different growth rates (such as polynomials and exponentials). The most important class of problems is PTIME (P, for short), the class of problems solved by a deterministic Turing machine working in time polynomial in the input length. The notion of *feasible* (or *tractable*) *decision problems*, which is clearly of practical importance, is often identified with the class P (see, e.g., [28, p. 402] and [36, Section 2.2]). Nowadays, this *thesis* of tractable problems is widely accepted by many complexity theorists and serves as a convincing criterion that allows us to dichotomize decision problems into tractable and intractable ones.

In addition to P, there are hundreds of complexity classes; see the *Complexity Zoo* [1] for an exhaustive list of complexity classes. In some important complexity classes, such as NP (i.e., the class of problems decidable in nondeterministic polynomial time), one can define the concept of *com-*

[11] The reader who is familiar with the basics on computational complexity theory may skip this subsection and the next subsection.

plete problems under a specific family of resource-bounded reductions. For instance, NP-complete problems are intuitively understood as the "hardest" ones in NP under some suitable notion of (typically, polynomial-time) reductions. Since Cook's historic achievement [27], the discovery of the first NP-complete problem (i.e., satisfiability of Boolean formulas), a plethora of problems have been shown, one after another, to be complete for NP; see [42] for an extensive list of NP-complete problems. Nevertheless, nobody has successfully found a feasible algorithm for NP-complete problems. This empirical fact reinforces the belief that NP-complete problems are not feasibly solvable.

According to the *P=?NP polls* held by Gasarch [43, 44, 45], many complexity theorists believe that NP-complete problems are intractable. It would thus be fair to say that NP-complete problems are probably intractable. On the other hand, some decidable problems (e.g., ExpTime-complete problems) are actually proven to be outside of P and therefore sometimes said to be *truly intractable* or *provably intractable*; see, e.g., [109, Appendix B].

In this sense, the decision problems we will address in the main chapters deserve to be truly intractable ones. They are however not in ExpTime and do not even fall within the class Elementary of *elementary recursive problems*, i.e., problems solvable by a Turing machine working in time bounded by a tower of exponentials with fixed height. In light of this, we refer to nonelementary recursive problems, which might be undecidable, as *extremely intractable problems*.

Fast-growing complexity hierarchy

A nice framework for capturing extremely intractable problems is the *ordinal-indexed hierarchy* $(\mathbf{F}_\alpha)_\alpha$ *of fast-growing complexity classes*, a groundbreaking concept introduced by Sylvain Schmitz in his paper [105]. However, this hierarchical notion, due to its novelty, is not as well-known as, for example, the *polynomial hierarchy* of Meyer and Stockmeyer [88]. For this reason, we briefly mention in advance Schmitz's motivation for introducing his hierarchy.

Long before the work of Schmitz, some important decision problems were shown to have a nonelementary complexity; e.g., the equivalence problem for star-free expression [113] and the provability problem for a sort of relevance logic [122]. In 1979, Statman [112] proved that the worst-case complexity of the β-equivalence problem for simply typed λ-calculus goes beyond elementary recursive time. Also, he pointed out that this problem is actually solvable in $\mathscr{F}_3^*$, the $\{0,1\}$-valued version of the third layer $\mathscr{F}_3$ of Löb and Wainer's *extended Grzegorczyk hierarchy* $(\mathscr{F}_\alpha)_\alpha$ [85]; see also [105, Section 3.1.2].

However, the problem of β-equivalence is not complete for $\mathscr{F}_3^*$ under any seemingly appropriate resource-bounded reduction, as shown by Schmitz; see [105, Section 5.2] for details. He also wrote:

> Unfortunately, the classes in the extended Grzegorczyk hierarchy are not quite satisfying for some interesting problems, which are nonelementary (or or [sic] nonprimitive recursive or nonmultiply recursive, etc), but only *barely* so. The issue is that complexity classes like, e.g., $\mathscr{F}_3^*$, which is the first class to contain nonelementary problems, are very large: i.e., $\mathscr{F}_3^*$ contains problems that require space $F_3^{100}(n)$[12], more than a hundredfold compositions of towers of exponentials. As a result, hardness for $\mathscr{F}_3^*$ cannot be obtained for many classical examples of nonelementary problems. ([105, p. 3:5], emphasis original)

Schmitz's purpose was to propose a better paradigm to properly classify various nonelementary problems (e.g., β-equivalence in simply-typed λ-calculus) not finely captured within the existing framework. Indeed, his hierarchy of fast-growing complexity classes is well-tailored to overcome the aforementioned difficulty (i.e., the absence of complete problems) faced by some classical paradigms, such as the extended Grzegorczyk hierarchy; the reader may refer to [105, Section 2.3] for more information on this topic.

Some landmarks of the Schmitz hierarchy: TOWER, ACK, and HACK

Among lots of classes comprising Schmitz's hierarchy, of particular interest to us are $\mathbf{F}_3$, $\mathbf{F}_\omega$, and $\mathbf{F}_{\omega^\omega}$, also known as TOWER, ACKERMANN (ACK, for short), and HYPERACKERMANN (HACK, for short), respectively. Roughly speaking, TOWER (resp. ACK, HACK) is the lowest-level class among the fast-growing complexity classes that contain nonelementary (resp. nonprimitive recursive, nonmultiply recursive) problems.

Schmitz defined the notion of completeness in each level in his hierarchy. The crux is the fact that there exist various natural complete problems in some typical layers (like TOWER, ACK, and HACK) of the $(\mathbf{F}_\alpha)_\alpha$, unlike in the extended Grzegorczyk hierarchy; see [105, Section 6]. Intuitively, complete problems in TOWER (resp. ACK, HACK) are nonelementary (resp.

[12] It denotes the 100 iterations of F_3, which is the third-level "nonelementary" function of the *fast-growing functions* $(F_\alpha)_\alpha$ of Löb and Wainer [85].

nonprimitive recursive, nonmultiply recursive) problems that lie close to the boundary line between elementary (resp. primitive recursive, multiply recursive) problems and nonelementary (resp. nonprimitive recursive, nonmultiply recursive) problems. For instance, β-equivalence in simply typed λ-calculus is now known to be TOWER-complete; see, e.g., [91] for details. In view of the notion of tractability that we have mentioned earlier, it is reasonable to say that TOWER-complete, ACK-complete, and HACK-complete problems are quite far from tractable ones.

1.4 An observation: two types of decision problems

Again, let us concentrate our attention on a discussion of substructural logic. We alert the reader that the word "decision problem" is often ambiguous in the context of substructural logic. Indeed, there are substructural logics that are decidable with respect to a certain type of decision problem, but undecidable with respect to other types of problems. Below we discuss this somewhat tricky point.

The most standard decision problem for a logic may well be the *provability problem*, which asks whether a given term is provable (without any nonlogical axioms). Obviously, it is possible to generalize the provability problem. The *deducibility problem* (also known as the problem of *finite consequence relation*; see, e.g., [39, Chapter 4]) asks for a given finite set Φ of terms (i.e., a finite theory) and a given term s whether s is derivable in the logical system under consideration using nonlogical axioms in Φ. The deducibility problem coincides with the usual provability problem if a given theory is empty. In this sense, the deducibility problem can be viewed as an uncontrived generalization of the corresponding provability problem.

Let us consider intuitionistic propositional logic (i.e., LJ) as an example. As a corollary of the celebrated *Hauptsatz*, Gentzen [47] obtained perhaps the first decidability proof of provability in intuitionistic propositional logic. After a while, Ladner [73] proved the PSPACE decidability of this problem, and eventually, Statman [111] proved that the PSPACE upper bound is optimal. We have a further question: what about the complexity of deducibility? In conclusion, deducibility in LJ remains PSPACE-complete. The simple reason for this is that LJ enjoys the deduction theorem, which directly serves as an efficient reduction from deducibility into provability. From the complexity-theoretic perspective, the deduction theorem can thus be seen as a phenomenon that *trivializes* the deducibility problem for a logic.

However, the deduction theorem does not hold for many substructural logics.[13] Thus, we often need to rigorously distinguish the deducibility problem from the provability problem.[14] The logic FL is a prominent example of substructural logics that are decidable (in PSpace) with respect to provability (see [70, 68, 64]) but undecidable with respect to deducibility (see [17]).

1.5 State of the art

With the discussion from the previous section in mind, we now consider the following question:

> *How do basic structural rules affect the (un)decidability and algorithmic complexity of a logical system?*

In Table 1.1, the reader may find some clues to the answer to this question. As a case study, we discuss below how (combinations of) basic structural rules affect the complexity of a logic.

Contraction

In many cases, the presence of contraction has a great effect on the computational complexity of a logic. Whereas cut elimination guarantees that provability in the basic substructural logics *without* contraction is decidable (see, e.g. [39, Section 4.2.1] for details), it is often unclear whether there is a decision procedure for provability in a logic *with* contraction. It is because the contraction rule is the only basic structural rule that makes the size of an upper sequent bigger than that of a lower sequent. For instance, the (un)decidability of provability in $\mathsf{FL_c}$ (i.e., FL plus the rule of contraction) was a longstanding open problem (see, e.g., [39, Exercise 14 in p. 241]) until Chvalovský and Horčík [21] solved it negatively in 2016.

[13] As Jean-Baptiste Joinet justifiably pointed out to the author in private communication, the description here is, to be precise, imprecise since every theorem is exactly a proven mathematical statement. For convenience, however, we often use such expressions in this thesis.

[14] Consider, for instance, $\mathsf{FL_e}$ with the theory consisting only of a fixed variable x. In $\mathsf{FL_e}$, we may deduce the term 1 without using any assumptions. It however is impossible to construct a derivation for the term $x \to 1$ in $\mathsf{FL_e}$ since $\mathsf{FL_e}$ lacks the rule of left-weakening. The standard deduction theorem thus fails to hold for $\mathsf{FL_e}$. Nevertheless, all substructural logics (i.e., axiomatic extensions of FL) admit the *parametrized local deduction theorem* [40], a very weak form of the standard deduction theorem. Unfortunately, it does not serve as a resource-bounded reduction.

	Provability	Deducibility
FL	PSPACE-c [70, 68, 64]	RE-c [17]
$\mathsf{FL_e}$	PSPACE-c [81, 64]	RE-c [81]
$\mathsf{FL_c}$	RE-c [21]	RE-c [61, 21]
$\mathsf{FL_i}$	PSPACE-c [64]	HACK-c [51]
$\mathsf{FL_o}$	PSPACE-c [64]	RE-c [17]
$\mathsf{FL_w}$	PSPACE-c [64]	HACK-c [51]
$\mathsf{FL_{ec}}$	ACK-c [122]	ACK-c [122]
$\mathsf{FL_{ei}}$	PSPACE-c [64]	**TOWER-c** (Cor. 3.3.6)
$\mathsf{FL_{eo}}$	PSPACE-c [64]	RE-c [81]
$\mathsf{FL_{ew}}$	PSPACE-c [64]	**TOWER-c** (Cor. 3.3.6)
$\mathsf{FL_{ci}}$ $(= \mathsf{FL_{eci}})$	PSPACE-c[15], cf. [111, 65, 16]	PSPACE-c, cf. [111, 65, 16]
$\mathsf{FL_{co}}$	RE-c [21]	RE-c [21]
$\mathsf{FL_{eco}}$	ACK-c[16]	ACK-c
LJ $(= \mathsf{FL_{ecw}} = \mathsf{FL_{cw}})$	PSPACE-c [111]	PSPACE-c [111]
LK	coNP-c (folklore, cf. [27])	coNP-c (folklore, cf. [27])

Table 1.1: Complexity status of basic substructural logics and classical propositional logic, i.e., Gentzen's LK [46, 47]; "$\mathcal{C}$-c" is an abbreviation for "complete for the class $\mathcal{C}$."

Under certain situations, however, as observed by Kripke [72], who trailblazed the decidability proof of relevance logics, including $\mathsf{FL_{ec}}$ (i.e., $\mathsf{FL_c}$ plus the rule of exchange), the contraction rule seems to be closely related to the decidability results rather than the undecidability results. Some recent results underpin this speculation. Bringing the combinatorial argument of Kripke [72] into the framework of hypersequent calculus, Ramanayake [100]

[15] The author is not aware of the result being explicitly announced anywhere. The polynomial space lower bound is obvious from [111]. For the upper bound, the polynomial space proof-search algorithm developed by Hudelmaier [65] should work also for $\mathsf{FL_{ci}}$. In [16] Buss and Iemhoff also obtained an alternative proof of the PSPACE-decidability of the system PJ for intuitionistic propositional logic; however, in our terminology, PJ seems to be equivalent to $\mathsf{FL_{ci}}$ rather than LJ. At all events, we do not further touch on this topic since it does not seem to provide us with any new insights.

[16] It seems almost obvious from Urquhart's result [122]; see Section 4.2 for the proof.

proved in 2020 that a wide range of axiomatic extensions (i.e., *amenable extensions*) of $\mathsf{FL_{ec}}$ are decidable concerning provability. Shortly thereafter, refining further Ramanayake's approach, Balasubramanian, Lang, and Ramanayake [8] showed that provability in any amenable extension of $\mathsf{FL_{ec}}$ is decidable in HAck.[17]

Left-weakening

It is worth noting that the basic substructural logics with the rule of left-weakening are decidable not only with respect to provability but also with respect to deducibility. Most of these results, i.e., the decidability of deducibility in $\mathsf{FL_i}$, $\mathsf{FL_w}$, $\mathsf{FL_{ei}}$, and $\mathsf{FL_{ew}}$, were (essentially) shown by Blok and van Alten [14, 15].

In view of these decidability results, the reader might expect that the left-weakening rule also causes the decidability of various substructural logics. This speculation is correct to some extent. For instance, Galatos and Jipsen [38] and Horčík [62] algebraically proved that a wide range of axiomatic extensions of $\mathsf{FL_i}$ (i.e., FL plus the rule of left-weakening) are decidable with respect to deducibility. More recently, Balasubramanian, Lang, and Ramanayake [8] proved the HAck-decidability of provability in analytic hyperstructural rule extensions of $\mathsf{FL_{ei}}$ via a proof-theoretic approach. These earlier papers corroborate a powerful effect of the left-weakening rule.

Exchange

Compared to the rules of contraction and left-weakening, the effect of the exchange rule seems to be less significant. However, the exchange rule sometimes "moderates" the complexity of a substructural logic. As mentioned in the subsection on Contraction, while provability in $\mathsf{FL_c}$ is already undecidable (cf. [21]), $\mathsf{FL_{ec}}$ is decidable regarding both provability and deducibility (cf. [72]).[18] A similar phenomenon occurs in the huge complexity gap, discovered recently between $\mathsf{FL_w}$ and $\mathsf{FL_{ew}}$; see Section 1.6 for detailed information on this topic.

[17] Thanks to the deduction theorem for axiomatic extensions of $\mathsf{FL_{ec}}$ (cf. [39, Corollary 2.15]), the HAck upper bound should also hold for deducibility.

[18] Remarkably, Urquhart [122] proved that checking the provability in $\mathsf{FL_{ec}}$ of a given term exceeds primitive recursive time; in terms of the Schmitz hierarchy, the problem is Ack-complete; see [105, Section 6]. The same holds for the deducibility problem due to the deduction theorem for $\mathsf{FL_{ec}}$; see [39, Corollary 2.15].

Right-weakening

The rule is the least interesting among the four basic structural rules. There seem to be no cases where the rule of (o) has a significant impact on the computational complexity.

1.6 Contributions

What was established by Chvalovský and Horčík [21] was not only the undecidability of $\mathsf{FL_c}$-provability but also the complete complexity classification of provability in basic substructural logics. However, many open questions remain regarding deducibility in substructural logics. The subject matter of this thesis is to present the extreme intractability of deducibility in various substructural logics through the lens of the fast-growing complexity hierarchy.

TOWER-completeness of deducibility in $\mathsf{FL_{ei}}$ and $\mathsf{FL_{ew}}$

Our first interest lies in the basic substructural logics without contraction. Thus far, these contraction-free logics have drawn much attention, especially in the context of the set theory of nonclassical logic, as they provide the basis for a consistent set theory with the unrestricted comprehension axiom; see Footnote 9 for more on this subject. However, we concentrate exclusively on their algorithmic complexities.

For a long time, the sharp complexity bounds for deducibility in $\mathsf{FL_i}$, $\mathsf{FL_w}$, $\mathsf{FL_{ei}}$, and $\mathsf{FL_{ew}}$ had remained unexplored.[19] In the early 2000s, as we have mentioned earlier, Blok and van Alten [14, 15] proved these problems to be decidable. In 2011, they were shown to be at least PSPACE-hard by Horčík and Terui [64]. Until recently, however, no further progress had been made.

In Chapter 3, we show that the deducibility problems for $\mathsf{FL_{ei}}$ and $\mathsf{FL_{ew}}$ are TOWER-complete.

Contribution A. *The deducibility problems for* $\mathsf{FL_{ei}}$ *and* $\mathsf{FL_{ew}}$ *are* TOWER-*complete. The same holds for deducibility in the implicational fragment of* $\mathsf{FL_{ew}}$*, i.e., BCK-logic [66].*

[19] The deducibility problem for $\mathsf{FL_{ew}}$ seems to have been recognized as an open question for at least a decade. Indeed, this issue was taken up by Bílková as an open question at Logic Colloquium 2008; see [12].

To establish Contribution A, we employ some useful ideas (e.g., the deduction theorem for linear logic [6, 81], !-prenex sequents [118], and a negative translation for linear logic [76]) from the linear logic literature. In particular, our proof depends heavily on the automata-theoretic approach, which traces back to the powerful techniques developed by Lazić and Schmitz [78]; see the introduction part of Chapter 3 for more information on the technical background on our contribution.

It is also worth mentioning that Greati and Ramanayake [51] recently proved the marvelous HACK-completeness for deducibility in $\mathsf{FL_i}$ and $\mathsf{FL_w}$.[20] The result, in conjunction with Contribution A, completely uncovered the precise complexity status of deducibility in basic substructural logics and shattered the optimism that there might exist elementary recursive algorithms for $\mathsf{FL_i}$, $\mathsf{FL_w}$, $\mathsf{FL_{ei}}$, and $\mathsf{FL_{ew}}$.

Various substructural logics requiring a nonelementary time

In contrast to the fine-grained complexity analysis of only a few contraction-free logics in Chapter 3, we present in Chapter 4 a simple classification of a large range of substructural logics in terms of the algorithmic complexity of deducibility. It is evident from the context of nonclassical logic that efforts to classify logics from the complexity-theoretic standpoint are nothing new and rather abound; see, e.g., [73, 20, 67, 64, 100, 8, 26] for this research direction.

We first provide a broad criterion under which deducibility in a substructural logic is not elementary recursive.

Contribution B. *The deducibility problem for any substructural logic weaker than* $\mathsf{FL_{ew}}$ *is* TOWER*-hard.*

Obviously, it is not the case that this "barely nonelementary" lower bound is the best lower bound for every substructural logic between FL and $\mathsf{FL_{ew}}$; recall that, for instance, $\mathsf{FL_w}$-deducibility is HACK-complete (see [51]). Given this situation, it would not be surprising if we may discover between FL and $\mathsf{FL_{ew}}$ many other logics requiring a greater amount of computational resources. Indeed, we show that the TOWER lower bound is tremendously far from the best lower bound as far as logics weaker than $\mathsf{FL_w}$ are concerned:

[20] In [51], Greati and Ramanayake did not explicitly declare the HACK-completeness of $\mathsf{FL_i}$-deducibility. It is however clear at a glance that their proof subsumes this complexity result.

Contribution C. *The deducibility problem for every substructural logic weaker than* $\mathsf{FL_w}$ *is* HACK*-hard.*

Furthermore, we also give a sufficient condition under which deducibility in a substructural logic is nonprimitive recursive. Unlike the other contributions described so far, the claim below seems to be closely related to *relevance logics* [2, 87], which were born out of some reasonable requirements by philosophers; see also Footnote 9 in this chapter.

Contribution D. *The deducibility problem for any substructural logic weaker than* $\mathsf{FL_{eco}}$ *is* ACK*-hard.*

Contributions B, C, and D are even more interesting when compared to the work of Horčík and Terui [64], who presented uniform complexity lower bounds for various substructural logics. They proved that provability in any consistent substructural logic is coNP-hard and that the problem is PSPACE-hard if a logic at hand enjoys the disjunction property.[21] The fact that the lower bounds we provide are much higher than the modest ones (i.e., coNP and PSPACE) given by Horčík and Terui supports, from the quantitative perspective, that deducibility is by far a more complex concept than provability. It is also worth mentioning that our approach is purely proof-theoretic, whereas Horčík and Terui made effective use of an algebraic manner in order to show the disjunction property for a broad class of substructural logics.[22]

Our proofs of Contributions B, C, and D are based on a very simple observation similar to the idea that Lincoln, Scedrov, and Shankar [82] employed in the undecidability proof of multiplicative intuitionistic linear logic with second-order quantifiers; see the introduction part of Chapter 4 for details. Some of our results with high importance are highlighted in Table 1.1.

A further result

Contributions B, C, and D entail quite a few consequences. We present one of them in this subsection; some other corollaries are detailed in Section 4.3.

[21] This is also a substructural generalization of the similar result for superintuitionistic logics, which states that every nontrivial superintuitionistic logic with the disjunction property is hard for PSPACE with respect to provability; see Chagrov and Zakharyaschev's book [20].

[22] As described in detail in Chapter 4, our proofs in Chapter 4 rely heavily on: Contribution A; the complexity result by Urquhart [122]; and the complexity result of Greati and Ramanayake [51].

	Provability	Deducibility
FL_m^n $(n > m \geq 1)$	RE-c [21]	RE-c [61, 21]
FL_m^n $(m > n \geq 2)$	PSPACE-c [64]	RE-c [61]
FL_m^1 $(m \geq 2)$	PSPACE-c [64]	?[24] (**HACK-hard** by Cor. 4.1.9)
FL_1^0 $(= \mathsf{FL_i} = \mathsf{FL}_m^0)$	PSPACE-c [64]	HACK-c [51]
$\mathsf{FL_e}_m^n$ $(n > m \geq 1)$	? (ACK-hard [110], dec. [124])	? (ACK-hard [110], dec. [124])
$\mathsf{FL_e}_1^2$ $(= \mathsf{FL_{ec}})$	ACK-c [122]	ACK-c [122]
$\mathsf{FL_e}_m^n$ $(m > n \geq 0)$	PSPACE-c [64]	? (**TOWER-hard** by Cor. 4.1.6, dec. [124])
$\mathsf{FL_e}_1^0$ $(= \mathsf{FL_{ei}} = \mathsf{FL_e}_m^0)$	PSPACE-c [64]	**TOWER-c** (Cor. 3.3.6)

Table 1.2: Current complexity status of FL_m^n and $\mathsf{FL_e}_m^n$; "dec." is an abbreviation for "decidable."

The rule of contraction (resp. left-weakening) is a structural rule for decreasing (resp. increasing) the number of terms appearing on the left-hand side of a sequent. Is there a way to capture these rules from a common perspective? One plausible way is to consider the following (m, n)-*knotted rule*, first devised by Hori, Ono, and Schellinx [58]:

$$\frac{\Gamma, s^{(n)}, \Delta \Rightarrow \sigma}{\Gamma, s^{(m)}, \Delta \Rightarrow \sigma}\ (\mathsf{knot_m^n})$$

where $m \geq 1$, $n \geq 0$, $m \neq n$, and $s^{(n)}$ denotes the sequence of the n copies of s.[23] Note that the (m, n)-knotted rule is equivalent to the rule of left-weakening (resp. contraction) if $m = 1$ and $n = 0$ (resp. $m = 1$ and $n = 2$). In this respect, knotted rules can thus be viewed as a natural generalization of contraction and left-weakening.

Below, FL_m^n (resp. $\mathsf{FL_e}_m^n$) denotes the substructural logic FL (resp. $\mathsf{FL_e}$) plus the (m, n)-knotted rule. In 2005, van Alten [124] proved that the deducibility problem for $\mathsf{FL_e}_m^n$ is decidable for every knotted pair (m, n). After a

[23] Clearly, (m, n)-knotted rules do not make sense if $m = n$. Also, it is easy to see that the logics $\mathsf{FL_e}_m^n$ and FL_m^n are inconsistent in the cases where $m = 0$ and $n \geq 1$, cf. [58]. Thus we exclude those cases.

[24] In [62] Horčík proved the decidability of the case where $m = 2$. Furthermore, he also pointed out in [60] a surprising relationship between the decision problems for FL_m^1 and the Burnside problem for groups. It seems that he proved that deducibility in FL_m^1 is decidable in the cases where $m = 3, 4, 5, 7$; however, to the best of our knowledge, the detailed proof has not been published anywhere.

decade, Horčík [61] proved that the deducibility problem for FL_m^n is undecidable whenever $m > n \geq 2$ or $n > m \geq 1$. Horčík and Terui [64] showed that the provability problems for FL_m^n and $\mathsf{FL}_{\mathsf{e}m}^n$ are PSpace-hard for each knotted pair (m, n) and that in particular, these problems are PSpace-complete in the cases where $m > n \geq 0$. In 2019, St. John showed in his PhD thesis [110] that provability in $\mathsf{FL}_{\mathsf{e}m}^n$ is already Ack-hard for $n > m \geq 1$.[25]

The exact complexity of deducibility in $\mathsf{FL}_{\mathsf{e}m}^n$ remains unknown for almost every combination of m and n. Nevertheless, we can show a nonelementary lower bound for deducibility in $\mathsf{FL}_{\mathsf{e}m}^n$ for every knotted pair (m, n), combining Contribution B with the result of St. John [110] (or Contribution D). More precisely, we obtain:

Contribution E. *For each knotted pair (m, n), the deducibility problem for $\mathsf{FL}_{\mathsf{e}m}^n$ is not elementary recursive since the following hold:*

1. *The deducibility problem for $\mathsf{FL}_{\mathsf{e}m}^n$ is* Tower*-hard for $m > n \geq 0$.*
2. *The deducibility (provability) problem for $\mathsf{FL}_{\mathsf{e}m}^n$ is* Ack*-hard for $n > m \geq 1$.*

In Table 1.2, we summarize the latest complexity status of knotted extensions of FL and FL_e; the main results, including those not detailed here, are colored.

Tower-completeness of provability in elementary affine logic

The scope of the thesis also extends to *elementary affine logic* (ELL_w); see, e.g., [4, 5]. This logic, an affine variant of Girard's *elementary linear logic* (ELL) [50], is known to capture elementary recursive functions via the *proofs-as-programs correspondence* (also known as the *Curry-Howard isomorphism*). In conjunction with the fact that elementary recursive functions, in the first place, were not of much interest to logicians in computer science, ELL_w was often undervalued compared to "logics of polynomial time computation" (such as *light linear logic* [50]), which were actively developed at the same time; see [7, Section 1] for more information on this situation. After a while, however, it became clear that ELL_w and its variants possess some other intriguing computational features despite their simple formulation; we may mention the following topics (1) and (2).

(1) *Characterization of P and the exponential hierarchy.* Baillot [7] showed that an intuitionistic variant of ELL_w, when equipped with fixpoints, can

[25] We can obtain the same result as a corollary of Contribution D and a sort of deduction theorem for FL_m^n with $n > m \geq 1$.

capture even the class P and the exponential hierarchy, exceeding Girard's original design philosophy. This supports the view that elementary affine logic possesses a more refined expressive power than originally expected.

(2) *A naïve set theory of elementary time computation.* In the spirit of Grišin [52, 53] and Girard [50], Terui [120] developed *elementary affine set theory*, i.e., (intuitionistic) elementary affine logic augmented with the principle of naïve comprehension. He showed that elementary affine set theory is consistent and that a numeric function is provably total in this set theory if and only if it is an elementary function. This result provides an alternative characterization of elementary time computation from the point of view of naïve set theory.

Unfortunately, the above topics concerning implicit computational complexity are beyond our focus. Since elementary affine logic has attracted attention in the context of type systems rather than that of nonclassical logic, we may consider several kinds of decision problems (e.g., type-checking, typability, and type-inhabitation) as is customary for type theorists.[26] Among them, we are only interested in the provability (or type inhabitation) problem. Dal Lago and Martini [33] settled the finite model property for $\mathsf{ELL_w}$, thus showing that provability in $\mathsf{ELL_w}$ is decidable; however, no known complexity bounds had been settled for the problem. We establish the precise complexity of $\mathsf{ELL_w}$-provability.

Contribution F. *Provability in* $\mathsf{ELL_w}$ *is* TOWER*-complete.*

The same is true for intuitionistic variants of $\mathsf{ELL_w}$; see Section 3.3.2 for details. With a little ingenuity, one can furthermore prove the TOWER-hardness for provability in *intuitionistic light affine logic* (ILAL) [4], a fine-tuned system characterizing polynomial time computation; see [117, Section 7] for details. We refer the reader to Terui [118, Chapter 7] for a decidability proof of ILAL-provability.

1.7 Organization of this thesis

In Chapter 2, we give a simple account of various concepts that are indispensable for the technical arguments in Chapters 3 and 4. In Chapter 3, we show that the deducibility problems for some substructural logics without contraction are TOWER-complete (Contribution A) and that the provability problems for some variants of elementary affine logic are also complete for

[26] For instance, Coppola and Martini [30] proved that typability in a fragment of intuitionistic elementary affine logic is decidable.

Tower (Contribution F).[27] In Chapter 4, we present some simple criteria under which a substructural logic is of huge complexity with respect to deducibility (Contributions B, C, and D). As a corollary, we show that various decidable problems are of nonelementary complexity (Contribution E). In the last chapter, we conclude this thesis by posing some open questions.

[27] Chapter 3 is an edited and polished version of the article [117], which appeared in the proceedings of the 31st Annual Conference on Computer Science Logic, and includes some new results, such as the undecidability of provability in elementary linear logic. However, we exclude from this chapter some minor results of [117], like e.g., the Ack-hardness of deducibility in BCI-logic [66]. The interested reader may find such results in Section 7 of [117].

Chapter 2

Preliminaries

In this chapter, we provide a quick review of the following three topics: (i) substructural logics (i.e., *full Lambek calculus* [40, 39] and *linear logic* [48, 49]); (ii) counter machines (i.e., *alternating branching vector addition systems with states and with full zero tests* [78]); (iii) some huge complexity classes (i.e., $(\mathbf{F}_\alpha)_\alpha$ of Schmitz [105]). We assume that the reader is skilled in some rudimentary notions and results from the realms of proof theory, recursion theory, and computational complexity theory. For instance, we often make use of the following concepts without any explanation:

1. some prominent properties for sequent calculus, e.g., cut elimination and its corollaries (such as subformula property and conservativity);
2. the (F)DTIME notation, some typical classes of decision problems (e.g., NP, PSPACE, RE), many-one (possibly resource bounded) reductions, and complete problems;
3. some number-theoretic functions obtained by variants of primitive recursion, e.g., elementary functions and primitive recursive functions.

The reader may consult Ono [97] for a concise exposition of Topic (1); Arora and Barak [3] and Rogers, Jr. [102] for a full discussion of Topic (2); and Odifreddi [92] for the basics on Topic (3).

Organization of this chapter. In Sections 2.1 and 2.2, we review sequent calculi for various logical systems. They share one common feature; each of them is an extension of full Lambek calculus (FL). In Subsection 2.1, we therefore start with a brief discussion about a sequent calculus for FL and then define various substructural logics, i.e., axiomatic extensions of FL. In

Subsection 2.2, we concentrate on proof theory of *commutative substructural logics*, i.e., linear logic and its variants.

In Section 2.3, we review the notion of *alternating branching vector addition systems with states and with full zero tests* ($\mathsf{ABVASS}_{\bar{0}}$s, for short), which is perhaps unfamiliar to most mathematical logicians. In a nutshell, $\mathsf{ABVASS}_{\bar{0}}$s are extensions of Hopcroft and Pansiot's *vector addition systems with states* (VASSes, for short; i.e., n-counter machines without zero tests) [57] by three kinds of instructions: (i) split rules, (ii) fork rules, (iii) full zero test rules. Moreover, we give an account for two kinds of operational semantics of $\mathsf{ABVASS}_{\bar{0}}$s, i.e., the standard operational semantics and the lossy operational semantics.

Section 2.4 is devoted to a brief explanation of Schmitz's *fast-growing complexity classes* $(\mathbf{F}_\alpha)_\alpha$ [105]. For this purpose, we first review some basics on ordinal terms, the *fast-growing functions* $(F_\alpha)_\alpha$ [85], and the *extended Grzegorczyk hierarchy* $(\mathscr{F}_\alpha)_\alpha$ [85]. Using nonelementary functions falling within $(F_\alpha)_\alpha$, we next review the hierarchy $(\mathbf{F}_\alpha)_\alpha$ of Schmitz and summarize some selected notions and properties of $(\mathbf{F}_\alpha)_\alpha$.

In Section 2.5, we describe some types of decision problems (i.e., provability, deducibility, reachability, and lossy-reachability), following the terminology of logic and vector addition systems. Moreover, we summarize some known Tower-, Ack-, and HAck-complete problems, which are indispensable for our proofs in Chapters 3 and 4.

2.1 Sequent calculi for substructural logics

In this section, we review some basics on *full Lambek calculus* (FL) and its various axiomatic extensions. Our exposition in this section primarily based on: Galatos and Ono [40, Section 3]; Galatos, Jipsen, Kowalski, and Ono [39, Chapter 2]; Horčík and Terui [64, Section 2]; and Horčík [59, Section 1].

2.1.1 Languages, terms, and sequents

Depending on the situation, we employ various distinct languages (i.e., a finite set of logical connective symbols). In what follows, the notation $\mathcal{L}_{\mathsf{FL}}$ stands for a language $\{\circ, \backslash, /, \wedge, \vee, 1, 0\}$, where 0 (*zero*), 1 (*one*) are constant (i.e., nullary operation) symbols, and $\circ$ (*product*), $\backslash$ (*left division*), $/$ (*right division*), $\wedge$ (*conjunction*), $\vee$ (*disjunction*) are binary operation symbols.

A *noncommutative intuitionistic language*[1] is just a nonempty subset of

[1] Each of the logical systems that we employ in this thesis is built on a noncommutative

Linear logic (Girard [48])	$\multimap$	$\multimapinv$	$\otimes$	$\parr$	$\&$	$\oplus$	$\mathbf{1}$	$\bot$	$!$	$?$
Linear logic (Troelstra [121])	$\multimap$	$\multimapinv$	$\star$	$+$	$\sqcap$	$\sqcup$	$\mathbf{1}$	$\mathbf{0}$	$!$	$?$
Substructural logic (this thesis)	$\backslash$	$/$	$\circ$	$+$	$\wedge$	$\vee$	1	0	$\Box$	$\Diamond$

Figure 2.1: Correspondence between the Girard-style notation, the Troelstra-style notation, and our notation

$\mathcal{L}_{\mathsf{FL}}$. For instance, we write $\mathcal{L}_{\mathsf{FL}^+}$ for the 0-free sublanguage of $\mathcal{L}_{\mathsf{FL}}$, i.e., $\{\circ, \backslash, /, \wedge, \vee, 1\}$. In what follows, we sometimes use the terminology derived from linear logic. However, our notation usually comes from the tradition of substructural logic rather than that of linear logic; see Figure 2.1 for the correspondence between the notation of substructural logic and the notation of linear logic.

Let $\mathcal{K}$ be a noncommutative intuitionistic language. We fix a denumerable set V of variables. We stipulate that the set V is uniquely fixed whichever languages we choose. We construct *noncommutative intuitionistic $\mathcal{K}$-terms* (*intuitionistic $\mathcal{K}$-terms*, for short) from V using the operation symbols in $\mathcal{K}$. For instance, intuitionistic $\mathcal{L}_{\mathsf{FL}}$-terms are defined as follows:

- Every variable is an intuitionistic $\mathcal{L}_{\mathsf{FL}}$-term.
- 1, 0 are intuitionistic $\mathcal{L}_{\mathsf{FL}}$-terms.
- If s and t are intuitionistic $\mathcal{L}_{\mathsf{FL}}$-terms, so are $s \circ t$, $s \backslash t$, s / t, $s \wedge t$, $s \vee t$.

We use the expression of the form $\sim s$ (resp. $-s$) as a shorthand for the intuitionistic term of the form $s \backslash 0$ (resp. $0 / s$). The set of intuitionistic $\mathcal{K}$-terms is denoted by $Tm_{\mathcal{K}}$.

Henceforth, as far as a noncommutative intuitionistic language is concerned, symbols $x, y, z, \ldots$ (resp. $s, t, u, v, \ldots$) range over V (resp. $Tm_{\mathcal{K}}$). Given a term s, we inductively define the term s^n by $s^0 = 1$, $s^{n+1} = s^n \circ s$. Similarly, we write $s^{(n)}$ for the sequence that consists of n copies of s.

Following the usual convention in proof theory, we stipulate that Greek capital letters $\Gamma, \Delta, \Sigma, \ldots$ stand for finite sequences of intuitionistic $\mathcal{K}$-terms.

language or a commutative language. Throughout Chapter 4, we study the logical systems that are based on noncommutative languages. On the other hand, all the logical systems in Chapter 3 are built on commutative languages. We provide a detailed account for commutative languages in Section 2.2.

Given an intuitionistic $\mathcal{K}$-term s, we simply write s for the sequence consisting only of exactly one occurrence of s. In what follows, the expression Γ, Δ denotes the concatenation of Γ and Δ.

Let $\mathcal{K}$ be a noncommutative intuitionistic language. If 0 is a member of $\mathcal{K}$, a *noncommutative intuitionistic $\mathcal{K}$-sequent* (*intuitionistic $\mathcal{K}$-sequent*, for short) is just an ordered pair of Γ and σ, where Γ is a finite sequence of intuitionistic $\mathcal{K}$-terms and σ is an intuitionistic $\mathcal{K}$-term or the empty sequence ε. In what follows, every intuitionistic $\mathcal{K}$-sequent (Γ, σ) is always expressed as $\Gamma \Rightarrow \sigma$; in particular, we usually write $\Gamma \Rightarrow$ if σ is empty. If $\mathcal{K}$ does not contain 0, by an intuitionistic $\mathcal{K}$-sequent we mean an expression of the shape $\Gamma \Rightarrow s$, where Γ is a finite sequence of intuitionistic $\mathcal{K}$-terms and s is always an intuitionistic $\mathcal{K}$-term. We stipulate that the right-hand side of a sequent always consists of a single term whenever considering the intuitionistic languages without 0.

2.1.2 Full Lambek calculus and basic substructural logics

Our starting point is *full Lambek calculus* (FL, for short) [40, 39]. The language of FL is $\mathcal{L}_{\mathsf{FL}}$. The sequent calculus for FL consists of the inference rules depicted in Figure 2.2.

We define the notion of derivation in the usual manner. Let Φ be a set of intuitionistic $\mathcal{L}_{\mathsf{FL}}$-terms and $\Gamma \Rightarrow \sigma$ be an intuitionistic $\mathcal{L}_{\mathsf{FL}}$-sequent. A *derivation in* FL *of* $\Gamma \Rightarrow \sigma$ *from* Φ is a finite binary tree T labeled by intuitionistic $\mathcal{L}_{\mathsf{FL}}$-sequents satisfying the following conditions.

- Each leaf of T is one of the following:
  1. $\Rightarrow t$, where $t \in \Phi$,
  2. $s \Rightarrow s$, i.e., an instance of (init),
  3. $\Rightarrow 1$,
  4. $0 \Rightarrow$.
- Each inner node of T is obtained from its children by applying a suitable instance of an inference rule (r) from Figure 2.2.

For brevity, we write $\Phi \vdash_{\mathsf{FL}} \Gamma \Rightarrow \sigma$ if there is a derivation in FL of $\Gamma \Rightarrow \sigma$ from Φ. In the case where $\emptyset \vdash_{\mathsf{FL}} \Gamma \Rightarrow \sigma$, we say that $\Gamma \Rightarrow \sigma$ is *provable* in FL and simply write $\vdash_{\mathsf{FL}} \Gamma \Rightarrow \sigma$. We often make use of the expression $\Phi \vdash_{\mathsf{FL}} s$ as an abbreviation for $\Phi \vdash_{\mathsf{FL}} \Rightarrow s$.

Let S be a set of intuitionistic $\mathcal{L}_{\mathsf{FL}}$-terms. The *axiomatic extension* of FL by S is the sequent calculus obtained from FL by adding the initial sequent

$$\frac{\Gamma \Rightarrow s \qquad \Sigma, s, \Delta \Rightarrow \sigma}{\Sigma, \Gamma, \Delta \Rightarrow \sigma}\ (\mathrm{cut}) \qquad \frac{}{s \Rightarrow s}\ (\mathrm{init})$$

$$\frac{\Gamma, \Delta \Rightarrow \sigma}{\Gamma, 1, \Delta \Rightarrow \sigma}\ (1 \Rightarrow) \qquad \frac{}{\Rightarrow 1}\ (\Rightarrow 1)$$

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (\Rightarrow 0) \qquad \frac{}{0 \Rightarrow}\ (0 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow s \qquad \Sigma, t, \Delta \Rightarrow \sigma}{\Sigma, \Gamma, s \backslash t, \Delta \Rightarrow \sigma}\ (\backslash \Rightarrow) \qquad \frac{s, \Gamma \Rightarrow t}{\Gamma \Rightarrow s \backslash t}\ (\Rightarrow \backslash)$$

$$\frac{\Gamma \Rightarrow s \qquad \Sigma, t, \Delta \Rightarrow \sigma}{\Sigma, t / s, \Gamma, \Delta \Rightarrow \sigma}\ (/ \Rightarrow) \qquad \frac{\Gamma, s \Rightarrow t}{\Gamma \Rightarrow t / s}\ (\Rightarrow /)$$

$$\frac{\Gamma, s, t, \Delta \Rightarrow \sigma}{\Gamma, s \circ t, \Delta \Rightarrow \sigma}\ (\circ \Rightarrow) \qquad \frac{\Gamma \Rightarrow s \qquad \Delta \Rightarrow t}{\Gamma, \Delta \Rightarrow s \circ t}\ (\Rightarrow \circ)$$

$$\frac{\Gamma, s_i, \Delta \Rightarrow \sigma}{\Gamma, s_1 \wedge s_2, \Delta \Rightarrow \sigma}\ (\wedge \Rightarrow),\ \text{where } i = 1, 2 \qquad \frac{\Gamma \Rightarrow s \qquad \Gamma \Rightarrow t}{\Gamma \Rightarrow s \wedge u}\ (\Rightarrow \wedge)$$

$$\frac{\Gamma, s, \Delta \Rightarrow \sigma \qquad \Gamma, t, \Delta \Rightarrow \sigma}{\Gamma, s \vee t, \Delta \Rightarrow \sigma}\ (\vee \Rightarrow) \qquad \frac{\Gamma \Rightarrow s_i}{\Gamma \Rightarrow s_1 \vee s_2}\ (\Rightarrow \vee),\ \text{where } i = 1, 2$$

Figure 2.2: Inference rules of FL; the metavariables s, s_1, s_2, t range over intuitionistic $\mathcal{L}_{\mathsf{FL}}$-terms, Γ, Δ over finite sequence of $\mathcal{L}_{\mathsf{FL}}$-terms, σ denotes an intuitionistic $\mathcal{L}_{\mathsf{FL}}$-term or the empty sequence.

$\Rightarrow \iota(s)$ for every s in S and every substitution ι. Here, terms in S are often called *axioms*. We write FL_S for the axiomatic extension of FL by S.

See Figure 2.3 for the list of typical axioms. The axioms of exchange (denoted by e), left-weakening, (denoted by i), right-weakening (denoted by o), and contraction (denoted by c) are often regarded as the most fundamental axioms, as they exactly correspond to the earliest structural rules introduced by Gentzen [46, 47].[2]

Following the standard terminology of substructural logic (see, e.g., [39, p. 88]), we refer to an axiomatic extension of FL (by some set S of terms) as a *substructural logic*. A substructural logic is called a *basic substructural logic* [40] if it is an axiomatic extension of FL by a set S of terms, where S is a (possibly empty) subset of $\{e, c, i, o\}$. For each substructural logic L, the

[2] For background on these axioms/rules, see Section 1.2.

Axiom (term)	Notation of axiom	Corresp. struct. rule	Name
$(x \circ y) \backslash (y \circ x)$	(e)	$\dfrac{\Gamma, s, t, \Delta \Rightarrow \sigma}{\Gamma, t, s, \Delta \Rightarrow \sigma}\,(\mathsf{e})$	axiom/rule of exchange [46, 47]
$x \backslash x^2$	(c)	$\dfrac{\Gamma, s, s, \Delta \Rightarrow \sigma}{\Gamma, s, \Delta \Rightarrow \sigma}\,(\mathsf{c})$	axiom/rule of contraction [46, 47]
$x \backslash 1$	(i)	$\dfrac{\Gamma, \Delta \Rightarrow \sigma}{\Gamma, s, \Delta \Rightarrow \sigma}\,(\mathsf{i})$	axiom/rule of left-weakening [46, 47]
$0 \backslash x$	(o)	$\dfrac{\Gamma \Rightarrow}{\Gamma \Rightarrow s}\,(\mathsf{o})$	axiom/rule of right-weakening [46, 47]
$x^m \backslash x^n$	$(knot^n_m)$	$\dfrac{\Gamma, s^{(n)}, \Delta \Rightarrow \sigma}{\Gamma, s^{(m)}, \Delta \Rightarrow \sigma}\,(\mathsf{knot^n_m})$	(m, n)-knotted axiom/rule [58, 124]

Figure 2.3: Axiom schemes and corresponding structural rules; symbols in italic represent axioms (e.g., e) and symbols in sans-serif represent structural rules (e.g., e).

deducibility relation for L (denoted by $\vdash_{\mathsf{L}}$) is defined in an obvious way.

In many cases, it is convenient to consider the notion of rule extensions [64] of FL rather than axiomatic extensions of FL. Let R be a set of inference rules.[3] The *rule extension* of FL by R is the sequent calculus obtained from FL by adding all the inference rules from R, and is denoted by FL_R. Typical (structural) inference rules are the rules of (e), (i), (o) and (c), which exactly correspond to the axioms (e), (i), (o) and (c), respectively; see Figure 2.3. Given $S \subseteq \{e, c, i, o\}$, we write R^S for the set of structural rules corresponding to S. It is easy to see that the following holds:

Fact 2.1.1 (E.g., Galatos, Jipsen, Kowalski, and Ono [39], Lemma 2.5)**.** *Let S be a subset of $\{e, i, o, c\}$ and $\Phi \cup \{s\}$ be a set of intuitionistic $\mathcal{L}_{\mathsf{FL}}$-terms.*

$$\Phi \vdash_{\mathsf{FL}_S} s \text{ if and only if } \Phi \vdash_{\mathsf{FL}_{R^S}} s.$$

In view of this, FL_R can be also viewed as a basic substructural logic for any subset R of $\{\mathsf{e}, \mathsf{i}, \mathsf{o}, \mathsf{c}\}$. In what follows, when considering a basic substructural logic FL_R, we always express R as a sequence of e, i, o and c; for instance, we use the notation $\mathsf{FL}_{\mathsf{eco}}$ as a shorthand for $\mathsf{FL}_{\{\mathsf{e},\mathsf{c},\mathsf{o}\}}$. If the

[3] In this thesis, we employ the definition of inference rules, given by Horčík and Terui [64, Section 2.1].

rules of (i) and (o) both belong to R, these rules are conventionally merged into the notation (w). For instance, by the notation $\mathsf{FL_{ew}}$ we always mean the basic substructural logic $\mathsf{FL}_{\{\mathsf{e},\mathsf{i},\mathsf{o}\}}$.

A *knotted pair* [58, 124] is an ordered pair of natural numbers m and n such that $m \geq 1$, $n \geq 0$, $m \neq n$. Given a knotted pair (m, n), the (m, n)-*knotted axiom* [58, 124] is the term of the form $x^m \setminus x^n$; see also Figure 2.3. Obviously, the axioms of left-weakening and contraction are special cases of knotted axioms. Similarly, the rules of (i) and (c) are nothing but the $(1, 0)$- and $(1, 2)$-knotted rules, respectively. For a knotted pair (m, n), we always write FL^n_m (resp. $\mathsf{FL_e}{}^n_m$) for the rule extension of FL (resp. $\mathsf{FL_e}$) by the (m, n)-knotted rule. For each knotted pair (m, n), the (m, n)-knotted axiom precisely corresponds to the rule of $(\mathsf{knot^n_m})$ in the following sense:

Fact 2.1.2 (E.g., Ciabattoni, Galatos, and Terui [24]). *Let $\Phi \cup \{s\}$ be a set of $\mathcal{L}_{\mathsf{FL}}$-terms.*

1. $\Phi \vdash_{\mathsf{FL}_{\{(knot^n_m)\}}} s$ *if and only if* $\Phi \vdash_{\mathsf{FL}^n_m} s$.
2. $\Phi \vdash_{\mathsf{FL}_{\mathsf{e}\{(knot^n_m)\}}} s$ *if and only if* $\Phi \vdash_{\mathsf{FL_e}{}^n_m} s$.

Given substructural logics L and L', we say that L *is weaker than* L' if $\Phi \vdash_{\mathsf{L}} s$ implies $\Phi \vdash_{\mathsf{L}'} s$ for every set Φ of intuitionistic $\mathcal{L}_{\mathsf{FL}}$-terms and for every intutionistic $\mathcal{L}_{\mathsf{FL}}$-term s. Moreover, given substructural logics L and L', L'', we say that L *is between* L' *and* L'' if L' is weaker than L and L is weaker than L''. It is quite easy to check the following:

Fact 2.1.3 (E.g., Hori, Ono, and Schellinx [58], Section 2). *Let (m, n) be a knotted pair.*

1. FL^n_m *and* $\mathsf{FL_e}{}^n_m$ *are substructural logics weaker than* $\mathsf{FL_{ei}}$ *for* $m > n \geq 0$.
2. FL^n_m *and* $\mathsf{FL_e}{}^n_m$ *are substructural logic weaker than* $\mathsf{FL_{ec}}$ *for* $n > m \geq 1$.

2.2 Sequent calculi with implicit exchange

2.2.1 Languages, terms, and sequents in commutative logics

In the previous section, we have defined various substructural logics, including *commutative substructural logics*, i.e., substructural logics with the rule of (e). It is however more convenient in many cases to define commutative substructural logics, not by introducing the rule of (e) explicitly, but by considering the lefthand side of a sequent as a multiset of terms. Employing the latter manner, for instance, we can express the rule of (cut) by using a bit simpler form:

$$\frac{\Gamma \Rightarrow s \qquad s, \Delta \Rightarrow \sigma}{\Gamma, \Delta \Rightarrow \sigma}\ (\mathrm{cut})$$

where Γ and Δ are not metavariables for *sequences* of terms but those for *multisets* of terms.

Recall that intuitionistic terms $s \backslash u$ and $u \,/\, s$ are provably equivalent in commutative substructural logics. This means that it is no longer necessary to distinguish between left- and right-divisions in commutative substructural logics. In view of this, many logicians make use of a binary connective $\to$ instead of $\backslash$ and $/$ when discussing about commutative substructural logics. In accordance with this convention, the rules of $(\Rightarrow \backslash)$ and $(\Rightarrow /)$ are merged into the following rule $(\Rightarrow\to)$:

$$\frac{s, \Gamma \Rightarrow t}{\Gamma \Rightarrow s \to t}\ (\Rightarrow\to)$$

Also, the rules of $(\backslash \Rightarrow)$ and $(/ \Rightarrow)$ are merged into:

$$\frac{\Gamma \Rightarrow s \qquad t, \Delta \Rightarrow \sigma}{\Gamma, \Delta, s \to t \Rightarrow \sigma}\ (\to\Rightarrow)$$

All the intuitionistic (i.e., single-succedent) sequent calculi discussed in this section (and Chapter 3) are built on a *commutative intuitionistic language*, i.e., a language that contains $\to$ instead of $\backslash$ and $/$, rather than a noncommutative intuitionistic language, described in the previous section. Below is a nonexhaustive list of commutative intuitionistic languages:

$$\begin{array}{lcllcl}
\mathcal{L}_{\mathsf{FL}_\mathsf{e}^+} & := & \{\circ, \to, \wedge, \vee, 1\} & \qquad \mathcal{L}_{\mathsf{ILL}} & := & \{\circ, \to, \wedge, \vee, 1, \Box\} \\
\mathcal{L}_{\mathsf{FL}_\mathsf{e}} & := & \{\circ, \to, \wedge, \vee, 1, 0\} & \qquad \mathcal{L}_{\mathsf{ILZ}} & := & \{\circ, \to, \wedge, \vee, 1, 0, \Box\}
\end{array}$$

Let $\mathcal{K}$ be a commutative intuitionistic language. We construct *commutative intuitionistic $\mathcal{K}$-terms* from variables in V using connective symbols from $\mathcal{K}$ in the obvious way. A *commutative intuitionistic $\mathcal{K}$-sequent* is an expression of the form $\Gamma \Rightarrow \sigma$ where Γ is a finite multiset (not finite sequence) of intuitionistic $\mathcal{K}$-terms and σ is an intuitionistic $\mathcal{K}$-term or is the empty multiset, denoted by ε. A commutative intuitionistic $\mathcal{K}$-terms (resp. commutative intuitionistic $\mathcal{K}$-sequents) is merely referred to as an intuitionistic $\mathcal{K}$-term (resp. intuitionistic $\mathcal{K}$-sequent) whenever it is clear from the context that $\mathcal{K}$ is a commutative language.

When discussing about commutative logical systems, we always assume the following.

$$\frac{\Gamma \Rightarrow s \qquad s, \Delta \Rightarrow \sigma}{\Gamma, \Delta \Rightarrow \sigma}\,(\mathrm{cut}) \qquad \frac{}{s \Rightarrow s}\,(\mathrm{init})$$

$$\frac{\Gamma \Rightarrow \sigma}{1, \Gamma \Rightarrow \sigma}\,(1 \Rightarrow) \qquad \frac{}{\Rightarrow 1}\,(\Rightarrow 1)$$

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\,(\Rightarrow 0) \qquad \frac{}{0 \Rightarrow}\,(0 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow s \qquad t, \Delta \Rightarrow \sigma}{s \to t, \Gamma, \Delta \Rightarrow \sigma}\,(\to\Rightarrow) \qquad \frac{s, \Gamma \Rightarrow t}{\Gamma \Rightarrow s \to t}\,(\Rightarrow\to)$$

$$\frac{s, t, \Gamma \Rightarrow \sigma}{s \circ t, \Gamma \Rightarrow \sigma}\,(\circ \Rightarrow) \qquad \frac{\Gamma \Rightarrow s \qquad \Delta \Rightarrow t}{\Gamma, \Delta \Rightarrow s \circ t}\,(\Rightarrow \circ)$$

$$\frac{s_i, \Gamma \Rightarrow \sigma}{s_1 \wedge s_2, \Gamma \Rightarrow \sigma}\,(\wedge \Rightarrow),\ \text{where } i = 1, 2 \qquad \frac{\Gamma \Rightarrow s \qquad \Gamma \Rightarrow t}{\Gamma \Rightarrow s \wedge u}\,(\Rightarrow \wedge)$$

$$\frac{s, \Gamma \Rightarrow \sigma \qquad t, \Gamma \Rightarrow \sigma}{s \vee t, \Gamma \Rightarrow \sigma}\,(\vee \Rightarrow) \qquad \frac{\Gamma \Rightarrow s_i}{\Gamma \Rightarrow s_1 \vee s_2}\,(\Rightarrow \vee),\ \text{where } i = 1, 2$$

Figure 2.4: Inference rules of $\mathsf{FL_e}$ without the rule of explicit exchange; the metavariables s, s_1, s_2, t range over intuitionistic $\mathcal{L}_{\mathsf{FL_e}}$-terms, Γ, Δ over finite multisets of intuitionistic $\mathcal{L}_{\mathsf{FL_e}}$-terms, σ denotes an $\mathcal{L}_{\mathsf{FL_e}}$-term or the empty multiset.

- Greek capital letters $\Gamma, \Delta, \Sigma, \dots$ stand for finite multiset of terms.
- The expression "Γ, Δ" denotes the multiset sum of Γ and Δ.

For instance, we can reformulate $\mathsf{FL_e}$ as a single-succedent sequent calculus over the language $\mathcal{L}_{\mathsf{FL_e^+}}$. It consists of the inference rules depicted in Figure 2.4. The treatment of commutative substructural logics that we have described in this subsection is standard in the linear logic community; see, e.g., [96, Appendix].

2.2.2 Sequent calculus for intuitionistic linear logic

In this subsection, we review various variants of intuitionistic linear logic. Our exposition relies mainly on several papers on such systems: i.e., Troelstra [121]; Okada and Terui [96]; Danos and Joinet [34]; Coppola and Martini [29, 30]; Asperti, Coppola, and Martini [5]; and Lazić and Schmitz [78].

From the previous section, we recall the commutative intuitionistic language $\mathcal{L}_{\mathsf{ILZ}}$. Given a finite multiset $\Gamma = s_1, \dots, s_n$ of intuitionistic $\mathcal{K}$-terms, where $\Box \in \mathcal{K} \subseteq \mathcal{L}_{\mathsf{ILZ}}$, we write $\Box\Gamma$ for the multiset $\Box s_1, \ldots, \Box s_n$. (If Γ is empty, so is $\Box\Gamma$.) *Intuitionistic linear logic with zero* (denoted by ILZ) [121] is an intuitionistic sequent calculus over the language $\mathcal{L}_{\mathsf{ILZ}}$. The inference rules of ILZ consists of those of $\mathsf{FL_e}$ (see Figure 2.4) and the following additional rules for *exponential modality*[4] "$\Box$."

$$\frac{s, \Gamma, \Rightarrow \sigma}{\Box s, \Gamma, \Rightarrow \sigma}\,(\Box \Rightarrow) \qquad\qquad \frac{\Box\Gamma \Rightarrow s}{\Box\Gamma \Rightarrow \Box s}\,(\Rightarrow \Box)$$

$$\frac{\Gamma \Rightarrow \sigma}{\Box s, \Gamma \Rightarrow \sigma}\,(\Box \mathsf{i}) \qquad\qquad \frac{\Box s, \Box s, \Gamma \Rightarrow \sigma}{\Box s, \Gamma \Rightarrow \sigma}\,(\Box \mathsf{c})$$

Intuitionistic elementary linear logic with zero (IEZ, for short) [34] is also a sequent calculus over the language $\mathcal{L}_{\mathsf{ILZ}}$. It is obtained from $\mathsf{FL_e}$ by adding the rules of ($\Box$i) and ($\Box$c), and the following rule, called *functorial rule* in [33]:

$$\frac{\Gamma \Rightarrow s}{\Box\Gamma \Rightarrow \Box s}\,(\mathrm{fct})$$

Moreover, we review some important extensions of ILZ and IEZ. *Intuitionistic affine logic with zero* ($\mathsf{ILZ_i}$) [96, 78] is the sequent calculus obtained from $\mathsf{ILZ_i}$ by adding the rule of left-weakening (i). The sequent calculus $\mathsf{ILZ_i}$ is equivalent to ILZW in [78], which is confusingly different from the logic $\mathsf{ILZ_w}$ defined below. Similarly, *intuitionistic elementary affine logic with zero* ($\mathsf{IEZ_i}$, for short; see, e.g., [5, 29, 30]) is the extension of IEZ by the rule of (i). We also introduce a natural extension of $\mathsf{FL_{ew}}$ by the exponential modality. The notation $\mathsf{ILZ_w}$ stands for the extension of ILZ by the rules of left- and right-weakening rules, i.e., the rules of (i) and (o).[5] We stress that there is also an unfortunate notational conflict between this thesis and [117]; the logic $\mathsf{ILZ_w}$ is equivalent to the system, denoted by $\mathbf{ILZW'}$ in [117]. The notion of deducibility relation over each of the logics ILZ, $\mathsf{ILZ_i}$, $\mathsf{ILZ_w}$, IEZ, and $\mathsf{IEZ_i}$ is defined in the obvious way.

Let L be one of the sequent calculi for ILZ, $\mathsf{ILZ_i}$, $\mathsf{ILZ_w}$, IEZ and $\mathsf{IEZ_i}$, and $\mathcal{K}$ be a sublanguage of $\mathcal{L}_{\mathsf{ILZ}}$. The $\mathcal{K}$*-fragment* of L means the sequent calculus that is comprised only of the inference rules of L concerning the connectives from $\mathcal{K}$. For instance, the $\{\circ, \to, \Box\}$-fragment of $\mathsf{ILZ_i}$ is comprised of the inference rules of (init), (cut), (i), ($\circ \Rightarrow$), ($\Rightarrow \circ$), ($\to\Rightarrow$), ($\Rightarrow\to$), ($\Box \Rightarrow$), ($\Rightarrow \Box$), ($\Box$i), ($\Box$c). We provide in Figure 2.5 a list of various fragments of ILZ, $\mathsf{ILZ_i}$, $\mathsf{ILZ_w}$, IEZ and $\mathsf{IEZ_i}$. Of course, we can naturally define the

[4]The name of *exponential modality* comes from linear logic [48].

[5]For our purposes in Chapter 4, we reformulate $\mathsf{ILZ_w}$ in Chapter 4 as a sequent calculus with the rule of explicit exchange.

Notation	Description	Name
BCI	the $\{\to\}$-fragment of ILZ	*BCI-logic* (e.g., [66])
BCK	the $\{\to\}$-fragment of $\mathsf{ILZ_i}$	*BCK-logic* (e.g.,[66])
ILL	the $\mathcal{L}_{\mathsf{ILL}}$-fragment of ILZ	*intuitionistic linear logic without* 0 (e.g., [121, 96])
$\mathsf{ILL_i}$	the $\mathcal{L}_{\mathsf{ILL}}$-fragment of $\mathsf{ILZ_i}$	*intuitionistic affine logic without* 0 (e.g., [96, 71])
IEL	the $\mathcal{L}_{\mathsf{ILL}}$-fragment of IEZ	*intuitionistic elementary linear logic without* 0 (e.g., [50, 34])
$\mathsf{IEL_i}$	the $\mathcal{L}_{\mathsf{ILL}}$-fragment of $\mathsf{IEZ_i}$	*intuitionistic elementary affine logic without* 0 (e.g., [5, 29, 30])
$\mathsf{FL}^+_{\mathsf{ei}}$	the $\mathcal{L}_{\mathsf{FL}^+_{\mathsf{e}}}$-fragment of $\mathsf{ILZ_i}$	*the positive fragment of* $\mathsf{FL_{ei}}$ (e.g., [40, 39])
$\mathsf{FL_{ei}}$	the $\mathcal{L}_{\mathsf{FL_e}}$-fragment of $\mathsf{ILZ_i}$	*full Lambek calculus with* (e) *and* (i) (e.g., [40, 39])
$\mathsf{FL_{ew}}$	the $\mathcal{L}_{\mathsf{FL_e}}$-fragment of $\mathsf{ILZ_w}$	*full Lambek calculus with* (e), (i), *and* (o) (e.g., [40, 39])

Figure 2.5: Typical fragments of ILZ, $\mathsf{ILZ_i}$, $\mathsf{ILZ_w}$, IEZ and $\mathsf{IEZ_i}$.

deducibility relations in these fragments.

2.2.3 Sequent calculus for classical linear logic

For our purposes in Chapter 3, we also review a right-hand sided calculi for *classical linear logic* (LL) and its variants, mainly following Girard [48, 49, 50], Lafont [74], and Dal Lago and Martini [33]. Although the argument here is still based on the fixed set V of variables, we also introduce the disjoint bijective copy V^- of V, i.e., $V^- = \{x^-, y^-, z^-, \ldots\}$. In the context of right-hand sided systems, we sometimes call elements of V^- (resp. V) *negative* (resp. *positive*) variables. The language $\mathcal{L}_{\mathsf{LL}}$ of LL consists of logical connectives $\circ$, $+$, $\wedge$, $\vee$, 1, 0, $\Box$, $\Diamond$ of arity 2, 2, 2, 2, 0, 0, 1, 1, respectively.

Let $\mathcal{K}$ be a sublanguage of $\mathcal{L}_{\mathsf{LL}}$. Terms over $\mathcal{K}$ (i.e., classical $\mathcal{K}$-terms) are defined in a slightly different way from intuitionistic terms, which we have described in the previous subsection. *Classical $\mathcal{K}$-terms* are built from positive and negative variables by using logical connectives in $\mathcal{K}$. For example, the set of classical $\{\circ, +, \Box, \Diamond\}$-terms is generated as follows:

- Positive variables $x, y, z, \ldots$ are classical $\{\circ, +, \Box, \Diamond\}$-terms.
- Negative variables $x^-, y^-, z^-, \ldots$ are classical $\{\circ, +, \Box, \Diamond\}$-terms.
- If s and t are classical $\{\circ, +, \Box, \Diamond\}$-terms, so are $s \circ t$, $s + t$, $\Box s$ and $\Diamond s$.

For any classical $\mathcal{L}_{\mathsf{LL}}$-term s, the classical $\mathcal{L}_{\mathsf{LL}}$-term s^- is defined by the standard *de Morgan duality*:

$$
\begin{array}{lllllll}
x^- & := & x^- & \qquad & (x^-)^- & := & x \\
(s \wedge t)^- & := & s^- \vee t^- & & (s \vee t)^- & := & s^- \wedge t^- \\
(s \circ t)^- & := & s^- + t^- & & (s + t)^- & := & s^- \circ t^- \\
(\Box s)^- & := & \Diamond s^- & & (\Diamond s)^- & := & \Box s^- \\
(1)^- & := & 0 & & (0)^- & := & 1
\end{array}
$$

We can confirm by easy induction that $s = s^{--}$ for any classical $\mathcal{L}_{\mathsf{LL}}$-term s. A *classical $\mathcal{K}$-sequent* is an expression of the shape $\Rightarrow \Gamma$, where Γ is a finite multiset of classical $\mathcal{K}$-terms. The sequent calculus for LL is a right-hand sided sequent calculus over the language $\mathcal{L}_{\mathsf{LL}}$ and consists of the inference rules presented in Figure 2.6.

As is the case with intuitionistic linear logic, there are some important variants of LL. *Classical elementary linear logic* (ELL, for short) [50, 33] is the sequent calculus obtained from LL by removing the rules of ($\Box$) and ($\Diamond$) and by adding the following rule of (fct):

$$\frac{\Rightarrow \Gamma, s}{\Rightarrow \Diamond\Gamma, \Box s}\ (\mathrm{fct})$$

Classical affine logic [74, 71] (resp. *classical elementary affine logic* [33]) is merely LL (resp. ELL) with the right-hand sided version of the weakening rule:

$$\frac{\Rightarrow \Gamma}{\Rightarrow \Gamma, s}\ (\mathsf{w})$$

In what follows, $\mathsf{LL_w}$ (resp. $\mathsf{ELL_w}$) denotes classical affine logic (resp. *classical elementary affine logic*).

Given a sublanguage $\mathcal{K}$ of $\mathcal{L}_{\mathsf{LL}}$, the $\mathcal{K}$-fragments of LL, $\mathsf{LL_w}$, ELL, and $\mathsf{ELL_w}$ are also defined in almost the same way as the intuitionistic fragments. Following the notation commonly used in the substructural logic literature (see, e.g., [39]), we write $\mathsf{InFL_e}$ (resp. $\mathsf{InFL_{ew}}$) for the $\{\circ, +, \wedge, \vee, 1, 0\}$-fragment of LL (resp. $\mathsf{LL_w}$). In the linear logic community, the logic $\mathsf{InFL_e}$ (resp. $\mathsf{InFL_{ew}}$) is usually called the *multiplicative additive fragment of linear logic* (resp. *multiplicative additive fragment of affine logic*); see, e.g., [74, 96].

Let $\mathcal{K}$ be a sublanguage of $\mathcal{L}_{\mathsf{LL}}$, L be the $\mathcal{K}$-fragment of LL, $\mathsf{LL_w}$, ELL or $\mathsf{ELL_w}$, and Φ be a set of classical $\mathcal{K}$-terms. We write $\Phi \vdash_{\mathsf{L}} \Rightarrow \Gamma$ if $\Rightarrow \Gamma$ is

$$\frac{}{\Rightarrow s, s^-}\ (\text{init}) \qquad \frac{}{\Rightarrow 1}\ (1) \qquad \frac{\Rightarrow \Gamma, s \qquad \Rightarrow s^-, \Delta}{\Rightarrow \Gamma, \Delta}\ (\text{cut})$$

$$\frac{\Rightarrow \Gamma}{\Rightarrow \Gamma, 0}\ (0) \qquad \frac{\Rightarrow \Gamma, s \qquad \Rightarrow \Delta, t}{\Rightarrow \Gamma, \Delta, s \circ t}\ (\circ) \qquad \frac{\Rightarrow \Gamma, s, t}{\Rightarrow \Gamma, s + t}\ (+)$$

$$\frac{\Rightarrow \Gamma, s \qquad \Rightarrow \Gamma, t}{\Rightarrow \Gamma, s \wedge t}\ (\wedge) \qquad \frac{\Rightarrow \Gamma, s}{\Rightarrow \Gamma, s \vee t}\ (\vee_1) \qquad \frac{\Rightarrow \Gamma, t}{\Rightarrow \Gamma, s \vee t}\ (\vee_2)$$

$$\frac{\Rightarrow \Gamma, s}{\Rightarrow \Gamma, \Diamond s}\ (\Diamond) \qquad \frac{\Rightarrow \Diamond\Gamma, s}{\Rightarrow \Diamond\Gamma, \Box s}\ (\Box) \qquad \frac{\Rightarrow \Gamma}{\Rightarrow \Gamma, \Diamond s}\ (\Diamond\mathsf{w}) \qquad \frac{\Rightarrow \Gamma, \Diamond s, \Diamond s}{\Rightarrow \Gamma, \Diamond s}\ (\Diamond\mathsf{c})$$

Figure 2.6: Inference rules of LL; s, t denote classical $\mathcal{L}_{\mathsf{LL}}$-terms, and Γ, Δ denote finite multisets of classical $\mathcal{L}_{\mathsf{LL}}$-terms.

derivable in the right-hand sided sequent system obtained from L by adding all the initial sequents of the form $\Rightarrow s$, where $s \in \Phi$.

Cut elimination is the most important property of the sequent calculi described so far.

Fact 2.2.1 (See, e.g., [48, 121, 81, 93, 94, 95, 74, 96, 33])**.** *Each of* ILZ*,* $\mathsf{ILZ_i}$*,* $\mathsf{ILZ_w}$*,* IEZ*,* $\mathsf{IEZ_i}$*,* LL*,* ELL*,* $\mathsf{LL_w}$*, and* $\mathsf{ELL_w}$ *admits cut elimination. So does each of the sequent calculi listed in Figure 2.5.*

Proof. There are some methods to prove cut elimination for these systems. The reader may refer to Lincoln, Mitchell, Scedrov, and Shankar [81, Appendix A] for a standard Gentzen-style proof and to Okada [93, 94, 95] for an algebraic proof. □

The author is not aware of anyone who explicitly proved cut elimination for, e.g., $\mathsf{ILZ_w}$; however, it is not difficult for skilled linear and/or substructural logicians to show this claim. We omit the proof since it is unlikely to provide us with helpful insights.

2.3 Alternating branching vector addition systems with states

Our proofs in Chapters 3 and 4 depend heavily on the complexity results of *alternating branching vector addition systems with states and with full zero test rules* ($\mathsf{ABVASS}_{\bar{0}}$s, for short), introduced by Lazić and Schmitz [78]. In this section, we therefore review some notions concerning $\mathsf{ABVASS}_{\bar{0}}$, borrowing the notation and the terminology from [78, Section 3].

Let d be in $\mathbb{N}$. In what follows, we write $\bar{\mathsf{v}}_1, \bar{\mathsf{v}}_2, \ldots$ for d-dimensional vectors over $\mathbb{N}^d$. We write:

- $\bar{\mathsf{v}}(i)$ for the value of the i-th element of $\bar{\mathsf{v}}$,
- $\bar{\mathsf{e}}_i$ for the *i-th unit vector* in $\mathbb{N}^d$, i.e., the vector $\bar{\mathsf{v}}$ such that $\bar{\mathsf{v}}(i) = 1$ and $\bar{\mathsf{v}}(j) = 0$ for every j other than i,
- $\bar{0}$ for the *null vector*, i.e., vector with zero in every coordinate.

An *alternating branching vector addition system with states and full zero tests* ($\mathsf{ABVASS}_{\bar{0}}$, for short) [78] is a structure $\mathcal{M} = \langle Q, d, U, S, F, Z \rangle$ where:

- Q is a finite set,
- d is in $\mathbb{N}$,
- $U \subseteq Q \times \mathbb{Z}^d \times Q$ (where U is finite),
- $S \cup F \subseteq Q^3$, and $Z \subseteq Q^2$.

Following the usual convention in the context of vector addition systems, we call Q a *state space* and d a *dimension*. In terms of the operational semantics described below, each element of U (resp. S, F, Z) is referred to as a *unary* (resp. *split*, *fork*, *full zero test*) rule of $\mathcal{M}$. For brevity, we write $q \xrightarrow{\bar{\mathsf{u}}} q'$ for $(q, \bar{\mathsf{u}}, q') \in U$, $q \to q_1 \;\mathsf{F}\; q_2$ for $(q, q_1, q_2) \in F$, $q \to q_1 \;\mathsf{S}\; q_2$ for $(q, q_1, q_2) \in S$, and $q \xrightarrow{\mathsf{Z}} q'$ for $q \xrightarrow{\mathsf{Z}} q' \in Z$. A *configuration* of $\mathcal{M}$ is just an ordered pair of a state and a d-dimensional vector.

We furthermore describe how a given $\mathsf{ABVASS}_{\bar{0}}$ behaves under the standard operational semantics. Let $\mathcal{M} = \langle Q, d, U, S, F, Z \rangle$ be an $\mathsf{ABVASS}_{\bar{0}}$. The *standard operational semantics* for $\mathcal{M}$ is given by a deduction system over $Q \times \mathbb{N}^d$, depicted in Figure 2.7. Consider a subset $\mathscr{L}$ of Q. An $\mathscr{L} \times \{\bar{0}\}$-*leaf-covering deduction tree* in $\mathcal{M}$ is a finite binary tree T labeled by configurations of $\mathcal{M}$ satisfying the following conditions:

- every leaf of T is in $\mathscr{L} \times \{\bar{0}\}$,
- each node in T other than leaves is obtained from its children by an application of one of the deduction rules in $U \cup S \cup F \cup Z$.

The following figure denotes a deduction tree T with root $\langle q, \bar{\mathsf{v}} \rangle$:

$$\frac{\langle q; \bar{\mathsf{v}} \rangle}{\langle q'; \bar{\mathsf{v}} + \bar{\mathsf{u}} \rangle} \; (q \xrightarrow{\bar{\mathsf{u}}} q') \qquad\qquad \frac{\langle q; \bar{0} \rangle}{\langle q'; \bar{0} \rangle} \; (q \xrightarrow{\mathsf{Z}} q')$$

$$\frac{\langle q; \bar{\mathsf{v}} \rangle}{\langle q_1; \bar{\mathsf{v}} \rangle \qquad \langle q_2; \bar{\mathsf{v}} \rangle} \; (q \to q_1 \; \mathsf{F} \; q_2) \qquad\qquad \frac{\langle q, \bar{\mathsf{v}}_1 + \bar{\mathsf{v}}_2 \rangle}{\langle q_1, \bar{\mathsf{v}}_1 \rangle \qquad \langle q_2, \bar{\mathsf{v}}_2 \rangle} \; (q \to q_1 \; \mathsf{S} \; q_2)$$

Figure 2.7: The standard operational semantics for an $\mathsf{ABVASS}_{\bar{0}}$ $\mathcal{M} = \langle Q, d, U, S, F, Z \rangle$; here, $q \xrightarrow{\mathsf{Z}} q' \in Z$, $q \xrightarrow{\bar{\mathsf{u}}} q' \in U$, $q \to q_1 \; \mathsf{S} \; q_2 \in S$, and $q \to q_1 \; \mathsf{F} \; q_2 \in F$. The notation "+" stands for componentwise addition.

$$\begin{array}{c} \langle q; \bar{\mathsf{v}} \rangle \\ T \end{array}$$

By "$\mathcal{M}, \mathscr{L} \rightsquigarrow \langle q; \bar{\mathsf{v}} \rangle$," we mean that there exists an $\mathscr{L} \times \{\bar{0}\}$-leaf-covering deduction tree with root $\langle q; \bar{\mathsf{v}} \rangle$.

Remark 2.3.1. For those who are not familiar with vector addition systems and their extensions, we make a supplementary comment on $\mathsf{ABVASS}_{\bar{0}}$s. A usual (deterministic) counter machine of Minsky [90] is known to have three kinds of instructions; increment, decrement, and zero test. Given a configuration of a counter machine, these instructions produce exactly one configuration and therefore a computation of the machine is represented as a list of configurations. On the other hand, an $\mathsf{ABVASS}_{\bar{0}}$, an extension of a counter machine (without zero tests), has two distinguishing instructions, called fork and split rules. By an application of fork or split instructions, a configuration of an $\mathsf{ABVASS}_{\bar{0}}$ is *ramified* into two configurations. Accordingly, a computation of an $\mathsf{ABVASS}_{\bar{0}}$ is represented, not as a list of configurations, but as a binary tree structure whose nodes are configurations. In this sense, a computation of an $\mathsf{ABVASS}_{\bar{0}}$ is analogous to a derivation in sequent calculus. This, as we will see in Chapter 3, is more than just an analogy.

Besides, fork instructions work similarly to additive inference rules of sequent calculus like $(\Rightarrow \wedge)$ and $(\vee \Rightarrow)$, and split instructions work similarly to multiplicative inference rules of sequent calculus such as $(\Rightarrow \circ)$ and $(\to\Rightarrow)$. This insight leads to the reduction of Lazić and Schmitz [78], which allows us to carry out proof-search of linear logic in an $\mathsf{ABVASS}_{\bar{0}}$.[6]

Now we review another important semantics for $\mathsf{ABVASS}_{\bar{0}}$s. Given an $\mathsf{ABVASS}_{\bar{0}}$ $\mathcal{M} = \langle Q, d, U, S, F, Z \rangle$, the *lossy operational semantics* for $\mathcal{M}$

[6]Full zero test rules are used for simulating some modal inference rules requiring the side condition.

is obtained from the standard operational semantics for $\mathcal{M}$ by adding the following new deduction rule, defined independently of the syntax of $\mathcal{M}$:

$$\frac{\langle q; \bar{\mathsf{v}} + \bar{\mathsf{e}}_i \rangle}{\langle q; \bar{\mathsf{v}} \rangle} \ (\mathsf{loss})$$

for every $q \in Q$ and for every $1 \leq i \leq d$. Given a subset $\mathscr{L}$ of Q, the notion of $\mathscr{L} \times \{\bar{0}\}$-leaf-covering lossy deduction tree in $\mathcal{M}$ is defined in the obvious way. Similarly, we use the expression "$\mathcal{M}, \mathscr{L} \rightsquigarrow_\ell \langle q; \bar{\mathsf{v}} \rangle$" if there is an $\mathscr{L} \times \{\bar{0}\}$-leaf-covering lossy deduction tree with root $\langle q; \bar{\mathsf{v}} \rangle$.

2.4 Fast-growing complexity classes

The purpose of this section is to review some basic facts about the fast-growing complexity hierarchy $(\mathbf{F}_\alpha)_\alpha$ of Schmitz [105]. For the reader who works (or will work) on his paper [105], we employ the same notations as those in [105] as much as possible.

2.4.1 Ordinal terms and fast-growing functions

We begin with the basics on ordinal terms; the exposition below depends on [108, Appendix C], [109, Appendices A.1 and A.2] and [54, Section 4.1]. Every ordinal below ϵ_0, the smallest solution of the equation $\alpha = \omega^\alpha$, can be represented as an *ordinal term* built up from the following Backus Naur Form grammar:

$$\alpha, \beta ::= 0 \mid \omega^\alpha \mid \alpha + \beta.$$

We write $\alpha.n$ as a shorthand for $\overbrace{\alpha + \cdots + \alpha}^{n \text{ times}}$. For brevity, we often write 1 for ω^0 and ω for ω^1. While each ordinal term α is a purely syntactic object in itself, it is associated with the (set-theoretic) ordinal $o(\alpha)$ in an obvious way; however, the map $\alpha \mapsto o(\alpha)$ is not injective. We may define a well-founded quasi-ordering over ordinal terms by setting $\alpha \leq \beta$ if $o(\alpha) \leq o(\beta)$.

More to the point, ordinals below ϵ_0 are uniquely written as ordinal terms in *Cantor normal form* (CNF, for short):

$$\alpha = \omega^{\beta_1} + \cdots + \omega^{\beta_k}$$

where $\alpha > \beta_1 \geq \cdots \geq \beta_k \geq 0$ and all β_i's are in CNF. In the representation:

- $\alpha = 0$ if $k = 0$.
- If $k > 0$ and $\beta_k = 0$, α is called a *successor*.

► If $k > 0$ and $\beta_k > 0$, α is called a *limit*.

Every limit in CNF is of the form either $\beta + \omega^{\gamma+1}$ or $\beta + \omega^{\lambda}$, where λ is a limit.

We write $\mathrm{CNF}(\epsilon_0)$ for the set of ordinal terms in CNF. Throughout the following discussion, by *ordinal terms* we mean those in $\mathrm{CNF}(\epsilon_0)$, which is in bijection with set-theoretic ordinals $< \epsilon_0$.

Given a limit ordinal term λ, we assign to it a particular *fundamental sequence* $\lambda[0] < \lambda[1] < \cdots < \lambda[n] < \cdots$ with supremum λ. It is defined inductively as follows:

$$(\beta + \omega^{\gamma+1})[n] := \beta + \omega^{\gamma}.(n+1), \qquad (\beta + \omega^{\lambda'})[n] := \beta + \omega^{\lambda'[n]},$$

where λ' is a limit. For instance, the fundamental sequence for ω^2 is:

$$\omega < \omega.2 < \omega.3 < \omega.4 \cdots .$$

Following, e.g., Schmitz [105, Section 2.2], we next review a variant of the *fast-growing functions* of Löb and Wainer [85] by defining $F_\alpha \colon \mathbb{N} \to \mathbb{N}$ for each ordinal term α.

$$\begin{aligned}
F_0(n) &:= n+1, \\
F_{\alpha+1}(n) &:= F_\alpha^{\omega[n]}(n) = \underbrace{F_\alpha(F_\alpha(\cdots(F_\alpha}_{n+1 \text{ times}}(n))\cdots)), \\
F_\lambda(n) &:= F_{\lambda[n]}(n),
\end{aligned}$$

where λ is a limit.

Clearly, F_0 and F_1 are linear functions. The map F_2 grows exponentially, i.e., $F_2 = 2^{n+1}(n+1) - 1$. The function F_3 is no longer elementary recursive; indeed, the growth rate of F_3 goes beyond that of the tower of exponentials. The diagonal function F_ω (resp. F_{ω^ω}) is a sort of Ackermannian function (resp. hyper-Ackermannian function), which dominates every primitive recursive (resp. multiply recursive) function.

We furthermore review the well-known *extended Grzegorczyk hierarchy* $(\mathscr{F}_\alpha)_\alpha$ [85]. Instead of introducing this hierarchy through recursion schemes, we define it from the complexity-theoretic point of view; see [105, Section 2.2] and [125] for details. For an ordinal term $\alpha \geq 2$, let $\mathscr{F}_\alpha$ (resp. $\mathscr{F}^*_\alpha$) denote the set of functions (resp. problems) computable in time $O(F_\alpha^k(n))$ for some k, i.e.,

$$\mathscr{F}_\alpha := \bigcup_{k<\omega} \mathrm{FDT{\small IME}}(F_\alpha^k(n)), \qquad \mathscr{F}_\alpha^* := \bigcup_{k<\omega} \mathrm{DT{\small IME}}(F_\alpha^k(n)).$$

Moreover, we define:

$$\mathscr{F}_{<\alpha} := \bigcup_{\gamma<\alpha} \mathscr{F}_\gamma, \qquad \mathscr{F}_{<\alpha}^* := \bigcup_{\gamma<\alpha} \mathscr{F}_\gamma^*.$$

In particular, $\mathscr{F}_{<3} = \mathscr{F}_2$ (resp. $\mathscr{F}_{<\omega}$, $\mathscr{F}_{<\omega^\omega}$) coincides with the set FElem (resp. FPR, FMR) of elementary recursive (resp. primitive recursive, multiply recursive) functions; see [105, Section 2.2] and [125] for details. The reader may also consult [92, Sections VIII.7–9] for general background on elementary recursive, primitive recursive, and multiply recursive functions.

2.4.2 Ordinal-indexed hierarchy of fast-growing complexity classes

Now we are ready to define the *fast-growing complexity hierarchy* $(\mathbf{F}_\alpha)_{\alpha\in\mathrm{CNF}(\epsilon_0)}$ of Schmitz [105, Section 2.3]:

$$\mathbf{F}_\alpha := \bigcup_{f\in\mathscr{F}_{<\alpha}} \mathrm{DT{\small IME}}(F_\alpha(f(n))).$$

In particular,

$$\begin{array}{lclcl}
\mathbf{F}_3 & = & \mathrm{T{\small OWER}} & := & \bigcup_{e\in\mathscr{F}_{<3}=\mathrm{FE{\small LEM}}} \mathrm{DT{\small IME}}(F_3(e(n))), \\
\mathbf{F}_\omega & = & \mathrm{A{\small CK}} & := & \bigcup_{p\in\mathscr{F}_{<\omega}=\mathrm{FPR}} \mathrm{DT{\small IME}}(F_\omega(p(n))), \\
\mathbf{F}_{\omega^\omega} & = & \mathrm{HA{\small CK}} & := & \bigcup_{m\in\mathscr{F}_{<\omega^\omega}=\mathrm{FMR}} \mathrm{DT{\small IME}}(F_{\omega^\omega}(m(n))).
\end{array}$$

Let Elementary (resp. Primitive-Recursive, Multiply-Recursive) denote the set of problems decidable by a deterministic Turing machine running in time bounded by an elementary recursive (resp. primitive recursive, multiply recursive) function. Below, we summarize some characteristics of the hierarchy of Schmitz. The following two lemmas say that his high-rise building is strict, i.e., does not collapse at any level. This is in sharp contrast to the situation that nobody is aware of whether the polynomial hierarchy $(\Sigma_k^{\mathrm{P}}, \Pi_k^{\mathrm{P}}, \Delta_k^{\mathrm{P}})_{k\in\mathbb{N}}$ of Meyer and Stockmeyer [88] is strict.

Fact 2.4.1 (Schmitz [105], Proposition 5.2 and Theorem 5.3)**.** *For each* $2 \leq \alpha < \beta$*:*

$$\mathbf{F}_\alpha \subsetneq \mathbf{F}_\beta,$$

and

$$\mathscr{F}^*_\alpha \subsetneq \mathbf{F}_\beta.$$

In particular:

$$\text{Elementary} = \mathscr{F}^*_{<3} = \mathscr{F}^*_2 \subsetneq \mathbf{F}_3.$$

Fact 2.4.2 (Schmitz [105], Corollary 5.4)**.** *For each limit* λ*,*

$$\mathscr{F}^*_{<\lambda} = \bigcup_{\gamma<\lambda} \mathbf{F}_\gamma \subsetneq \mathbf{F}_\lambda.$$

In particular,

$$\text{Primitive-Recursive} = \mathscr{F}^*_{<\omega} = \bigcup_{\alpha<\omega} \mathbf{F}_\alpha \subsetneq \mathbf{F}_\omega,$$

$$\text{Multiply-Recursive} = \mathscr{F}^*_{<\omega^\omega} = \bigcup_{\alpha<\omega^\omega} \mathbf{F}_\alpha \subsetneq \mathbf{F}_{\omega^\omega}.$$

In summary, the following inclusion relations hold:

Fact 2.4.3 (Schmitz [105], Section 5.2)**.**

$$\bigcup_{k<\omega} k\text{-ExpTime} = \text{Elementary} \subsetneq \text{Tower} = \mathbf{F}_3$$

$$\mathbf{F}_3 \subsetneq \mathbf{F}_4 \subsetneq \cdots \subsetneq \mathbf{F}_k \subsetneq \mathbf{F}_{k+1} \subsetneq \cdots \subsetneq \bigcup_{\alpha<\omega} \mathbf{F}_\alpha = \text{Primitive-Recursive}$$

$$\text{Primitive-Recursive} \subsetneq \text{Ack} = \mathbf{F}_\omega \subsetneq \mathbf{F}_{\omega+1} \subsetneq \cdots \subsetneq \mathbf{F}_{\omega.2} \subsetneq \cdots \subsetneq \mathbf{F}_{\omega^2}$$

$$\mathbf{F}_{\omega^2} \subsetneq \cdots \subsetneq \mathbf{F}_{\omega^k} \subsetneq \mathbf{F}_{\omega^{k+1}} \subsetneq \cdots \subsetneq \bigcup_{\alpha<\omega^\omega} \mathbf{F}_\alpha = \text{Multiply-Recursive}$$

$$\text{Multiply-Recursive} \subsetneq \mathbf{F}_{\omega^\omega} = \text{HAck}.$$

Another important property of $(\mathbf{F}_\alpha)_\alpha$ (for $\alpha \geq 3$) is the property of *robustness*. In particular, we obtain the same complexity class as $\mathbf{F}_\alpha$ for each $\alpha \geq 3$ even when we replace DTime with NTime or DSpace; see [105, Section 4.2.1] for more on this subject.

To conclude this section, we define the notions of reduction and completeness, following [105, Section 4.2]. Let Σ_1, Σ_2 be alphabets and X, Y be languages over Σ_1, Σ_2, i.e., subsets of Σ_1^*, Σ_2^*, respectively; here, Σ_i^* denotes the set of finite strings over Σ_i. Given a function $f \colon \mathbb{N} \to \mathbb{N}$, we say that X is *many-one reducible to* Y *in time* f if there is a map $g \colon \Sigma_1^* \to \Sigma_2^*$ such that:

- ► g is computable by a deterministic Turing machine that runs within a number of steps bounded by f in the input size;
- ► $x \in X$ if and only if $g(x) \in Y$ for every $x \in \Sigma_1^*$.

We also say that X is *many-one $\mathscr{F}_{<\alpha}$-reducible* (*$\mathscr{F}_{<\alpha}$-reducible*, for short) to Y if X is many-one reducible to Y in time f for some f in $\mathscr{F}_{<\alpha}$. A class $\mathcal{C}$ of problems is said to be *closed under $\mathscr{F}_{<\alpha}$ reductions* if for every language X and Y: X is in $\mathcal{C}$ whenever (i) Y is in $\mathcal{C}$ and (ii) X is $\mathscr{F}_{<\alpha}$-reducible to Y.

The main results in Section 3.2 are grounded on the following closure property:

Fact 2.4.4 (Schmitz [105], Theorem 4.7)**.** *For each α, $\mathbf{F}_\alpha$ is closed under many-one $\mathscr{F}_{<\alpha}$-reductions. Hence:*

- ► *The class* TOWER *is closed under many-one elementary (i.e., $\mathscr{F}_{<3}$) reductions.*
- ► *The class* ACK *is closed under many-one primitive recursive (i.e., $\mathscr{F}_{<\omega}$) reductions.*
- ► *The class* HACK *is closed under many-one multiply recursive (i.e., $\mathscr{F}_{<\omega^\omega}$) reductions.*

As in [105], we define the notion of TOWER-hardness (resp. ACK-hardness, HACK-hardness) with respect to elementary recursive (resp. primitive recursive, multiply recursive) many-one reductions.[7] It can be easily seen that TOWER-complete (resp. ACK-complete, HACK-complete) problems belong to $\mathbf{F}_3 \setminus \mathscr{F}^*_{<3}$ (resp. $\mathbf{F}_\omega \setminus \mathscr{F}^*_{<\omega}$, $\mathbf{F}_{\omega^\omega} \setminus \mathscr{F}^*_{<\omega^\omega}$).

2.5 Decision problems beyond ELEMENTARY

There are many TOWER-, ACK-, and HACK-complete problems in various fields of theoretical computer science; the reader can find in [105, Section 6] a nice catalog of such nonelementary problems. From the realms of nonclassical logic and vector additive systems, we select a few such decision problems that are necessary for our purposes.

[7] It is not necessary to adhere to many-one $\mathscr{F}_{<\alpha}$-reductions; for $\alpha \geq 3$, the notion of $\mathbf{F}_\alpha$-hardness can also be defined via, for instance, *$\mathscr{F}_{<\alpha}$-Turing reductions*; see [105, Section 4.2.3]. However, we consider only many-one reductions throughout the thesis.

2.5.1 Decision problems for logics

Although there are many variations of decision problems for a logical system (e.g., the rule admissibility problem [104, 67] and the cut elimination problem [112, 86]), we restrict our focus to the following two types of problems:

Decision problem (The provability problem for L; see, e.g., Section 1.5.2 and Chapter 4 of [39])**.** *The provability problem for a logic* L *consists of the following:*

- *Input: A term* s.
- *Question: Does it hold that* $\vdash_{\mathsf{L}} s$*?*

Decision problem (The deducibility problem for L; see, e.g., Chapter 4 of [39])**.** *The deducibility problem for a logic* L *consists of the following:*

- *Input: A single term* s *and a finite set* T *of terms.*
- *Question: Does it hold that* $T \vdash_{\mathsf{L}} s$*?*

Some examples of Tower-, Ack-, and HAck-complete problems were discovered in the contexts of linear and substructural logic. The following prominent results have decisive roles in the proofs of our main results.

Fact 2.5.1 (Lazić and Schmitz [78], Corollary 5.4 and Fact 4.2)**.** *The provability problems for* $\mathsf{ILZ_i}$ *and* $\mathsf{LL_w}$ *are* Tower*-complete.*

Fact 2.5.2 (Urquhart [122], Sections 9 and 11)**.** *The provability problem for (the* 0*-free fragment of)* $\mathsf{FL_{ec}}$ *is* Ack*-complete.*

Fact 2.5.3 (Greati and Ramanayake [51], Theorem 1)**.** *The deducibility problems for* $\mathsf{FL_w}$ *and* $\mathsf{FL_i}$ *are* HAck*-complete.*

2.5.2 Decision problems for counter systems

In Section 2.3, we have reviewed two types of operational semantics for $\mathsf{ABVASS}_{\bar{0}}$s, i.e., the standard semantics and the lossy semantics. Accordingly, we consider two sorts of decision problems for $\mathsf{ABVASS}_{\bar{0}}$s.

Decision problem (Reachability in $\mathsf{ABVASS}_{\bar{0}}$s; Section 3.2.1 of [78])**.** *The reachability problem for* $\mathsf{ABVASS}_{\bar{0}}$*s is given as follows:*

- *Input: An* $\mathsf{ABVASS}_{\bar{0}}$ $\mathcal{M} = \langle Q, d, U, S, F, Z\rangle$*, a subset* $\mathscr{L}$ *of* Q*, and a state* q_{root} *in* Q.

- *Question: Does it hold that $\mathcal{M}, \mathscr{L} \rightsquigarrow \langle q_{\mathsf{root}}, \bar{0}\rangle$?*

Simulating zero test instructions of *Minsky machines* [90] by fork rules, Lincoln, Mitchell, Scedrov, and Shankar [81] proved that the reachability problem for *alternating VASSes* (i.e., $\mathsf{ABVASS}_{\bar{0}}$s without split rules and full-zero-test rules) is undecidable; see also [78, Section 3.3.1] for more on this topic.[8]

Fact 2.5.4 (Lincoln, Mitchell, Scedrov, and Shankar [81], Lemma 3.4)**.** *Reachability in alternating VASSes is* RE*-complete. So is reachability in $\mathsf{ABVASS}_{\bar{0}}$s.*

One can also define a decision problem for $\mathsf{ABVASS}_{\bar{0}}$s with the lossy semantics.

Decision problem (Lossy reachability in $\mathsf{ABVASS}_{\bar{0}}$s; see Section 3.2.2 of [78])**.** *The lossy reachability problem for $\mathsf{ABVASS}_{\bar{0}}$s is given as follows:*

- *Input: An $\mathsf{ABVASS}_{\bar{0}}$ $\mathcal{M} = \langle Q, d, U, S, F, Z\rangle$, a subset $\mathscr{L}$ of Q, and a state q_{root} in Q.*
- *Question: Does it hold that $\mathcal{M}, \mathscr{L} \rightsquigarrow_{\ell} \langle q_{\mathsf{root}}, \bar{0}\rangle$?*

In sharp contrast to the undecidability of reachability in $\mathsf{ABVASS}_{\bar{0}}$s, Lazić and Schmitz [78] showed the following theorem, one of the most important facts for our main results.

Fact 2.5.5 (Lazić and Schmitz [78], Theorem 3.6)**.** *Lossy reachability in $\mathsf{ABVASS}_{\bar{0}}$s is* Tower*-complete.*

In [78], it is also shown that the lossy reachability problem remains Tower-complete even when we impose a great restriction on $\mathsf{ABVASS}_{\bar{0}}$s. An $\mathsf{ABVASS}_{\bar{0}}$ $\mathcal{M} = \langle Q, d, U, S, F, Z\rangle$ is said to be *ordinary* if for every unary rule $q \xrightarrow{\bar{\mathsf{u}}} q'$ in U, the vector $\bar{\mathsf{u}}$ is of the form either $\bar{\mathsf{e}}_i$ or $-\bar{\mathsf{e}}_i$ for some $i \in \{1, \ldots, d\}$; see [78, Section 3.4.2] for details. A *branching vector addition system with states* (BVASS, for short) [35, 78, 106] is an $\mathsf{ABVASS}_{\bar{0}}$ equipped only with unary and split rules. Branching VASSes are also known as *vector addition tree automata*, investigated by de Groote, Guillaume, and Salvati [35]. Our proofs in Section 3.3 depend on the following:

Fact 2.5.6 (Lazić and Schmitz [78], Lemma 3.5 and Theorem 3.6)**.** *Lossy reachability in (ordinary) BVASSes is* Tower*-complete.*

[8] Alternating VASSs are called *and-branching counter machines without zero tests* in [81].

As a side remark, it is, to date, unknown whether reachability (not lossy-reachability) in BVASSes is decidable. As observed in [35], this is closely related to a famous open question in linear logic; see also Chapter 5 for details.

Chapter 3

TOWER-complete problems in logics without contraction

> Every kind of human knowledge has unanswerable questions at its end: and until the insoluble problems are revealed, we plume ourselves upon our progress.
>
> Augustus De Morgan
> *On Infinity; and on the Sign of Equality*, 1871

In this chapter, we establish the tight TOWER complexity for some substructural logics that lack the rule of (c). To be precise, we show that the following decision problems are TOWER-complete:

- the deducibility problems for $\mathsf{FL_{ew}}$ and related systems,
- the provability problems for $\mathsf{ELL_{w}}$ and related systems.

As we have cautioned in the previous chapter, all the intuitionistic logical systems in this chapter are based on commutative languages.

Organization of Chapter 3. In Section 3.1, we define a negative translation of classical linear logic into intuitionistic linear logic. Our translation is a modification of the *parametric negative translation*, provided by Laurent [76]. In order to establish various complexity results, we repeatedly use the translation in Sections 3.2 and 3.3.

In Section 3.2, we show that various decision problems for contraction-free logics are in TOWER. In Subsection 3.2.1, we show that there is an (elementary time) reduction from provability in $\mathsf{IEL_i}$ to lossy reachability in $\mathsf{ABVASS}_{\bar{0}}$s. Our reduction is just a slightly modified version of Lazić and Schmitz's encoding [78] of provability in $\mathsf{ILZ_i}$ into lossy reachability in $\mathsf{ABVASS}_{\bar{0}}$s. Combining this reduction with the translation from Section 3.1, we also obtain the Tower-membership of $\mathsf{ELL_w}$-provability. In Subsection 3.2.2, we show that the deducibility problem for various modality-free fragments of affine logic (e.g., $\mathsf{FL_{ei}}$, $\mathsf{FL_{ew}}$) are decidable in Tower. However, it seems to be difficult to construct a direct reduction from each of these problems into the lossy reachability problem for $\mathsf{ABVASS}_{\bar{0}}$s. This is because: (i) a sequent calculus (e.g., $\mathsf{FL_{ew}}$) augmented with nonlogical axioms lacks cut elimination; (ii) our argument here depends critically on cut elimination. Therefore, in order to settle the Tower-membership of deducibility in $\mathsf{FL_{ew}}$, we take the following strategy.

1. We first show that there is an efficient translation from deducibility in $\mathsf{FL_{ew}}$ to provability in a sort of intuitionistic affine logic that admits cut elimination, i.e., $\mathsf{ILZ_w}$. This translation is merely an affine variant of the deduction theorem for full propositional linear logic, shown by Avron [6, Section 4.1].[1]

2. Following the idea of Lazić and Schmitz [78], we next construct a reduction of $\mathsf{ILZ_w}$-provability into $\mathsf{ABVASS}_{\bar{0}}$-lossy-reachability.

Combining Steps (1) and (2), we can show that deducibility in $\mathsf{FL_{ew}}$ is decidable in Tower, as desired.[2]

In Section 3.3, we show the Tower-hardness for the decision problems, shown in Section 3.2 to be decidable in Tower. To this end, we make use of an elegant encoding of ordinary BVASSes into classical linear logic. Whereas this encoding directly comes from Lazić and Schmitz [78], it originally dates back to the idea used by Lincoln, Mitchell, Scedrov, and Shankar [81] in the proof of the undecidability of LL-provability. In light of the encoding of Lazić and Schmitz, we check the Tower-hardness of the problem that asks whether a very restricted form of sequent (called *a* $\Diamond$*-prenex* $\{\circ, +\}$*-sequent* [118]) is provable in $\mathsf{LL_w}$ and prove that the intuitionistic version of that problem (i.e., the provability problem for $\Box$-prenex $\{\rightarrow\}$-sequents in the $\{\Box, \rightarrow\}$-fragment of $\mathsf{ILL_i}$) is also hard for Tower. We next show that provability of $\Box$-prenex $\{\rightarrow\}$-sequents is polynomial-time reducible to deducibility in some contraction-free logics, such as $\mathsf{FL_{ew}}$ and BCK, thus

[1] Lincoln, Mitchell, Scedrov, and Shankar [81, Section 3.2] also used the same translation in the undecidability proof of LL-provability.

[2] The same strategy works for other modality-free logics without the rule of (c).

obtaining the TOWER-hardness for those problems. In passing, we also show that some variants of elementary affine logic are TOWER-hard with respect to provability.

Our idea furthermore leads to a reduction from ABVASS-reachability to provability in (some variants of) elementary linear logic. As a side effect, we also show that the provability problems for ELL and IEL are undecidable, i.e., RE-complete.

3.1 Simple translation of classical linear logic to intuitionistic linear logic

The aim of this section is to provide a polynomial-time translation from LL-provability (resp. $\mathsf{LL_w}$-provability, ELL-provability, $\mathsf{ELL_w}$-provability) into ILL-provablity (resp. $\mathsf{ILL_i}$-provability, IEL-provability, $\mathsf{IEL_i}$-provability). To this end, we modify the *parametric negative translation* by Laurent; see [76, Definition 2.2]. Our proof below is also very similar to Laurent's proof; see [76, Section 2].

Let u be an intuitionistic $\mathcal{L}_{\mathsf{ILZ}}$-term and s be an classical $\mathcal{L}_{\mathsf{LL}}$-term. In what follows, we write $\neg_u t$ for $t \to u$. We define the intuitionistic $\mathcal{L}_{\mathsf{ILZ}}$-term $s^{\langle u \rangle}$ as follows:

$$
\begin{array}{lcl@{\qquad\qquad}lcl}
x^{\langle u\rangle} & := & \neg_u x & (x^-)^{\langle u\rangle} & := & x \\
1^{\langle u\rangle} & := & \neg_u 1 & 0^{\langle u\rangle} & := & 1 \\
(s\circ t)^{\langle u\rangle} & := & \neg_u s^{\langle u\rangle} \to t^{\langle u\rangle} & (s+t)^{\langle u\rangle} & := & \neg_u(s^{\langle u\rangle} \to \neg_u t^{\langle u\rangle}) \\
(s\wedge t)^{\langle u\rangle} & := & s^{\langle u\rangle} \vee t^{\langle u\rangle} & (s\vee t)^{\langle u\rangle} & := & \neg_u(\neg_u s^{\langle u\rangle} \vee \neg_u t^{\langle u\rangle}) \\
(\Box s)^{\langle u\rangle} & := & \neg_u \Box \neg_u s^{\langle u\rangle} & (\Diamond s)^{\langle u\rangle} & := & \Box s^{\langle u\rangle}
\end{array}
$$

where x is a variable.

We define a trivial translation of ILZ into LL. The translation merely transforms a term of the shape $s \to t$ into a term of the shape $s^- + t$ (see, e.g., [76, Section 2.2.2]). For each intuitionistic $\mathcal{L}_{\mathsf{ILZ}}$-term s, we define the classical $\mathcal{L}_{\mathsf{LL}}$-term $\underline{s}$ by:

$$
\begin{array}{lcl@{\qquad\qquad}lcl}
\underline{x} & := & x & \underline{c} & := & c \\
\underline{s \to t} & := & \underline{s}^- + \underline{t} & \underline{s \diamond t} & := & \underline{s} \diamond \underline{t} \\
\underline{\Box s} & := & \Box \underline{s} & & &
\end{array}
$$

where c denotes 1 or 0, and $\diamond$ is in $\{\circ, \wedge, \vee\}$. The following lemma is trivial:

Lemma 3.1.1. *Let $\Gamma \Rightarrow s$ be an intuitionistic $\mathcal{L}_{\mathsf{ILL}}$-sequent.*

1. *If $\vdash_{\mathsf{ILL}} \Gamma \Rightarrow s$ then $\vdash_{\mathsf{LL}} \Rightarrow \underline{\Gamma}^-, \underline{s}$.*
2. *If $\vdash_{\mathsf{ILL_i}} \Gamma \Rightarrow s$ then $\vdash_{\mathsf{LL_w}} \Rightarrow \underline{\Gamma}^-, \underline{s}$.*
3. *If $\vdash_{\mathsf{IEL}} \Gamma \Rightarrow s$ then $\vdash_{\mathsf{ELL}} \Rightarrow \underline{\Gamma}^-, \underline{s}$.*
4. *If $\vdash_{\mathsf{IEL_i}} \Gamma \Rightarrow s$ then $\vdash_{\mathsf{ELL_w}} \Rightarrow \underline{\Gamma}^-, \underline{s}$.*

Proof. Follows by routine induction on the length of the derivation. $\Box$

We define the notion of substitution for classical $\mathcal{L}_{\mathsf{LL}}$-terms. In this section, by a *substitution* we mean a map from V to the set of classical $\mathcal{L}_{\mathsf{LL}}$-terms. In an obvious way, any substitution is naturally extended to a map on the set of classical $\mathcal{L}_{\mathsf{LL}}$-terms:

$$\begin{array}{lcl@{\qquad\qquad}lcl} \rho(x^-) & := & \rho(x)^- & \rho(c) & := & c \\ \rho(s \natural t) & := & \rho(s) \natural \rho(t) & \rho(\sharp s) & := & \sharp \rho(s) \end{array}$$

where $c \in \{1, 0\}$, $\natural \in \{\circ, +, \wedge, \vee\}$ and $\sharp \in \{\Box, \Diamond\}$.

Lemma 3.1.2. *Let $\Rightarrow \Gamma$ be a classical $\mathcal{L}_{\mathsf{LL}}$-sequent, ρ be a substitution, and Φ be a finite set of classical $\mathcal{L}_{\mathsf{LL}}$-terms.*

1. *If $\Phi \vdash_{\mathsf{LL}} \Rightarrow \Gamma$, then $\rho[\Phi] \vdash_{\mathsf{LL}} \Rightarrow \rho[\Gamma]$.*
2. *If $\Phi \vdash_{\mathsf{LL_w}} \Rightarrow \Gamma$, then $\rho[\Phi] \vdash_{\mathsf{LL_w}} \Rightarrow \rho[\Gamma]$.*
3. *If $\Phi \vdash_{\mathsf{ELL}} \Rightarrow \Gamma$, then $\rho[\Phi] \vdash_{\mathsf{ELL}} \Rightarrow \rho[\Gamma]$.*
4. *If $\Phi \vdash_{\mathsf{ELL_w}} \Rightarrow \Gamma$, then $\rho[\Phi] \vdash_{\mathsf{ELL_w}} \Rightarrow \rho[\Gamma]$.*

Proof. Follows by straightforward induction on the height of derivations. $\Box$

Lemma 3.1.3. *Let s be a classical $\mathcal{L}_{\mathsf{LL}}$-term and u be an intuitionistic $\mathcal{L}_{\mathsf{ILL}}$-term.*

1. *$\vdash_{\mathsf{ILL}} s^{\langle u \rangle}, (s^-)^{\langle u \rangle} \Rightarrow u$.*
2. *$\vdash_{\mathsf{ILL_i}} s^{\langle u \rangle}, (s^-)^{\langle u \rangle} \Rightarrow u$.*
3. *$\vdash_{\mathsf{IEL}} s^{\langle u \rangle}, (s^-)^{\langle u \rangle} \Rightarrow u$.*

4. $\vdash_{\mathsf{IEL_i}} s^{\langle u \rangle}, (s^-)^{\langle u \rangle} \Rightarrow u.$

Proof. By induction on the structure of s. □

For a classical $\mathcal{L}_{\mathsf{LL}}$-term c, we write Φ_c for the set $\{c^-, c+1\}$.

Lemma 3.1.4. *Let s be a classical $\mathcal{L}_{\mathsf{LL}}$-term and u be an intuitionistic $\mathcal{L}_{\mathsf{ILL}}$-term.*

1. $\Phi_{\underline{u}} \vdash_{\mathsf{LL}} \Rightarrow \underline{s^{\langle u \rangle}}, s.$
2. $\Phi_{\underline{u}} \vdash_{\mathsf{LL_w}} \Rightarrow \underline{s^{\langle u \rangle}}, s.$
3. $\Phi_{\underline{u}} \vdash_{\mathsf{ELL}} \Rightarrow \underline{s^{\langle u \rangle}}, s.$
4. $\Phi_{\underline{u}} \vdash_{\mathsf{ELL_w}} \Rightarrow \underline{s^{\langle u \rangle}}, s.$

Proof. By induction on s. □

Remark 3.1.5. As far as affine cases are considered, the same argument works well even when we define $\Phi_c = \{c^-\}$.

Using Lemma 3.1.3, we show:

Lemma 3.1.6. *Let $\Rightarrow \Gamma$ be a classical $\mathcal{L}_{\mathsf{LL}}$-sequent and u be an intuitionistic $\mathcal{L}_{\mathsf{ILL}}$-term.*

1. *If* $\vdash_{\mathsf{LL}} \Rightarrow \Gamma$ *then* $\vdash_{\mathsf{ILL}} \Gamma^{\langle u \rangle} \Rightarrow u.$
2. *If* $\vdash_{\mathsf{LL_w}} \Rightarrow \Gamma$ *then* $\vdash_{\mathsf{ILL_i}} \Gamma^{\langle u \rangle} \Rightarrow u.$
3. *If* $\vdash_{\mathsf{ELL}} \Rightarrow \Gamma$ *then* $\vdash_{\mathsf{IEL}} \Gamma^{\langle u \rangle} \Rightarrow u.$
4. *If* $\vdash_{\mathsf{ELL_w}} \Rightarrow \Gamma$ *then* $\vdash_{\mathsf{IEL_i}} \Gamma^{\langle u \rangle} \Rightarrow u.$

Proof. By induction on the length of the *cut-free* derivation of $\Rightarrow \Gamma$; see Fact 2.2.1. In the case where the rule of (init) is applied last, we use Lemma 3.1.3. For the induction step, we only consider the cases where the rules of $(+)$ and $(\circ)$ are applied last in the derivation.

- ▶ If $\Rightarrow \Gamma, s+t$ is obtained from $\Rightarrow \Gamma, s, t$ by an application of $(+)$ in the right-hand sided system, by the ind. hyp., $\Gamma^{\langle u \rangle}, s^{\langle u \rangle}, t^{\langle u \rangle} \Rightarrow u$ is provable in the intuitionistic sequent calculus. Thus we may construct the following intuitionistic derivation:

$$\dfrac{\dfrac{\dfrac{\text{ind. hyp.}}{\Gamma^{\langle u\rangle}, s^{\langle u\rangle}, t^{\langle u\rangle} \Rightarrow u}}{\Gamma^{\langle u\rangle}, s^{\langle u\rangle} \Rightarrow \neg_u t^{\langle u\rangle}}\,(\Rightarrow\rightarrow)}{\Gamma^{\langle u\rangle} \Rightarrow s^{\langle u\rangle} \rightarrow \neg_u t^{\langle u\rangle}}\,(\Rightarrow\rightarrow) \quad \dfrac{}{u \Rightarrow u}\,(\text{init})}{\Gamma^{\langle u\rangle}, \neg_u(s^{\langle u\rangle} \rightarrow \neg_u t^{\langle u\rangle}) \Rightarrow u}\,(\rightarrow\Rightarrow)$$

- ▶ In the case where $\Rightarrow \Gamma, \Delta, s \circ t$ is deduced from $\Rightarrow \Gamma, s$ and $\Rightarrow \Delta, t$ by an application of $(\circ)$ in the righthand sided system, by the ind. hyp., $\Gamma^{\langle u\rangle}, s^{\langle u\rangle} \Rightarrow u$ and $\Delta^{\langle u\rangle}, t^{\langle u\rangle} \Rightarrow u$ are both provable in the corresponding intuitionistic sequent calculus. We have:

$$\dfrac{\dfrac{\dfrac{\text{ind. hyp.}}{\Gamma^{\langle u\rangle}, s^{\langle u\rangle} \Rightarrow u}}{\Gamma^{\langle u\rangle} \Rightarrow \neg_u s^{\langle u\rangle}}\,(\Rightarrow\rightarrow) \quad \dfrac{\text{ind. hyp.}}{\Delta^{\langle u\rangle}, t^{\langle u\rangle} \Rightarrow u}}{\Gamma^{\langle u\rangle}, \Delta^{\langle u\rangle}, \neg_u s^{\langle u\rangle} \rightarrow t^{\langle u\rangle} \Rightarrow u}\,(\rightarrow\Rightarrow)$$

The remaining cases can be shown in a similar way. Observe that the cut admissibility is essential here. □

We now show that the translation $(\_)^{\langle x\rangle}$ from LL (resp. $\mathsf{LL_w}$, ELL, $\mathsf{ELL_w}$) into ILL (resp. $\mathsf{ILL_i}$, IEL, $\mathsf{IEL_i}$) is sound and faithful.

Theorem 3.1.7. *Let* $\Rightarrow \Gamma$ *be a classical* $\mathcal{L}_{\mathsf{LL}}$*-sequent and* x *be a fresh variable not in* Γ.

1. $\vdash_{\mathsf{LL}} \Rightarrow \Gamma$ *if and only if* $\vdash_{\mathsf{ILL}} \Gamma^{\langle x\rangle} \Rightarrow x$.
2. $\vdash_{\mathsf{LL_w}} \Rightarrow \Gamma$ *if and only if* $\vdash_{\mathsf{ILL_i}} \Gamma^{\langle x\rangle} \Rightarrow x$.
3. $\vdash_{\mathsf{ELL}} \Rightarrow \Gamma$ *if and only if* $\vdash_{\mathsf{IEL}} \Gamma^{\langle x\rangle} \Rightarrow x$.
4. $\vdash_{\mathsf{ELL_w}} \Rightarrow \Gamma$ *if and only if* $\vdash_{\mathsf{IEL_i}} \Gamma^{\langle x\rangle} \Rightarrow x$.

Proof. We only prove Statement (1), the other statements being similar. The proof is akin to that of [76, Proposition 2.16].

Left-to-right: immediate from Lemma 3.1.6.

Right-to-left: suppose that $\vdash_{\mathsf{ILL}} \Gamma^{\langle x\rangle} \Rightarrow x$. By Lemma 3.1.1, $\vdash_{\mathsf{LL}} (\underline{\Gamma^{\langle x\rangle}})^-, \underline{x}$; hence, $\Phi_{\underline{x}} \vdash_{\mathsf{LL}} \Rightarrow (\underline{\Gamma^{\langle x\rangle}})^-, \underline{x}$. By Lemma 3.1.4, $\Phi_{\underline{x}} \vdash_{\mathsf{LL}} \Rightarrow \underline{c^{\langle x\rangle}}, c$ for every classical $\mathcal{L}_{\mathsf{LL}}$-term c. Moreover, we stress that $\Phi_{\underline{x}} \vdash_{\mathsf{LL}} \Rightarrow \underline{x}^-$. By applying the rule of cut, we thus have a derivation for $\Phi_{\underline{x}} \vdash_{\mathsf{LL}} \Rightarrow \Gamma$. Define a substitution ρ by $\rho(x) = 0$ and $\rho(y) = y$ for each variable y such that $y \neq x$. By

Lemma 3.1.2, $\rho(\Phi_{\underline{x}}) \vdash_{\mathsf{LL}} \Rightarrow \rho(\Gamma)$. Observing that x is not in Γ, we see that $\rho(\Phi_{\underline{x}}) \vdash_{\mathsf{LL}} \Rightarrow \Gamma$. Notice that the terms in $\rho(\Phi_{\underline{x}})$ are already provable in LL without any assumptions. Thus $\vdash_{\mathsf{LL}} \Rightarrow \Gamma$, as desired. □

One may also show the following standard negative translation theorem. The proof is omitted here, as it is nearly the same as the proof of Theorem 3.1.7.

Theorem 3.1.8. *Let* $\Rightarrow \Gamma$ *be a classical* $\mathcal{L}_{\mathsf{LL}}$*-sequent.*

1. $\vdash_{\mathsf{LL}} \Rightarrow \Gamma$ *if and only if* $\vdash_{\mathsf{ILZ}} \Gamma^{\langle 0 \rangle} \Rightarrow$.
2. $\vdash_{\mathsf{LL_w}} \Rightarrow \Gamma$ *if and only if* $\vdash_{\mathsf{ILZ_i}} \Gamma^{\langle 0 \rangle} \Rightarrow$.
3. $\vdash_{\mathsf{ELL}} \Rightarrow \Gamma$ *if and only if* $\vdash_{\mathsf{IEZ}} \Gamma^{\langle 0 \rangle} \Rightarrow$.
4. $\vdash_{\mathsf{ELL_w}} \Rightarrow \Gamma$ *if and only if* $\vdash_{\mathsf{IEZ_i}} \Gamma^{\langle 0 \rangle} \Rightarrow$.

As a simple application of Theorem 3.1.7, we show the following:

Corollary 3.1.9. *The provability problem for* BCK *is* NP*-complete.*

Proof. In proving the membership in NP of BCK-provability, cut elimination is crucial; the proof idea is due to, e.g., Lincoln, Mitchell, Scedrov, and Shankar [81, Lemma 5.3].[3] Obviously, every cut-free derivation consists only of applications of (init), ($\Rightarrow\to$), ($\to\Rightarrow$) and (i). Using this fact, we can easily confirm that every sequent provable in BCK has a polynomial size cut-free derivation; thus, BCK-provability is decidable in NP.[4]

We show that the NP upper bound is optimal. To this end, we show that provability in the $\{\circ, +\}$-fragment of $\mathsf{LL_w}$ is polynomial time reducible into provability in BCK. The former is known to be NP-complete thanks to Lincoln, Mitchell, Scedrov, and Shankar [81] and Kanovich [69]. Let s be a classical $\{\circ, +\}$-term and x a fresh variable not in s. It suffices to check that $\Rightarrow s$ is provable in the $\{\circ, +\}$-fragment of $\mathsf{LL_w}$ iff $\vdash_{\mathsf{BCK}} \neg_x s^{\langle x \rangle}$. Since $\mathsf{LL_w}$ enjoys cut elimination, clearly $\mathsf{LL_w}$ is a conservative extension of its $\{\circ, +\}$-fragment. Thus $\Rightarrow s$ is provable in the $\{\circ, +\}$-fragment of $\mathsf{LL_w}$ iff it is provable in $\mathsf{LL_w}$. By Theorem 3.1.7, $\vdash_{\mathsf{LL_w}} s$ iff $\vdash_{\mathsf{ILL_i}} s^{\langle x \rangle} \Rightarrow x$. Note that the latter forms an intuitionistic $\{\to\}$-sequent. By the cut elimination theorem for $\mathsf{ILL_i}$, BCK is a conservative fragment of $\mathsf{ILL_i}$; that is, $\vdash_{\mathsf{ILL_i}} s^{\langle x \rangle} \Rightarrow x$ iff

[3] A similar argument is found in the literature of other nonclassical logics. For instance, Pentus [99] made use of the same idea in the proof of NP-completeness of provability in Lambek calculus.

[4] In [56, p. 71], Haniková already pointed out the NP-membership of provability in BCK.

$\vdash_{\mathsf{BCK}} s^{\langle x\rangle} \Rightarrow x$. Since the rule of $(\Rightarrow\rightarrow)$ is invertible, $\vdash_{\mathsf{BCK}} s^{\langle x\rangle} \Rightarrow x$ iff $\vdash_{\mathsf{BCK}} \neg_x s^{\langle x\rangle}$. To sum up, s is provable in the $\{\circ, +\}$-fragment of $\mathsf{LL_w}$ iff $\vdash_{\mathsf{BCK}} \neg_x s^{\langle x\rangle}$, as desired. $\square$

3.2 Tower upper bounds for contraction-free logics

3.2.1 The provability problem for elementary affine logic is in Tower

By slightly modifying the technique that Lazić and Schmitz [78, Section 4.1.2] employed in the proof of the Tower upper bound for $\mathsf{ILZ_i}$-provability, we show that there is an elementary time reduction from provability in $\mathsf{IEZ_i}$ to lossy reachability in $\mathsf{ABVASS}_{\bar{0}}$s.

Let c be an instance of the provability problem for $\mathsf{IEZ_i}$. We write:

- $\mathsf{Sub}(c)$ for the set of subterms of c,
- $\mathsf{Sub}^{\square}(c)$ for the set of terms in $\mathsf{Sub}(c)$ of the shape $\square t$,
- $\bullet$ for a new symbol not in $\mathsf{Sub}(c)$.

We define a map $(\_)^{\dagger}\colon \mathsf{Sub}(c) \cup \{\varepsilon\} \to \mathsf{Sub}(c) \cup \{\bullet\}$ as follows:

$$\sigma^{\dagger} := \begin{cases} s & \text{if } \sigma \text{ is of the form } s, \text{ where } s \text{ is in } \mathsf{Sub}(c),\\ \bullet & \text{if } \sigma \text{ is the empty multiset.}\end{cases}$$

A multiset over $\mathsf{Sub}(c)$ is just a function from $\mathsf{Sub}(c)$ to $\mathbb{N}$, i.e., an element of $\mathbb{N}^{\mathsf{Sub}(c)}$. Given a multiset m over $\mathsf{Sub}(c)$, by $m(s)$ we mean the multiplicity of a term s in m.

Let $d = |\mathsf{Sub}(c)|$. We fix an enumeration $s_1, \ldots, s_d$ of all the terms from $\mathsf{Sub}(c)$. Clearly, a multiset m over $\mathsf{Sub}(c)$ can be expressed as:

$$s_1^{m(s_1)}, \ldots, s_d^{m(s_d)}.$$

Given a multiset m over $\mathsf{Sub}(c)$, we write $\bar{\mathsf{v}}_m$ for the d-dimensional vector $\langle m(s_1), \ldots, m(s_d)\rangle$ in $\mathbb{N}^d$. In what follows, the notation $\bar{\mathsf{e}}_s$ denotes the unit vector corresponding to the multiset-singleton of a term s. Obviously, the empty multiset corresponds to $\bar{0} = \langle\underbrace{0, \ldots, 0}_{d}\rangle$. The notation $\eta(m)$ stands for the *support* of a multiset m, i.e., $\{s \in \mathsf{Sub}(c) \mid m(s) > 0\}$.

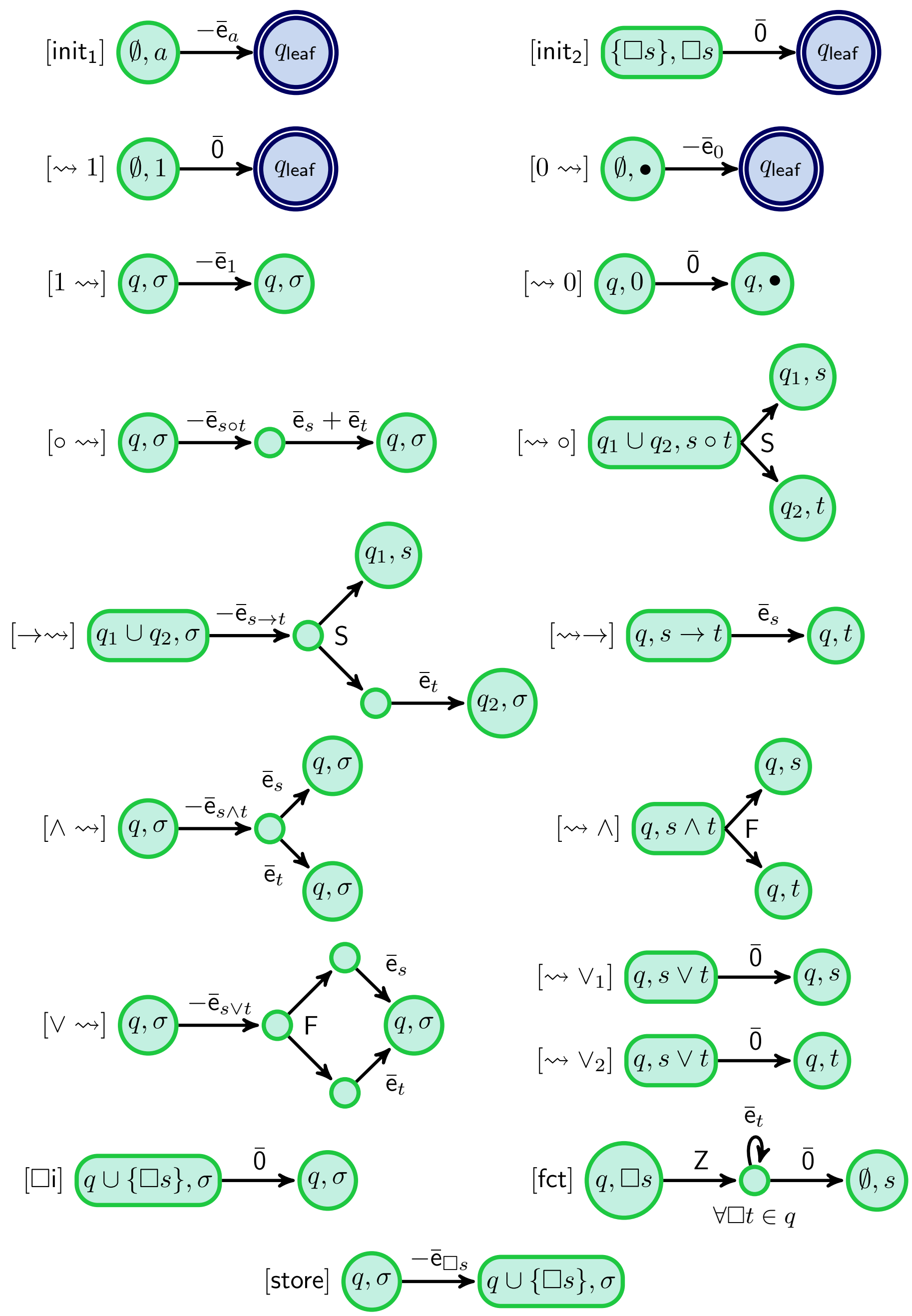

Figure 3.1: The definition of rules and intermediate states of $\mathcal{M}_c^{\mathsf{IEZ_i}}$; all terms are in $\mathsf{Sub}(c)$, q, q_1, q_2 are in $\mathcal{P}(\mathsf{Sub}^{\Box}(c))$, a is in $\mathsf{Sub}(c) \setminus \mathsf{Sub}^{\Box}(c)$, and σ is in $\mathsf{Sub}(c) \cup \{\bullet\}$. Small circles with no symbols stand for intermediate states.

Now we define a d-dimensional $\mathsf{ABVASS}_{\bar{0}}$ $\mathcal{M}_c^{\mathsf{IEZ_i}} = \langle Q, d, U, S, F, Z\rangle$, which simulates the proof-search of c. This machine is constructed as follows:

- The state space Q is the union of the following sets:
  - $\mathcal{P}(\mathsf{Sub}^{\Box}(c)) \times (\mathsf{Sub}(c) \cup \{\bullet\})$,
  - a singleton of a special leaf state q_{leaf},
  - a set of some intermediate states, i.e., blank circles depicted in Figure 3.1.
- The rules in U, S, F, and Z are defined as in Figure 3.1.

Remark 3.2.1. We comment on the core insight behind the construction of $\mathcal{M}_c^{\mathsf{IEZ_i}}$. Our machine is designed to simulate the inference rules of $\mathsf{IEZ_i}$ with the aid of a combination of unary, fork, split, and full zero test rules. For instance, the inference rule of $(\to\Rightarrow)$ in $\mathsf{IEZ_i}$ is implemented by the group $[\to\rightsquigarrow]$ of $\mathsf{ABVASS}_{\bar{0}}$ rules (see Figure 3.1). This group of $\mathsf{ABVASS}_{\bar{0}}$ rules is comprised of unary rules, split rules, and intermediate states. Roughly speaking, intermediate states are understood as auxiliary states introduced for simulating some of $\mathsf{IEZ_i}$ inference rules, i.e., $(\to\Rightarrow)$, $(\wedge\to)$, $(\vee\Rightarrow)$, (fct) and $(\circ\Rightarrow)$. Observe that our machine does not have the rules corresponding to the left-weakening (i). This rule is simulated by loss rules, which are unique to the lossy operational semantics for $\mathcal{M}_c^{\mathsf{IEZ_i}}$.

Remark 3.2.2. Our machine is just a slight modification of the $\mathsf{ABVASS}_{\bar{0}}$ $\mathcal{A}_F^I$ that Lazić and Schmitz defined in [78, Section 4.1.2]. The only critical difference between $\mathcal{A}_F^I$ and $\mathcal{M}_c^{\mathsf{IEZ_i}}$ is that $\mathcal{M}_c^{\mathsf{IEZ_i}}$ is equipped with the rules for [fct] instead of the rules for $[\rightsquigarrow\Box]$ and $[\Box\rightsquigarrow]$[5], which are of importance to our proof of the TOWER-membership of $\mathsf{FL_{ew}}$-deducibility; see Section 3.2.2 for details.

We next describe how we can interpret a given sequent as the corresponding configuration of $\mathcal{M}_c^{\mathsf{IEZ_i}}$. Clearly, every sequent constructed from $\mathsf{Sub}(c)$ is of the form $\Theta, \Gamma \Rightarrow \sigma$, where Θ is a finite multiset of terms in $\mathsf{Sub}^{\Box}(c)$, Γ is a finite multiset of terms from $\mathsf{Sub}(c) \setminus \mathsf{Sub}^{\Box}(c)$, and σ is in $\mathsf{Sub}(c) \cup \{\varepsilon\}$. We translate $\Theta, \Gamma \Rightarrow \sigma$ as the configuration $\langle \eta(\Theta), \sigma^{\dagger}; \bar{\mathsf{v}}_{\Gamma}\rangle$ of $\mathcal{M}_c^{\mathsf{IEZ_i}}$.

The following is the main theorem of this subsection:

Theorem 3.2.3. *For any σ in $\mathsf{Sub}(c) \cup \{\varepsilon\}$, any finite multiset Θ of terms from $\mathsf{Sub}^{\Box}(c)$, and any finite multiset Γ of terms from $\mathsf{Sub}(c) \setminus \mathsf{Sub}^{\Box}(c)$,*

$$\vdash_{\mathsf{IEZ_i}} \Theta, \Gamma \Rightarrow \sigma \textit{ if and only if } \mathcal{M}_c^{\mathsf{IEZ_i}}, \{q_{\mathsf{leaf}}\} \rightsquigarrow_{\ell} \langle \eta(\Theta), \sigma^{\dagger}; \bar{\mathsf{v}}_{\Gamma}\rangle.$$

[5]In [78], the rules for $[\rightsquigarrow\Box]$ and $[\Box\rightsquigarrow]$ are denoted by (!P) and (!D), respectively.

Prior to the proof of the main theorem, we prove some *ad hoc* lemmas. We first define the notion of regular deduction trees, implicitly introduced in [78, Section 3.2.2].

Let $\mathcal{M} = \langle Q, d, U, S, F, Z\rangle$ be an $\mathsf{ABVASS}_{\bar{0}}$ and $\mathscr{L}$ be a subset of Q. An $\mathscr{L} \times \{\bar{0}\}$*-leaf-covering regular lossy deduction tree* in $\mathcal{M}$ is constructed as follows:

- For each $q \in \mathscr{L}$,
$$\langle q; \bar{\mathsf{0}}\rangle$$
is an $\mathscr{L} \times \{\bar{0}\}$-leaf-covering regular lossy deduction tree with root $\langle q, \bar{0}\rangle$.

- For each $q \in \mathscr{L}$,
$$\frac{\langle q; \bar{\mathsf{e}}_i\rangle}{\langle q; \bar{\mathsf{0}}\rangle}\ (\mathsf{loss})$$
is an $\mathscr{L} \times \{\bar{0}\}$-leaf-covering regular lossy deduction tree with root $\langle q; \bar{\mathsf{e}}_i\rangle$.

- If T' is an $\mathscr{L} \times \{\bar{0}\}$-leaf-covering regular lossy deduction tree with root $\langle q'; \bar{\mathsf{v}}''\rangle$, T'' is an $\mathscr{L} \times \{\bar{0}\}$-leaf-covering regular lossy deduction tree with root $\langle q''; \bar{\mathsf{v}}''\rangle$, and $q \to q'\ \mathsf{S}\ q''$ is in S, then the following deduction tree is an $\mathscr{L} \times \{\bar{0}\}$-leaf-covering regular lossy deduction tree with root $\langle q, \bar{\mathsf{v}}' + \bar{\mathsf{v}}''\rangle$:
$$\frac{\langle q, \bar{\mathsf{v}}' + \bar{\mathsf{v}}''\rangle}{\begin{array}{cc}\langle q'; \bar{\mathsf{v}}'\rangle & \langle q''; \bar{\mathsf{v}}''\rangle \\ T' & T''\end{array}}\ (q \to q'\ \mathsf{S}\ q'')$$

- If T' is an $\mathscr{L} \times \{\bar{0}\}$-leaf-covering regular lossy deduction tree with root $\langle q'; \bar{\mathsf{v}}\rangle$, T'' is an $\mathscr{L} \times \{\bar{0}\}$-leaf-covering regular lossy deduction tree with root $\langle q''; \bar{\mathsf{v}}\rangle$, and $q \to q'\ \mathsf{F}\ q''$ is in F, then the following figure is an $\mathscr{L} \times \{\bar{0}\}$-leaf-covering regular lossy deduction tree with root $\langle q, \bar{\mathsf{v}}\rangle$:
$$\frac{\langle q, \bar{\mathsf{v}}\rangle}{\begin{array}{cc}\langle q'; \bar{\mathsf{v}}\rangle & \langle q''; \bar{\mathsf{v}}\rangle \\ T' & T''\end{array}}\ (q \to q'\ \mathsf{F}\ q'')$$

- If T is an $\mathscr{L} \times \{\bar{0}\}$-leaf-covering regular lossy deduction tree with root $\langle q'; \bar{\mathsf{v}} + \bar{\mathsf{u}}\rangle$ and $q \xrightarrow{\bar{\mathsf{u}}} q' \in U$, then the figure of the form:

$$\dfrac{\langle q; \bar{\mathsf{v}}\rangle}{\langle q'; \bar{\mathsf{v}} + \bar{\mathsf{u}}\rangle}\, (q \xrightarrow{\bar{\mathsf{u}}} q') \atop T$$

is an $\mathscr{L} \times \{\bar{0}\}$-leaf-covering regular lossy deduction tree with root $\langle q; \bar{\mathsf{v}}\rangle$.

- If T is an $\mathscr{L} \times \{\bar{0}\}$-leaf-covering regular lossy deduction tree with root $\langle q'; \bar{0}\rangle$ and $q \xrightarrow{\mathsf{Z}} q' \in Z$, then the following figure is an $\mathscr{L} \times \{\bar{0}\}$-leaf-covering regular lossy deduction tree with root $\langle q; \bar{0}\rangle$:

$$\dfrac{\langle q; \bar{0}\rangle}{\langle q', \bar{0}\rangle}\, (q \xrightarrow{\mathsf{Z}} q') \atop T$$

- If T is an $\mathscr{L} \times \{\bar{0}\}$-leaf-covering regular lossy deduction tree with root $\langle q; \bar{\mathsf{v}}\rangle$ whose last applied rule is either a loss rule or a full zero test rule, then for each $1 \leq i \leq d$ the figure below is an $\mathscr{L} \times \{\bar{0}\}$-leaf-covering regular lossy deduction tree with root $\langle q; \bar{\mathsf{v}} + \bar{\mathsf{e}}_i\rangle$.

$$\dfrac{\langle q, \bar{\mathsf{v}} + \bar{\mathsf{e}}_i\rangle}{\langle q, \bar{\mathsf{v}}\rangle}\, (\mathsf{loss}) \atop T$$

We write $\mathcal{M}, \mathscr{L} \rightsquigarrow_{\ell}^{reg} \langle q; \bar{\mathsf{v}}\rangle$ if there is an $\mathscr{L} \times \{\bar{0}\}$-leaf-covering regular lossy deduction tree of $\mathcal{M}$ with root $\langle q; \bar{\mathsf{v}}\rangle$.

Lemma 3.2.4 (Cf. Section 3.2.2 of Lazić and Schmitz [78])**.** *Suppose that $\mathcal{M}$ is an $\mathsf{ABVASS}_{\bar{0}}$, that T is an $\mathscr{L} \times \{\bar{0}\}$-leaf-covering lossy deduction tree in $\mathcal{M}$, which has the shape:*

$$\dfrac{\langle q; \bar{\mathsf{v}} + \bar{\mathsf{e}}_i\rangle}{\langle q; \bar{\mathsf{v}}\rangle}\, (\mathsf{loss}), \atop \vdots$$

and that the immediate subtree of T with root $\langle q; \bar{\mathsf{v}}\rangle$ is regular. Then $\mathcal{M}, \mathscr{L} \rightsquigarrow_{\ell}^{reg} \langle q; \bar{\mathsf{v}} + \bar{\mathsf{e}}_i\rangle$.

Proof. The proof is by induction on the height of T. Consider, e.g., the case where the immediate subtree ends with an application of a fork rule. It is of the following shape:

$$\dfrac{\dfrac{\langle q; \bar{\mathsf{v}} + \bar{\mathsf{e}}_i \rangle}{\langle q; \bar{\mathsf{v}} \rangle}\,(\mathsf{loss})}{\underset{T'}{\langle q'; \bar{\mathsf{v}} \rangle} \qquad \underset{T''}{\langle q''; \bar{\mathsf{v}} \rangle}}\,(q \to q' \mathsf{\ F\ } q'')$$

The above figure can be transformed into the following:

$$\dfrac{\langle q; \bar{\mathsf{v}} + \bar{\mathsf{e}}_i \rangle}{\dfrac{\langle q'; \bar{\mathsf{v}} + \bar{\mathsf{e}}_i \rangle}{\underset{T'}{\langle q'; \bar{\mathsf{v}} \rangle}}\,(\mathsf{loss}) \qquad \dfrac{\langle q''; \bar{\mathsf{v}} + \bar{\mathsf{e}}_i \rangle}{\underset{T''}{\langle q''; \bar{\mathsf{v}} \rangle}}\,(\mathsf{loss})}\,(q \to q' \mathsf{\ F\ } q'')$$

The height of the sub-deduction-tree with root $\langle q'; \bar{\mathsf{v}}' + \bar{\mathsf{e}}_i \rangle$ is less than that of T. Also, the height of the sub-deduction-tree with root $\langle q''; \bar{\mathsf{v}}'' + \bar{\mathsf{e}}_i \rangle$ is less than that of T. Observe that T' and T'' are regular; thus by the ind. hyp., $\mathcal{M}, \mathscr{L} \rightsquigarrow_\ell^{reg} \langle q'; \bar{\mathsf{v}} + \bar{\mathsf{e}}_i \rangle$ and $\mathcal{M}, \mathscr{L} \rightsquigarrow_\ell^{reg} \langle q''; \bar{\mathsf{v}} + \bar{\mathsf{e}}_i \rangle$. Applying the fork rule $q \to q' \mathsf{\ F\ } q''$, we may obtain a deduction tree for $\mathcal{M}, \mathscr{L} \rightsquigarrow_\ell^{reg} \langle q; \bar{\mathsf{v}} + \bar{\mathsf{e}}_i \rangle$, as desired. The remaining cases are similar. $\square$

Using the above lemma, we show:

Lemma 3.2.5 (Cf. Section 3.2.2 of Lazić and Schmitz [78])**.** *Let $\mathcal{M}$ be an $\mathit{ABVASS}_{\bar{0}}$ with a state space Q, $\mathscr{L}$ a subset of Q, and $\langle q; \bar{\mathsf{v}} \rangle$ a configuration of $\mathcal{M}$.*

If $\mathcal{M}, \mathscr{L} \rightsquigarrow_\ell \langle q; \bar{\mathsf{v}} \rangle$ then $\mathcal{M}, \mathscr{L} \rightsquigarrow_\ell^{reg} \langle q; \bar{\mathsf{v}} \rangle$.

Proof. The proof proceeds by induction on the height of the lossy deduction tree for $\mathcal{M}, \mathscr{L} \rightsquigarrow_\ell, \langle q; \bar{\mathsf{v}} \rangle$. Below we perform case analysis depending on which deduction rule is applied last in the deduction tree.

Clearly, the only critical case is the case where the last applied rule is a loss rule, i.e., the case where the last part of the lossy deduction tree is of the form:

$$\dfrac{\langle q; \bar{\mathsf{v}} + \bar{\mathsf{e}}_i \rangle}{\langle q; \bar{\mathsf{v}} \rangle}\,(\mathsf{loss}) \\ \vdots$$

By the ind. hyp., there is an $\mathscr{L} \times \{\bar{0}\}$-leaf-covering regular lossy deduction tree T with root $\langle q; \bar{\mathsf{v}} \rangle$ in $\mathcal{M}$. Thus we may construct the following deduction tree:

$$\dfrac{\langle q; \bar{\mathsf{v}} + \bar{\mathsf{e}}_i \rangle}{\underset{T}{\langle q; \bar{\mathsf{v}} \rangle}}\,(\mathsf{loss})$$

By Lemma 3.2.4, $\mathcal{M}, \mathscr{L} \rightsquigarrow_{\ell}^{reg} \langle q; \bar{\mathsf{v}} + \bar{\mathsf{e}}_i \rangle$. $\square$

We again concentrate our attention on the $\mathsf{ABVASS}_{\bar{0}}$ $\mathcal{M}_c^{\mathsf{IEZ_i}}$ constructed before. For each multiset m over $\mathsf{Sub}(c)$:

- the notation $\zeta(m)$ denotes the submultiset of terms in m of the form $\square s$;
- the notation $\xi(m)$ denotes the submultiset of remaining terms in m.

Obviously, $m = \zeta(m), \xi(m)$ and $\bar{\mathsf{v}}_m = \bar{\mathsf{v}}_{\zeta(m)} + \bar{\mathsf{v}}_{\xi(m)}$.

Employing the ideas from [78, Claims 4.3.1 and 4.3.2], we use the fairly *ad hoc* notion of standard deduction tree, which is important in the proofs of Theorems 3.2.3 and 3.2.13. A lossy deduction tree T in $\mathcal{M}_c^{\mathsf{IEZ_i}}$ is said to be *standard* if the deduction tree satisfies the following:

1. For each subtree T' of T, the last applied rule in T' is $[\mathsf{store}]$ provided that the root node of T' is of the shape $\langle q, \sigma; \bar{\mathsf{v}}_{\Gamma, \square s} \rangle$ for some q in $\mathcal{P}(\mathsf{Sub}^{\square}(c))$, some Γ in $\mathbb{N}^{\mathsf{Sub}(c)}$, some σ in $\mathsf{Sub}(c) \cup \{\bullet\}$, and some $\square s$ in $\mathsf{Sub}^{\square}(c)$.
2. For each subtree T' of T, the root node of T' belongs to $\mathcal{P}(\mathsf{Sub}^{\square}(c)) \times (\mathsf{Sub}(c) \cup \{\bullet\}) \times \mathbb{N}^d$ provided that the (loss) rule is applied last in T'.

Intuitively, a standard lossy deduction tree is a lossy deduction tree such that:

- every application of a store rule occurs as close to the root configuration as possible, and
- every application of (loss) occurs at configurations of the form $\langle q, \sigma; \bar{\mathsf{v}} \rangle$.

We write $\mathcal{M}_c^{\mathsf{IEZ_i}}, \{q_{\mathsf{leaf}}\} \rightsquigarrow_{\ell}^{st} \langle q, \sigma; \bar{\mathsf{v}} \rangle$ if there is a $\{(q_{\mathsf{leaf}}; \bar{0})\}$-leaf-covering standard lossy deduction tree of $\mathcal{M}_c^{\mathsf{IEZ_i}}$ with root $\langle q, \sigma; \bar{\mathsf{v}} \rangle$.

Lemma 3.2.6. *Let q be in $\mathcal{P}(\mathsf{Sub}^{\square}(c))$, Γ be in $\mathbb{N}^{\mathsf{Sub}(c)}$, and σ be in $\mathsf{Sub}(c) \cup \{\bullet\}$.*

$$\textit{If } \mathcal{M}_c^{\mathsf{IEZ_i}}, \{q_{\mathsf{leaf}}\} \rightsquigarrow_{\ell} \langle q, \sigma; \bar{\mathsf{v}}_\Gamma \rangle \textit{ then } \mathcal{M}_c^{\mathsf{IEZ_i}}, \{q_{\mathsf{leaf}}\} \rightsquigarrow_{\ell}^{st} \langle q, \sigma; \bar{\mathsf{v}}_\Gamma \rangle.$$

Proof. In light of Lemma 3.2.5, we can prove the statement by induction on the height of the $\{(q_{\mathsf{leaf}}, \bar{0})\}$-leaf-covering regular lossy deduction tree of $\mathcal{M}_c^{\mathsf{IEZ_i}}$ with root $\langle q, \sigma; \bar{\mathsf{v}}_\Gamma \rangle$. We perform a case analysis on the deduction rules that are applied last.

If $[\mathsf{init}_2]$ is applied last, the regular lossy deduction tree is of the form:

$$\dfrac{\langle \{\Box s\}, \Box s; \bar{\mathsf{v}}_\Gamma\rangle}{\langle q_{\mathsf{leaf}}; \bar{\mathsf{v}}_\Gamma\rangle}\,[\mathsf{init}_2]$$
$$\vdots$$

In this case, we may construct a desired deduction tree without using the induction hypothesis:

$$\dfrac{\dfrac{\dfrac{\dfrac{\langle \{\Box s\}, \Box s; \bar{\mathsf{v}}_\Gamma\rangle}{\langle \{\Box s\} \cup \eta(\zeta(\Gamma)), \Box s; \bar{\mathsf{v}}_{\xi(\Gamma)}\rangle}\,[\mathsf{store}]}{\langle \{\Box s\}, \Box s; \bar{\mathsf{v}}_{\xi(\Gamma)}\rangle}\,[\Box\mathsf{i}]}{\langle \{\Box s\}, \Box s; \bar{\mathsf{0}}\rangle}\,(\mathsf{loss})}{\langle q_{\mathsf{leaf}}, \bar{\mathsf{0}}\rangle}\,[\mathsf{init}_2]$$

where the double line labeled by the deduction rule $[r]$ denotes several (possibly zero) applications of $[r]$.

A more careful analysis is required in some other cases. Consider, for instance, the case where a unary rule of the form:

$$q, \sigma \xrightarrow{-\bar{\mathsf{e}}_{s\vee t}} \mathsf{Int}^{[\vee\rightsquigarrow]}(q, \sigma, s \vee t)$$

is applied last. Here, $\mathsf{Int}^{[\vee\rightsquigarrow]}(q, \sigma, s \vee t)$ stands for the intermediate state of $[\vee \rightsquigarrow]$ that corresponds to the state q, σ and the term $s \vee t$, cf. the rules for $[\vee \rightsquigarrow]$ of Figure 3.1. Therefore the last part of the regular lossy deduction tree is of the shape:

$$\dfrac{\langle q, \sigma; \bar{\mathsf{v}}_{\Gamma, s\vee t}\rangle}{\langle \mathsf{Int}^{[\vee\rightsquigarrow]}(q, \sigma, s \vee t); \bar{\mathsf{v}}_\Gamma\rangle}$$
$$\vdots$$

Obviously, the configuration $\langle \mathsf{Int}^{[\vee\rightsquigarrow]}(q, \sigma, s \vee t); \bar{\mathsf{v}}_\Gamma\rangle$ is not obtained by an application of a loss rule because the deduction tree in question is regular. Therefore the only rule that we can apply at $\mathsf{Int}^{[\vee\rightsquigarrow]}(q, \sigma, s \vee t)$ must be a fork rule of the form:

$$\mathsf{Int}^{\vee}(q, \sigma, s \vee t) \to \mathsf{Int}^{[\vee\rightsquigarrow]_l}(q, \sigma, s \vee t)\ \mathsf{F}\ \mathsf{Int}^{[\vee\rightsquigarrow]_r}(q, \sigma, s \vee t)$$

where $\mathsf{Int}^{[\vee\rightsquigarrow]_l}(q, \sigma, s \vee t)$ and $\mathsf{Int}^{[\vee\rightsquigarrow]_r}(q, \sigma, s \vee t)$ are the corresponding intermediate states comprising of $[\rightsquigarrow \vee]$ (cf. Figure 3.1). Hence the regular lossy deduction tree must be of the form:

$$\dfrac{\dfrac{\langle q,\sigma;\bar{\mathsf{v}}_{\Gamma,s\vee t}\rangle}{\langle \mathsf{Int}^{[\vee\rightsquigarrow]}(q,\sigma,s\vee t);\bar{\mathsf{v}}_\Gamma\rangle}}{\begin{array}{cc}\langle \mathsf{Int}^{[\vee\rightsquigarrow]_l}(q,\sigma,s\vee t);\bar{\mathsf{v}}_\Gamma\rangle & \langle \mathsf{Int}^{[\vee\rightsquigarrow]_r}(q,\sigma,s\vee t);\bar{\mathsf{v}}_\Gamma\rangle \\ \vdots & \vdots\end{array}}$$

By the regularity property of the deduction tree in question, the configuration $\langle \mathsf{Int}^{[\vee\rightsquigarrow]_l}(q,\sigma,s\vee t);\bar{\mathsf{v}}_\Gamma\rangle$ is not obtained by a loss rule. Also, it is of course impossible that the configuration $\langle \mathsf{Int}^{[\vee\rightsquigarrow]_r}(q,\sigma,s\vee t);\bar{\mathsf{v}}_\Gamma\rangle$ is obtained by a loss. Thus:

- The only rule that can be applied at $\mathsf{Int}^{[\vee\rightsquigarrow]_l}(q,\sigma,s\vee t)$ is:

$$\mathsf{Int}^{[\vee\rightsquigarrow]_l}(q,\sigma,s\vee t)\xrightarrow{\bar{\mathsf{e}}_s} q,\sigma.$$

- The only rule that can be applied at $\mathsf{Int}^{[\vee\rightsquigarrow]_r}(q,\sigma,s\vee t)$ is:

$$\mathsf{Int}^{[\vee\rightsquigarrow]_r}(q,\sigma,s\vee t)\xrightarrow{\bar{\mathsf{e}}_t} q,\sigma.$$

Hence the regular lossy deduction tree must be of the form:

$$\dfrac{\dfrac{\langle q,\sigma;\bar{\mathsf{v}}_{\Gamma,s\vee t}\rangle}{\langle \mathsf{Int}^{[\vee\rightsquigarrow]}(q,\sigma,s\vee t);\bar{\mathsf{v}}_\Gamma\rangle}}{\begin{array}{cc}\dfrac{\langle \mathsf{Int}^{[\vee\rightsquigarrow]_l}(q,\sigma,s\vee t);\bar{\mathsf{v}}_\Gamma\rangle}{\langle q,\sigma;\bar{\mathsf{v}}_{\Gamma,s}\rangle} & \dfrac{\langle \mathsf{Int}^{[\vee\rightsquigarrow]_r}(q,\sigma,s\vee t);\bar{\mathsf{v}}_\Gamma\rangle}{\langle q,\sigma;\bar{\mathsf{v}}_{\Gamma,t}\rangle} \\ \vdots & \vdots\end{array}}$$

We consider, e.g., the case where s and t are both in $\mathsf{Sub}^{\Box}(c)$; the other cases being similar. By the ind. hyp., we obtain a standard lossy deduction tree for $\mathcal{M}_c^{\mathsf{IEZ_i}},\{q_{\mathsf{leaf}}\}\rightsquigarrow_\ell^{st}\langle q,\sigma;\bar{\mathsf{v}}_{\Gamma,s}\rangle$. It must be of the form:

$$\dfrac{\langle q,\sigma;\bar{\mathsf{v}}_{\Gamma,s}\rangle}{\dfrac{}{\langle q\cup\eta(\zeta(\Gamma))\cup\{s\},\sigma;\bar{\mathsf{v}}_{\xi(\Gamma)}\rangle}}\,[\mathsf{store}]$$
$$T_1$$

where the sub-deduction-tree T_1 is standard. By using the ind. hyp., we have a standard lossy deduction tree for $\mathcal{M}_c^{\mathsf{IEZ_i}},\{q_{\mathsf{leaf}}\}\rightsquigarrow_\ell^{st}\langle q,\sigma;\bar{\mathsf{v}}_{\Gamma,t}\rangle$. It has the shape:

$$\dfrac{\langle q,\sigma;\bar{\mathsf{v}}_{\Gamma,t}\rangle}{\dfrac{}{\langle q\cup\eta(\zeta(\Gamma))\cup\{t\},\sigma;\bar{\mathsf{v}}_{\xi(\Gamma)}\rangle}}\,[\mathsf{store}]$$
$$T_2$$

Note that the sub-deduction-tree T_2 is standard. We may obtain a desired standard lossy deduction tree for $\mathcal{M}_c^{\mathsf{IEZ_i}}, \{q_{\mathsf{leaf}}\} \rightsquigarrow_\ell^{st} \langle q, \sigma; \bar{\mathsf{v}}_{\Gamma, s\vee t}\rangle$ as follows:

$$\dfrac{\dfrac{\dfrac{\langle q, \sigma; \bar{\mathsf{v}}_{\Gamma, s\vee t}\rangle}{\langle q\cup\eta(\zeta(\Gamma)), \sigma; \bar{\mathsf{v}}_{\xi(\Gamma), s\vee t}\rangle}\,[\mathsf{store}]}{\langle \mathsf{Int}^{[\vee\rightsquigarrow]}(q\cup\eta(\zeta(\Gamma)), \sigma, s\vee t); \bar{\mathsf{v}}_{\xi(\Gamma)}\rangle}}{\begin{array}{cc} \dfrac{\langle \mathsf{Int}^{[\vee\rightsquigarrow]_l}(q\cup\eta(\zeta(\Gamma)), \sigma, s\vee t); \bar{\mathsf{v}}_{\xi(\Gamma)}\rangle}{\dfrac{\langle q\cup\eta(\zeta(\Gamma)), \sigma; \bar{\mathsf{v}}_{\xi(\Gamma), s}\rangle}{\langle q\cup\eta(\zeta(\Gamma))\cup\{s\}, \sigma; \bar{\mathsf{v}}_{\xi(\Gamma)}\rangle}\,[\mathsf{store}]} & \dfrac{\langle \mathsf{Int}^{[\vee\rightsquigarrow]_r}(q\cup\eta(\zeta(\Gamma)), \sigma, s\vee t); \bar{\mathsf{v}}_{\xi(\Gamma)}\rangle}{\dfrac{\langle q\cup\eta(\zeta(\Gamma)), \sigma; \bar{\mathsf{v}}_{\xi(\Gamma), t}\rangle}{\langle q\cup\eta(\zeta(\Gamma))\cup\{t\}, \sigma; \bar{\mathsf{v}}_{\xi(\Gamma)}\rangle}\,[\mathsf{store}]} \\ T_1 & T_2 \end{array}}$$

The remaining cases are similar. ☐

It is high time to show the main claim we have already stated:

Theorem 3.2.3. *For any σ in $\mathsf{Sub}(c)\cup\{\varepsilon\}$, any finite multiset Θ of terms from $\mathsf{Sub}^{\Box}(c)$, and any finite multiset Γ of terms from $\mathsf{Sub}(c)\setminus\mathsf{Sub}^{\Box}(c)$,*

$$\vdash_{\mathsf{IEZ_i}} \Theta, \Gamma \Rightarrow \sigma \text{ if and only if } \mathcal{M}_c^{\mathsf{IEZ_i}}, \{q_{\mathsf{leaf}}\} \rightsquigarrow_\ell \langle \eta(\Theta), \sigma^\dagger; \bar{\mathsf{v}}_\Gamma\rangle.$$

Proof. Our proof is merely a minor variation of that of [78, Claims 4.3.1 and 4.3.2].

Left-to-right: by induction on the size of the cut-free derivation for $\vdash_{\mathsf{IEZ_i}} \Theta, \Gamma \Rightarrow \sigma$. We only consider the case where the last part of the derivation is of the shape:

$$\dfrac{\begin{array}{c}\vdots\\ \Theta, \Gamma \Rightarrow s\end{array}}{\Box\Theta, \Box\Gamma \Rightarrow \Box s}\,(\mathrm{fct})$$

where Θ is a finite multiset of terms over $\mathsf{Sub}^{\Box}(c)$, and Γ is a finite multiset of terms from $\mathsf{Sub}(c)\setminus\mathsf{Sub}^{\Box}(c)$. By appealing to the induction hypothesis, we obtain a lossy deduction tree for $\mathcal{M}_c^{\mathsf{IEZ_i}}, \{q_{\mathsf{leaf}}\} \rightsquigarrow_\ell \langle \eta(\Theta), s; \bar{\mathsf{v}}_\Gamma\rangle$. Applying store rules, we obtain a lossy deduction tree for $\mathcal{M}_c^{\mathsf{IEZ_i}}, \{q_{\mathsf{leaf}}\} \rightsquigarrow_\ell \langle \emptyset, s; \sum_{\Box t\in\eta(\Theta)} \bar{\mathsf{e}}_{\Box t} + \bar{\mathsf{v}}_\Gamma\rangle$. Applying the rules for [fct], we may construct a lossy deduction tree for $\mathcal{M}_c^{\mathsf{IEZ_i}}, \{q_{\mathsf{leaf}}\} \rightsquigarrow_\ell \langle \eta(\Box\Theta, \Box\Gamma), \Box s; \bar{\mathsf{v}}_\varepsilon\rangle$.

Right-to-left: in light of the standardization lemma, Lemma 3.2.6, we show this direction by induction on the size of the standard lossy deduction tree for $\mathcal{M}_c^{\mathsf{IEZ_i}}, \{q_{\mathsf{leaf}}\} \rightsquigarrow_\ell^{st} \langle \eta(\Theta), \sigma^\dagger; \bar{\mathsf{v}}_\Gamma\rangle$. Since most cases are analyzed in the proof of [78, Claims 4.3.1 and 4.3.2], we consider here the case of [fct], i.e., the case where the deduction tree is of the shape:

$$\cfrac{\langle \eta(\Theta), \Box s; \bar{0}\rangle}{\langle \mathsf{Int}^{[\mathsf{fct}]}(\eta(\Theta), \Box s); \bar{0}\rangle} \\ \vdots$$

where $\mathsf{Int}^{[\mathsf{fct}]}(\eta(\Theta), \Box s)$ denotes the corresponding intermediate state in [fct]. We highlight the following two points:

- The only deduction rules that can be applied at $\mathsf{Int}^{[\mathsf{fct}]}(\eta(\Theta), \Box s)$ are:
  - $\mathsf{Int}^{[\mathsf{fct}]}(\eta(\Theta), \Box s) \xrightarrow{\bar{\mathsf{e}}_t} \mathsf{Int}^{[\mathsf{fct}]}(\eta(\Theta), \Box s)$ (for each $\Box t \in \eta(\Theta)$), and
  - $\mathsf{Int}^{[\mathsf{fct}]}(\eta(\Theta), \Box s) \xrightarrow{\bar{0}} \emptyset, s$.
- Since the deduction tree in question is standard, loss rules are not applied at the state $\mathsf{Int}^{[\mathsf{fct}]}(\eta(\Theta), \Box s)$.

The deduction tree must thus end with the following form:

$$\cfrac{\cfrac{\cfrac{\langle \eta(\Theta), \Box s; \bar{0}\rangle}{\langle \mathsf{Int}^{[\mathsf{fct}]}(\eta(\Theta), \Box s); \bar{0}\rangle}}{\langle \mathsf{Int}^{[\mathsf{fct}]}(\eta(\Theta), \Box s); \bar{\mathsf{v}}_\Delta\rangle}}{\langle \emptyset, s; \bar{\mathsf{v}}_\Delta\rangle} \\ \vdots$$

Here the dotted line denotes (possibly zero) applications of the unary rules of the form $\mathsf{Int}^{[\mathsf{fct}]}(\eta(\Theta), \Box s) \xrightarrow{\bar{\mathsf{e}}_t} \mathsf{Int}^{[\mathsf{fct}]}(\eta(\Theta), \Box s)$, where $\Box t \in \eta(\Theta)$. Hence Δ is a multiset over $\mathsf{Sub}(c)$ and for every term u in $\eta(\Delta)$, $\Box u$ is in $\eta(\Theta)$. Since the lossy deduction tree is standard, the applications of store rules must successively occur as many times as possible immediately after the application of $\mathsf{Int}^{[\mathsf{fct}]}(\eta(\Theta), \Box s) \xrightarrow{\bar{0}} \emptyset, s$. The deduction tree is thus of the shape:

$$\cfrac{\cfrac{\cfrac{\cfrac{\langle \eta(\Theta), \Box s; \bar{0}\rangle}{\langle \mathsf{Int}^{[\mathsf{fct}]}(\eta(\Theta), \Box s); \bar{0}\rangle}}{\langle \mathsf{Int}^{[\mathsf{fct}]}(\eta(\Theta), \Box s); \bar{\mathsf{v}}_\Delta\rangle}}{\langle \emptyset, s; \bar{\mathsf{v}}_\Delta\rangle}}{\langle \eta(\zeta(\Delta)), s; \bar{\mathsf{v}}_{\xi(\Delta)}\rangle}\ [\mathsf{store}] \\ \vdots$$

By the ind. hyp., $\vdash_{\mathsf{IEZ_i}} \zeta(\Delta), \xi(\Delta) \Rightarrow s$; that is, $\vdash_{\mathsf{IEZ_i}} \Delta \Rightarrow s$. Applying the rule of (fct), we obtain a derivation for $\vdash_{\mathsf{IEZ_i}} \Box\Delta \Rightarrow \Box s$. Since every term in $\Box\Delta$ is in Θ, we may construct a derivation for $\vdash_{\mathsf{IEZ_i}} \Theta \Rightarrow \Box s$ by applying (i) and ($\Box$c) several times. $\Box$

As an immediate corollary of Theorem 3.2.3, we obtain:

$$\vdash_{\mathsf{IEZ_i}} c \text{ if and only if } \mathcal{M}_c^{\mathsf{IEZ_i}}, \{q_{\mathsf{leaf}}\} \rightsquigarrow_\ell \langle \emptyset, c; \bar{0} \rangle.$$

By Theorems 2.5.5 and 3.1.8, we conclude:

Corollary 3.2.7. *The provability problems for* $\mathsf{IEZ_i}$ *and* $\mathsf{ELL_w}$ *are decidable in* Tower*.*

It is clear that the same holds for fragments of $\mathsf{IEZ_i}$ and $\mathsf{ELL_w}$, e.g., the $\{\rightarrow, \Box\}$-fragment of $\mathsf{IEZ_i}$. In the next section, we will show that this small fragment is already hard for Tower.

3.2.2 The deducibility problems for $\mathsf{FL_{ei}}$ and $\mathsf{FL_{ew}}$ are in Tower

First, we define the notion of $\Box$-*prenex sequent*, which traces back to the notion of !-*prenex sequent* devised by Terui [118, Section 2.1.1]. Let $\mathcal{K}$ be a sublanguage of $\mathcal{L}_{\mathsf{ILZ}}$ such that $0 \in \mathcal{K}$. By a $\Box$-*prenex* $\mathcal{K}$-*sequent* we mean an intuitionistic $\mathcal{K} \cup \{\Box\}$-sequent of the form $\Box\Gamma, \Delta \Rightarrow \sigma$, where Γ and Δ are finite multisets of intuitionistic $\mathcal{K}$-terms, and σ is an intuitionistic $\mathcal{K}$-term or is empty. Similarly, let $\mathcal{K}$ be a sublanguage of $\mathcal{L}_{\mathsf{LL}}$. A $\Diamond$-*prenex* $\mathcal{K}$-*sequent* is a classical $\mathcal{K} \cup \{\Diamond\}$-sequent of the form $\Rightarrow \Diamond\Gamma, \Delta$, where Γ and Δ are finite multisets of classical $\mathcal{K}$-terms.

The two lemmas below are modified versions of the technical lemma shown by Terui [118, Proposition 2.6]. In the literature of algebraic logic, the reader can find a similar result, i.e., the *local deduction theorem* of Blok and Pigozzi [13].

Lemma 3.2.8. *Let* $\Box\Gamma, \Delta \Rightarrow \sigma$ *be a* $\Box$*-prenex* $\mathcal{L}_{\mathsf{FL_e}}$*-sequent.*

1. *If* $\vdash_{\mathsf{ILZ_i}} \Box\Gamma, \Delta \Rightarrow \sigma$ *then* $\vdash_{\mathsf{FL_{ei}}} \Gamma^{(n)}, \Delta \Rightarrow \sigma$ *for some* $n \geq 0$.
2. *If* $\vdash_{\mathsf{ILZ_w}} \Box\Gamma, \Delta \Rightarrow \sigma$ *then* $\vdash_{\mathsf{FL_{ew}}} \Gamma^{(n)}, \Delta \Rightarrow \sigma$ *for some* $n \geq 0$.

Here, $\Gamma^{(n)}$ *denotes the multiset sum of* n *copies of* Γ.

Proof. We only show Statement 1 by induction on the size of the cut-free derivation in $\mathsf{ILZ_i}$ for $\Box\Gamma, \Delta \Rightarrow \sigma$; the proof of Statement 2 being similar. We consider the case of ($\Rightarrow \circ$); the remaining cases are also easy to check. If the cut-free derivation is of the form:

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ \Box\Gamma, \Delta \Rightarrow s & \Box\Sigma, \Xi \Rightarrow t\end{array}}{\Box\Gamma, \Box\Sigma, \Delta, \Xi \Rightarrow s \circ t}\ (\Rightarrow \circ),$$

by the ind. hyp., $\vdash_{\mathsf{FL_{ei}}} \Gamma^{(n)}, \Delta \Rightarrow s$ for some n, and $\vdash_{\mathsf{FL_{ei}}} \Sigma^{(m)}, \Xi \Rightarrow t$ for some m. Using the rule of $(\Rightarrow \circ)$, $\vdash_{\mathsf{FL_{ei}}} \Gamma^{(n)}, \Sigma^{(m)}, \Delta, \Xi \Rightarrow s \circ t$. By the left-weakening rule, $\vdash_{\mathsf{FL_{ei}}} (\Gamma, \Sigma)^{(k)}, \Delta, \Xi \Rightarrow s \circ t$, where $k = \max\{m, n\}$. $\Box$

A similar argument works also for the involutive case:

Lemma 3.2.9. *Let $\Rightarrow \Diamond\Gamma, \Delta$ be a $\Diamond$-prenex $\{\circ, +, \wedge, \vee, 1, 0\}$-sequent.*

$$\textit{If } \vdash_{\mathsf{LL_w}} \Rightarrow \Diamond\Gamma, \Delta \textit{ then } \vdash_{\mathsf{InFL_{ew}}} \Rightarrow \Gamma^{(n)}, \Delta \textit{ for some } n \geq 0.$$

The following proposition is a variant of the deduction theorem for linear logic due to Avron [6, Section 4.1]; see also [81, Section 3.2]. It serves as an efficient translation of $\mathsf{FL_{ei}}$-deducibility (resp. $\mathsf{FL_{ew}}$-deducibility) into $\mathsf{ILZ_i}$-provability (resp. $\mathsf{ILZ_w}$-provability).

Lemma 3.2.10 (An affine variant of the deduction theorem of Avron [6])**.** *Let Φ be a finite set of intuitionistic $\mathcal{L}_{\mathsf{FL_e}}$-terms and $\Gamma \Rightarrow \sigma$ an intuitionistic $\mathcal{L}_{\mathsf{FL_e}}$-sequent.*

1. *$\Phi \vdash_{\mathsf{FL_{ei}}} \Gamma \Rightarrow \sigma$ if and only if $\vdash_{\mathsf{ILZ_i}} \Box\Phi, \Gamma \Rightarrow \sigma$.*
2. *$\Phi \vdash_{\mathsf{FL_{ew}}} \Gamma \Rightarrow \sigma$ if and only if $\vdash_{\mathsf{ILZ_w}} \Box\Phi, \Gamma \Rightarrow \sigma$.*

Proof. We only prove Statement 1, as one may show the second one in a similar way.

Right-to-left: suppose that $\vdash_{\mathsf{ILZ_i}} \Box\Phi, \Gamma \Rightarrow s$. Note that this sequent is a $\Box$-prenex $\mathcal{L}_{\mathsf{FL_e}}$-sequent. By Lemma 3.2.8, $\vdash_{\mathsf{FL_{ei}}} \Phi^{(n)}, \Gamma \Rightarrow s$ for some n. Due to the fact that $\Phi \vdash_{\mathsf{FL_{ei}}} \Rightarrow t$ for every $t \in \Phi$, we can derive the sequent $\Gamma \Rightarrow s$ in $\mathsf{FL_{ei}}$ from Φ by using the cut rule several times.

Left-to-right: the proof is merely by straightforward induction on the height of the derivation for $\Phi \vdash_{\mathsf{FL_{ei}}} \Gamma \Rightarrow \sigma$. If the derivation ends with an application of $(\to\Rightarrow)$, it must be of the form:

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ \Gamma \Rightarrow s & t, \Sigma \Rightarrow \sigma\end{array}}{s \to t, \Gamma, \Sigma \Rightarrow \sigma}\ (\to\Rightarrow)$$

By the ind. hyp., $\vdash_{\mathsf{ILZ_i}} \Box\Phi, \Gamma \Rightarrow s$ and $\vdash_{\mathsf{ILZ_i}} \Box\Phi, t, \Sigma \Rightarrow \sigma$. By applying the rule of $(\to\Rightarrow)$, we obtain a derivation for $\vdash_{\mathsf{ILZ_i}} \Box\Phi, \Box\Phi, s \to t, \Gamma, \Sigma \Rightarrow \sigma$. Using the rule of $(\Box\mathsf{c})$ repeatedly, we may construct a derivation for $\vdash_{\mathsf{ILZ_i}} \Box\Phi, s \to t, \Gamma, \Sigma \Rightarrow \sigma$. The rest of the proof is straightforward. $\Box$

It is easy to check that each of $\mathsf{FL}_{\mathsf{ei}}^{+}$-deducibility and BCK-deducibility is reducible to $\mathsf{ILZ}_{\mathsf{i}}$-provability. Moreover, there is also a simple reduction from deducibility in $\mathsf{InFL}_{\mathsf{ew}}$-deducibility to LL_{w}-provability.

Lemma 3.2.11. *Let Φ be a finite set of classical $\{\circ, +, \wedge, \vee, 1, 0\}$-terms and $\Rightarrow \Gamma$ a classical $\{\circ, +, \wedge, \vee, 1, 0\}$-sequent.*

$$\Phi \vdash_{\mathsf{InFL}_{\mathsf{ew}}} \Rightarrow \Gamma \textit{ if and only if } \vdash_{\mathsf{LL}_{\mathsf{w}}} \Rightarrow \Diamond\Phi^{\perp}, \Gamma.$$

Proof. Similar to the proof of Lemma 3.2.10. Use Lemma 3.2.9. □

By Fact 2.5.1, $\mathsf{ILZ}_{\mathsf{i}}$-provability and LL_{w}-provability belong to TOWER. By Lemmas 3.2.10 and 3.2.11, we hence obtain:

Corollary 3.2.12. *The class* TOWER *includes the following problems:*

- *the deducibility problem for* BCK*,*
- *the deducibility problem for* $\mathsf{FL}_{\mathsf{ei}}^{+}$*,*
- *the deducibility problem for* $\mathsf{FL}_{\mathsf{ei}}$*,*
- *the deducibility problem for* $\mathsf{InFL}_{\mathsf{ew}}$*.*

We now embark on a proof of the TOWER upper bound for deducibility in $\mathsf{FL}_{\mathsf{ew}}$. By Lemma 3.2.10, it is enough to show that provability in $\mathsf{ILZ}_{\mathsf{w}}$ is in TOWER.

Let c be an instance of the provability problem for $\mathsf{ILZ}_{\mathsf{w}}$. Modifying slightly the previous construction of $\mathcal{M}_c^{\mathsf{IEZ}_{\mathsf{i}}}$, we define the canonical $\mathsf{ABVASS}_{\bar{0}}$ $\mathcal{M}_c^{\mathsf{ILZ}_{\mathsf{w}}}$ of dimension $d = |\mathsf{Sub}(c)|$. To be precise:

- Q is the union of the following:
  - $\mathcal{P}(\mathsf{Sub}^{\Box}(c)) \times (\mathsf{Sub}(c) \cup \{\bullet\})$, where $\bullet$ is a fresh symbol.
  - $\{q_{\mathsf{leaf}}\}$, where q_{leaf} is a distinguished leaf state,
  - a set of some intermediate states that are indispensable to define the rules of $[\rightarrow\rightsquigarrow]$, $[\wedge \rightsquigarrow]$, $[\vee \rightsquigarrow]$ and $[\circ \rightsquigarrow]$, cf. Figure 3.1.
- The rules of $\mathcal{M}_c^{\mathsf{ILZ}_{\mathsf{w}}}$ consist of $[\mathsf{init}_1]$, $[\mathsf{init}_2]$, $[\mathsf{store}]$, $[1 \rightsquigarrow]$, $[\rightsquigarrow 1]$, $[0 \rightsquigarrow]$, $[\rightsquigarrow 0]$, $[\rightarrow\rightsquigarrow]$, $[\rightsquigarrow\rightarrow]$, $[\circ \rightsquigarrow]$, $[\rightsquigarrow \circ]$, $[\wedge \rightsquigarrow]$, $[\rightsquigarrow \wedge]$, $[\vee \rightsquigarrow]$, $[\rightsquigarrow \vee_1]$, $[\rightsquigarrow \vee_2]$, $[\Box\mathsf{i}]$, $[\mathsf{o}]$, $[\Box \rightsquigarrow]$ and $[\rightsquigarrow \Box]$. All of these rules except for $[\mathsf{o}]$, $[\Box \rightsquigarrow]$ and $[\rightsquigarrow \Box]$ are defined as in Figure 3.1.
- The rules of $[\mathsf{o}]$, $[\Box \rightsquigarrow]$ and $[\rightsquigarrow \Box]$ are defined as in Figure 3.2.

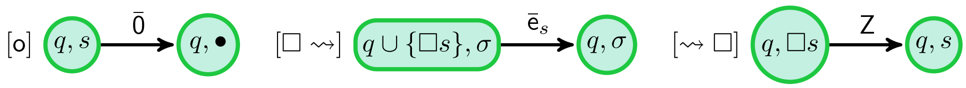

Figure 3.2: The rules of $[\mathsf{o}]$, $[\Box \rightsquigarrow]$ and $[\rightsquigarrow \Box]$.

Clearly, the $\mathsf{ABVASS}_{\bar{0}}$ rules in Figure 3.2 correspond to the inference rules of (o), $(\Box \Rightarrow)$ and $(\Rightarrow \Box)$ in $\mathsf{ILZ_w}$. Observe that $\mathcal{M}_c^{\mathsf{ILZ_w}}$ does not have the deduction rules corresponding to the inference rule for (fct). We now show that $\mathcal{M}_c^{\mathsf{ILZ_w}}$ can simulate the proof search of c in $\mathsf{ILZ_w}$ in the following sense:

Theorem 3.2.13. *Let σ be in $\mathsf{Sub}(c) \cup \{\varepsilon\}$, Θ a finite multiset of terms in $\mathsf{Sub}^{\Box}(c)$, and Γ a finite multiset of terms in $\mathsf{Sub}(c) \setminus \mathsf{Sub}^{\Box}(c)$.*

$$\vdash_{\mathsf{ILZ_w}} \Theta, \Gamma \Rightarrow \sigma \text{ if and only if } \mathcal{M}_c^{\mathsf{ILZ_w}}, \{q_{\mathsf{leaf}}\} \rightsquigarrow_\ell \langle \eta(\Theta), \sigma^\dagger; \bar{\mathsf{v}}_\Gamma \rangle.$$

Proof. Left-to-right direction is shown by induction on the height of the cut-free derivation in $\mathsf{ILZ_w}$ for $\Theta, \Gamma \Rightarrow \sigma$. If the last part of the derivation is of the form:

$$\cfrac{\begin{array}{c}\vdots\\ \Theta, \Gamma \Rightarrow\end{array}}{\Theta, \Gamma \Rightarrow s}\ (\mathsf{o}),$$

by the ind. hyp., $\mathcal{M}_c^{\mathsf{ILZ_w}}, \{q_{\mathsf{leaf}}\} \rightsquigarrow_\ell \langle \eta(\Theta), \bullet; \bar{\mathsf{v}}_\Gamma \rangle$. Applying the unary rule of $[\mathsf{o}]$, we obtain a lossy deduction tree for $\mathcal{M}_c^{\mathsf{ILZ_w}}, \{q_{\mathsf{leaf}}\} \rightsquigarrow_\ell \langle \eta(\Theta), s; \bar{\mathsf{v}}_\Gamma \rangle$.

Right-to-left: it is straightforward to show that the *standardization lemma* (Lemma 3.2.6) holds even for $\mathcal{M}_c^{\mathsf{ILZ_w}}$. We thus show this direction by induction on the length of the standard lossy deduction tree for $\mathcal{M}_c^{\mathsf{ILZ_w}}, \{q_{\mathsf{leaf}}\} \rightsquigarrow_\ell^{st} \langle \eta(\Theta), \sigma^\dagger; \bar{\mathsf{v}}_\Gamma \rangle$. If the standard lossy deduction tree is of the shape:

$$\begin{array}{c}\cfrac{\langle \eta(\Theta), s; \bar{\mathsf{v}}_\Gamma \rangle}{\langle \eta(\Theta), \bullet; \bar{\mathsf{v}}_\Gamma \rangle}\ [\mathsf{o}],\\ \vdots\end{array}$$

by the ind. hyp., $\vdash_{\mathsf{ILZ_w}} \Theta, \Gamma \Rightarrow$. We obtain a derivation in $\mathsf{ILZ_w}$ for $\Theta, \Gamma \Rightarrow s$ by applying the right-weakening rule.

The reader may refer to [78, Claims 4.3.1 and 4.3.2] for other typical cases. $\Box$

In particular, we obtain:

$$\vdash_{\mathsf{ILZ_w}} c \text{ if and only if } \mathcal{M}_c^{\mathsf{ILZ_w}}, \{q_{\mathsf{leaf}}\} \rightsquigarrow_\ell \langle \emptyset, c; \bar{0} \rangle.$$

By Lemma 3.2.10 and Theorem 2.5.5, we have:

Corollary 3.2.14. *The provability problem for* $\mathsf{ILZ_w}$ *is decidable in* TOWER.

Corollary 3.2.15. *The deducibility problem for* $\mathsf{FL_{ew}}$ *is decidable in* TOWER.

3.3 TOWER lower bounds for contraction-free logics

Following [78, Section 4.2], we make use of another reduction technique developed by Lazić and Schmitz. It is a reduction from the lossy reachability problem for ordinary BVASSes to the provability problem of $\lozenge$-prenex $\{\circ, +\}$-sequents in $\mathsf{LL_w}$.[6]

Consider an instance $(\mathcal{M}, \mathscr{L}, q_{\mathsf{root}})$ of the lossy reachability problem for ordinary BVASSes, where $\mathcal{M} = \langle Q, d, U, S, \emptyset, \emptyset\rangle$ and $\mathscr{L} \cup \{q_{\mathsf{root}}\} \subseteq Q$. Fix a set $Q \cup \{e_i \mid 1 \leq i \leq d\}$ of variables. For any $\langle q; \bar{\mathsf{v}}\rangle \in Q \times \mathbb{N}^d$, define:

$$\theta\langle q; \bar{\mathsf{v}}\rangle := q^-, (e_1^-)^{(\bar{\mathsf{v}}(1))}, \ldots, (e_d^-)^{(\bar{\mathsf{v}}(d))}.$$

We furthermore define the set T of nonlogical axioms that implement unary and split rules from $U \cup S$ as follows:

- $\Rightarrow q^-, q_1 \circ e_i$ for $q \xrightarrow{\bar{\mathsf{e}}_i} q_1 \in U$,
- $\Rightarrow q^-, e_i^-, q_1$ for $q \xrightarrow{-\bar{\mathsf{e}}_i} q_1 \in U$,
- $\Rightarrow q^-, q_1 + q_2$ for $q \to q_1 \mathsf{S}\, q_2 \in S$.

Note that every sequent from T is of the shape $\Rightarrow p_1^-, \ldots, p_n^-, t$, where $p_1^-, \ldots, p_n^-$ are negative variables and t is a classical $\{\circ, +\}$-term. For any sequent $S = \Rightarrow p_1^-, \ldots, p_n^-, t$ in T, we define $[S] = p_1 \circ \cdots \circ p_n \circ t^-$. We write $[T]$ for the finite multiset obtained from T by transforming each sequent S of T into $[S]$. Lazić and Schmitz [78] proved that:

$$\mathcal{M}, \mathscr{L} \rightsquigarrow_\ell \langle q; \bar{\mathsf{v}}\rangle \text{ if and only if } \vdash_{\mathsf{LL_w}} \Rightarrow \lozenge[T], \lozenge\mathscr{L}, \theta\langle q; \bar{\mathsf{v}}\rangle$$

for every $\langle q; \bar{\mathsf{v}}\rangle \in Q \times \mathbb{N}^d$. Specifically, the following thus holds:

Fact 3.3.1 (Lazić and Schmitz [78], Section 4.2)**.** $\mathcal{M}, \mathscr{L} \rightsquigarrow_\ell \langle q_{\mathsf{root}}; \bar{0}\rangle$ *if and only if* $\vdash_{\mathsf{LL_w}} \Rightarrow \lozenge[T], \lozenge\mathscr{L}, q_{\mathsf{root}}^-$.

[6] This reduction traces back to the technique, which Lincoln, Mitchell, Scedrov, and Shankar [81] used in the undecidability proof of LL-provability.

Note here that $\Rightarrow \Diamond[T], \Diamond\mathscr{L}, q^{-}_{\mathsf{root}}$ forms a $\Diamond$-prenex $\{\circ, +\}$-sequent. In view of Theorem 2.5.6, we thus obtain:

Fact 3.3.2 (Lazić and Schmitz [78], Section 4.2)**.** *The decision problem of determining whether a given $\Diamond$-prenex $\{\circ, +\}$-sequent is provable in $\mathsf{LL_w}$ is* Tower*-hard.*

3.3.1 Tower-hardness of deducibility in $\mathsf{FL_{ew}}$ and related systems

Using the complexity results shown by Lazić and Schmitz [78], we show that the deducibility problems for $\mathsf{FL_{ew}}$, $\mathsf{FL_{ei}}$, $\mathsf{FL^{+}_{ei}}$, BCK, and $\mathsf{InFL_{ew}}$ are hard for Tower.

A $\Box$*-prenex* $\{\rightarrow\}$*-sequent* is an intuitionistic $\{\rightarrow, \Box\}$-sequent of the shape $\Box\Gamma, \Delta \Rightarrow s$ where Γ, Δ are finite multiset of intuitionistic $\{\rightarrow\}$-terms and s is an intuitionistic $\{\rightarrow\}$-term. We now prove:

Theorem 3.3.3. *The decision problem of determining whether a given $\Box$-prenex $\{\rightarrow\}$-sequent is provable in the $\{\rightarrow, \Box\}$-fragment of $\mathsf{ILL_i}$ is* Tower*-hard.*

Proof. The following decision problem is Tower-hard by Fact 3.3.2.

- ▶ The problem of asking whether a given $\Diamond$-prenex $\{\circ, +\}$-sequent is provable in $\mathsf{LL_w}$.

It therefore suffices to construct a reduction from this problem to the one in the statement. Fix a $\Diamond$-prenex $\{\circ, +\}$-sequent $\Rightarrow \Diamond\Gamma, \Delta$ and a new variable x not occurring in the multiset $\Diamond\Gamma, \Delta$. By Theorem 3.1.7,

$$\vdash_{\mathsf{LL_w}} \Rightarrow \Diamond\Gamma, \Delta \text{ if and only if } \vdash_{\mathsf{ILL_i}} (\Diamond\Gamma, \Delta)^{\langle x \rangle} \Rightarrow x.$$

Observe that the latter is a $\Box$-prenex $\{\rightarrow\}$-sequent. Since $\mathsf{ILL_i}$ is a conservative extension of its $\{\rightarrow, \Box\}$-fragment, $\vdash_{\mathsf{ILL_i}} (\Diamond\Gamma, \Delta)^{\langle x \rangle} \Rightarrow x$ if and only if $(\Diamond\Gamma, \Delta)^{\langle x \rangle} \Rightarrow x$ is provable in the $\{\rightarrow, \Box\}$-fragment of $\mathsf{ILL_i}$. To sum up, $\vdash_{\mathsf{LL_w}} \Rightarrow \Diamond\Gamma, \Delta$ if and only if the sequent $(\Diamond\Gamma, \Delta)^{\langle x \rangle} \Rightarrow x$ is provable in the $\{\rightarrow, \Box\}$-fragment of $\mathsf{ILL_i}$. Clearly, the construction of $(\Diamond\Gamma, \Delta)^{\langle x \rangle} \Rightarrow x$ can be done in polynomial time. $\Box$

It is quite easy to check that the problem of determining whether a given $\Box$-prenex $\{\rightarrow\}$-sequent is provable in the $\{\rightarrow, \Box\}$-fragment of $\mathsf{ILZ_i}$ is polynomial time reducible to each of the deducibility problems for BCK, $\mathsf{FL^{+}_{ei}}$, $\mathsf{FL_{ei}}$ and $\mathsf{FL_{ew}}$. To this end, we recall the notion of *support set* from the previous section. For a finite multiset Γ of terms, let $\eta(\Gamma)$ denote the set of terms, each of which has the multiplicity strictly larger than 0.

Lemma 3.3.4. *Let $\Box\Gamma, \Delta \Rightarrow s$ be a $\Box$-prenex $\{\rightarrow\}$-sequent. The following are equivalent:*

1. $\Box\Gamma, \Delta \Rightarrow s$ *is provable in the* $\{\rightarrow, \Box\}$*-fragment of* $\mathsf{ILL_i}$,
2. $\eta(\Gamma) \vdash_{\mathsf{BCK}} \Delta \Rightarrow s$,
3. $\eta(\Gamma) \vdash_{\mathsf{FL}_{ei}^{+}} \Delta \Rightarrow s$,
4. $\eta(\Gamma) \vdash_{\mathsf{FL_{ei}}} \Delta \Rightarrow s$,
5. $\eta(\Gamma) \vdash_{\mathsf{FL_{ew}}} \Delta \Rightarrow s$.

Proof. ($1 \Rightarrow 2$): the following claim holds.

> ($\clubsuit$): *For any* $\Box$*-prenex* $\{\rightarrow\}$*-sequent* $\Box\Gamma, \Delta \Rightarrow s$*, if* $\Box\Gamma, \Delta \Rightarrow s$ *is provable in the* $\{\rightarrow, \Box\}$*-fragment of* $\mathsf{ILL_i}$ *then* $\vdash_{\mathsf{BCK}} \Gamma^{(n)}, \Delta \Rightarrow s$ *for some* n.

The proof is by induction on the size of cut-free derivation for $\Box\Gamma, \Delta \Rightarrow s$. Now suppose that $\Box\Gamma, \Delta \Rightarrow s$ is provable in the $\{\rightarrow, \Box\}$-fragment of $\mathsf{ILL_i}$. By Claim ($\clubsuit$), $\vdash_{\mathsf{BCK}} \Gamma^{(n)}, \Delta \Rightarrow s$ for some n. Since t is in $\eta(\Gamma)$ for every term t in Γ, several applications of (cut) yield a derivation for $\Delta \Rightarrow s$ in BCK with the theory $\eta(\Gamma)$.

($2 \Rightarrow 3$), ($3 \Rightarrow 4$), ($4 \Rightarrow 5$): all these implications are trivial.

($5 \Rightarrow 1$): observe that the following holds:

> ($\spadesuit$): *Let* Γ *be a finite multiset of* $\mathcal{L}_{\mathsf{FL_e}}$*-terms and* $\Delta \Rightarrow s$ *an intuitionistic* $\mathcal{L}_{\mathsf{FL_e}}$*-sequent.*
>
> $$\textit{If } \eta(\Gamma) \vdash_{\mathsf{FL_{ew}}} \Delta \Rightarrow s \textit{ then } \vdash_{\mathsf{ILZ_w}} \Box\Gamma, \Delta \Rightarrow s.$$

Suppose that $\eta(\Gamma) \vdash_{\mathsf{FL_{ew}}} \Delta \Rightarrow s$. By Claim ($\spadesuit$), $\vdash_{\mathsf{ILZ_w}} \Box\Gamma, \Delta \Rightarrow s$. Since $\mathsf{ILZ_w}$ enjoys cut elimination, clearly, $\mathsf{ILZ_w}$ is conservative over its $\{\rightarrow, \Box\}$-fragment. Thus there is a derivation for $\Box\Gamma, \Delta \Rightarrow s$ in the $\{\rightarrow, \Box\}$-fragment of $\mathsf{ILL_i}$. $\Box$

The above lemma guarantees that there is a polynomial-time reduction from the problem that determines if a given $\Box$-prenex $\{\rightarrow\}$-sequent is provable in the $\{\rightarrow, \Box\}$-fragment of $\mathsf{ILL_i}$ into each of the following problems:

- the deducibility problem for BCK,
- the deducibility problem for FL_{ei}^{+},

- the deducibility problem for $\mathsf{FL_{ei}}$,
- the deducibility problem for $\mathsf{FL_{ew}}$.

By Corollary 3.3.3, we thus conclude that the deducibility problems for BCK, $\mathsf{FL^+_{ei}}$, $\mathsf{FL_{ei}}$ and $\mathsf{FL_{ew}}$ are hard for TOWER. Similarly, we may easily show:

Lemma 3.3.5. *For any $\Diamond$-prenex $\{\circ, +, \wedge, \vee, 1, 0\}$-sequent $\Rightarrow \Diamond\Gamma, \Delta$,*

$$\vdash_{\mathsf{LL_w}} \Rightarrow \Diamond\Gamma, \Delta \textit{ if and only if } \eta(\Gamma^-) \vdash_{\mathsf{InFL_{ew}}} \Rightarrow \Delta.$$

Proof. Similar to the proof of Lemma 3.3.4. □

By Corollary 3.3.2 and Lemma 3.3.5, $\mathsf{InFL_{ew}}$-deducibility is also TOWER-hard. Recall that the deducibility problems for BCK, $\mathsf{FL^+_{ei}}$, $\mathsf{FL_{ei}}$, $\mathsf{FL_{ew}}$, and $\mathsf{InFL_{ew}}$ fall within TOWER (Corollaries 3.2.12 and 3.2.15). We therefore obtain:

Corollary 3.3.6. *The following are* TOWER*-complete problems:*

- *the deducibility problem for* BCK,
- *the deducibility problem for* $\mathsf{FL^+_{ei}}$,
- *the deducibility problem for* $\mathsf{FL_{ei}}$,
- *the deducibility problem for* $\mathsf{FL_{ew}}$,
- *the deducibility problem for* $\mathsf{InFL_{ew}}$.

3.3.2 TOWER-hardness of provability in elementary affine logic

In this subsection, we prove that the provability problems for $\mathsf{ELL_w}$ and its variants are TOWER-hard.

Lemma 3.3.7. *For any finite multiset Γ of classical $\{\circ, +, \wedge, \vee, 1, 0\}$-terms and any classical $\{\circ, +, \wedge, \vee, 1, 0\}$-term s,*

$$\vdash_{\mathsf{LL_w}} \Rightarrow \Diamond\Gamma, s \textit{ if and only if } \vdash_{\mathsf{ELL_w}} \Rightarrow \Diamond\Gamma, \Box s.$$

Proof. Left-to-right: suppose that $\vdash_{\mathsf{LL_w}} \Rightarrow \Diamond\Gamma, s$. By Lemma 3.2.9, $\vdash_{\mathsf{InFL_{ew}}} \Rightarrow \Gamma^{(n)}, s$ for some $n \geq 0$; so $\vdash_{\mathsf{ELL_w}} \Rightarrow \Gamma^{(n)}, s$ for some n. By the rules of (fct), (w), and ($\Diamond\mathsf{c}$), we obtain a derivation for $\Rightarrow \Diamond\Gamma, \Box s$, as desired.

Right-to-left: we assume that $\vdash_{\mathsf{ELL_w}} \Rightarrow \Diamond\Gamma, \Box s$. Clearly, $\vdash_{\mathsf{LL_w}} \Rightarrow \Diamond\Gamma, \Box s$, as $\mathsf{LL_w}$ is strictly stronger than $\mathsf{ELL_w}$. Since $\Rightarrow \Diamond s^-, s$ is provable in $\mathsf{LL_w}$, we may construct a derivation for $\vdash_{\mathsf{LL_w}} \Rightarrow \Diamond\Gamma, s$ by applying (cut). □

Given an instance $(\mathcal{M}, \mathscr{L}, q_{\mathsf{root}})$ of the lossy-reachability problem for ordinary BVASSes, the following holds by Theorem 3.3.1 and Lemma 3.3.7:

$$\mathcal{M}, \mathscr{L} \rightsquigarrow_{\ell} \langle q_{\mathsf{root}}; \bar{0}\rangle \text{ if and only if } \vdash_{\mathsf{ELL_w}} \Rightarrow \Diamond[T], \Diamond\mathscr{L}, \Box q^{-}_{\mathsf{root}}.$$

Since the latter sequent is built up from negative and positive variables using connectives from the language $\{\circ, +, \Box, \Diamond\}$, the following holds:

Corollary 3.3.8. *The decision problem of determining whether a given classical* $\{\circ, +, \Box, \Diamond\}$*-sequent is provable in* $\mathsf{ELL_w}$ *is* TOWER*-hard.*

Combining this with Corollary 3.2.7, we obtain:

Corollary 3.3.9. *Provability in* $\mathsf{ELL_w}$ *is* TOWER*-complete.*

One can easily check that the same is true for $\{\rightarrow, \Box\}$-fragment of $\mathsf{IEL_i}$:

Theorem 3.3.10. *The provability problem for the* $\{\rightarrow, \Box\}$*-fragment of* $\mathsf{IEL_i}$ *is* TOWER*-complete.*

Proof. The proof proceeds in the same way as that of Theorem 3.3.3. We fix a classical $\{\circ, +, \Box, \Diamond\}$-sequent $\Rightarrow \Gamma$ and a fresh variable x not occurring in Γ. By Theorem 3.1.7, $\vdash_{\mathsf{ELL_w}} \Rightarrow \Gamma$ iff $\vdash_{\mathsf{IEL_i}} \Gamma^{\langle x\rangle} \Rightarrow x$. It follows from cut elimination for $\mathsf{IEL_i}$ that $\mathsf{IEL_i}$ is a conservative over the $\{\rightarrow, \Box\}$-fragment; thus, $\vdash_{\mathsf{IEL_i}} \Gamma^{\langle x\rangle} \Rightarrow x$ iff $\Gamma^{\langle x\rangle} \Rightarrow x$ is provable in the $\{\rightarrow, \Box\}$-fragment of $\mathsf{IEL_i}$. Hence $\vdash_{\mathsf{ELL_w}} \Rightarrow \Gamma$ iff $\Gamma^{\langle x\rangle} \Rightarrow x$ has a derivation in the $\{\rightarrow, \Box\}$-fragment of $\mathsf{IEL_i}$. Thus there is a polynomial-time reduction from the problem determining if a given classical $\{\circ, +, \Box, \Diamond\}$-sequent is provable in $\mathsf{ELL_w}$ to the provability problem for the $\{\rightarrow, \Box\}$-fragment of $\mathsf{IEL_i}$. By Corollary 3.3.8, the latter is TOWER-hard. The TOWER-membership is obvious from Corollary 3.2.7. $\Box$

Remark on provability in elementary linear logic. In [78], Lazić and Schmitz also proved that the reachability problem for ordinary ABVASSes (i.e., $\mathsf{ABVASS_{\bar{0}}}$s without full zero test rules) is reducible into the provability problem for LL, thus obtaining another proof of the undecidability of provability in LL, first proven by Lincoln, Mitchell, Scedrov, and Shankar [81]. Our purpose here is to show that reachability in ordinary ABVASSes is many-one reducible to provability in a bit weaker version of LL, i.e., ELL. This implies the undecidability of ELL.

Let $(\mathcal{M}, \mathscr{L}, q_{\mathsf{root}})$ be an instance of the reachability problem for ordinary ABVASSes. We assume here that $\mathcal{M} = \langle Q, d, U, S, F, \emptyset\rangle$ is an ordinary ABVASS and that $\mathscr{L} \cup \{q_{\mathsf{root}}\}$ is a subset of Q. Given a configuration $\langle q; \bar{\mathsf{v}}\rangle$ of

$\mathcal{M}$, we define the corresponding sequent $\theta\langle q; \bar{\mathsf{v}}\rangle$ in exactly the same way as before. In the same method taken by Lazić and Schmitz [78, Section 4.2], we construct the set T of nonlogical axioms from rules in $U \cup S \cup F$ as follows:

- $\Rightarrow q^-, q_1 \circ e_i$ for $q \xrightarrow{\bar{\mathsf{e}}_i} q_1 \in U$,
- $\Rightarrow q^-, e_i^-, q_1$ for $q \xrightarrow{-\bar{\mathsf{e}}_i} q_1 \in U$,
- $\Rightarrow q^-, q_1 + q_2$ for $q \to q_1 \mathsf{S}\, q_2 \in S$,
- $\Rightarrow q^-, q_1 \vee q_2$ for $q \to q_1 \mathsf{F}\, q_2 \in F$.

Note that additive connectives make it possible to simulate the fork rules in LL. The following holds:

Fact 3.3.11 (Lazić and Schmitz [78], Section 4.2)**.** *$\mathcal{M}, \mathscr{L} \rightsquigarrow \langle q_{\mathsf{root}}; \bar{0}\rangle$ if and only if $\vdash_{\mathsf{LL}} \Rightarrow \Diamond[T], \Diamond\mathscr{L}, q^-_{\mathsf{root}}$.*

As a corollary, we have:

Fact 3.3.12 (Lazić and Schmitz [78], Section 4.2)**.** *There are log-space reductions:*

1. *from reachability in ordinary ABVASSes to the problem of asking whether a given $\Diamond$-prenex $\{\circ, +, \wedge, \vee\}$-sequent is provable in LL,*
2. *from reachability in ordinary BVASSes to the problem of asking whether a given $\Diamond$-prenex $\{\circ, +\}$-sequent is provable in LL.*

Given a multiset $\Gamma = s_1, \ldots, s_n$ of classical terms, $\Gamma^{\vee 0}$ denotes the multiset $s_1 \vee 0, \ldots, s_n \vee 0$.

Lemma 3.3.13. *For any finite multiset Γ of classical $\{\circ, +, \wedge, \vee, 1, 0\}$-terms and any classical $\{\circ, +, \wedge, \vee, 1, 0\}$-term s,*

$$\vdash_{\mathsf{LL}} \Rightarrow \Diamond\Gamma, s \textit{ if and only if } \vdash_{\mathsf{ELL}} \Rightarrow \Diamond(\Gamma^{\vee 0}), \Box s.$$

Proof. The proof is almost the same as that of Lemma 3.3.7. For starters, observe that the following claim holds.[7]

($\heartsuit$): *Let $\Rightarrow \Diamond\Gamma, \Delta$ be a $\Diamond$-prenex $\{\circ, +, \wedge, \vee, 1, 0\}$-sequent.*

If $\vdash_{\mathsf{LL}} \Rightarrow \Diamond\Gamma, \Delta$ then $\vdash_{\mathsf{InFL_e}} \Rightarrow (\Gamma^{\vee 0})^{(n)}, \Delta$ for some n.

[7]Claim ($\heartsuit$) is merely a classical version of [118, Proposition 2.6].

The proof proceeds by induction on the cut-free derivation in LL for $\Rightarrow \Diamond\Gamma, \Delta$.

Left-to-right: suppose that $\vdash_{\mathsf{LL}} \Rightarrow \Diamond\Gamma, s$. By Claim ($\heartsuit$), $\vdash_{\mathsf{InFL_e}} \Rightarrow (\Gamma^{\vee 0})^{(n)}, s$ for some n. Clearly, $\vdash_{\mathsf{ELL}} \Rightarrow (\Gamma^{\vee 0})^{(n)}, s$ for some n. By the rule of (fct), $\vdash_{\mathsf{ELL}} \Rightarrow \Diamond(\Gamma^{\vee 0})^{(n)}, \Box s$ for some n. We therefore obtain a derivation for $\vdash_{\mathsf{ELL}} \Rightarrow \Diamond(\Gamma^{\vee 0}), \Box s$ by applying ($\Diamond\mathsf{c}$) several times.

Right-to-left: suppose that $\vdash_{\mathsf{ELL}} \Rightarrow \Diamond(\Gamma^{\vee 0}), \Box s$. It is clear that $\vdash_{\mathsf{LL}} \Rightarrow \Diamond(\Gamma^{\vee 0}), \Box s$. For each classical term s, $\vdash_{\mathsf{LL}} \Rightarrow \Diamond s, \Box(s^- \wedge 1)$. Recall also that $\vdash_{\mathsf{LL}} \Rightarrow \Diamond s^-, s$ for each classical term s. Thus we may construct a derivation in LL for $\Rightarrow \Diamond\Gamma, s$, as desired. $\Box$

By Fact 3.3.11 and Lemma 3.3.13:

$$\mathcal{M}, \mathscr{L} \rightsquigarrow \langle q_{\mathsf{root}}; \bar{0}\rangle \text{ if and only if } \vdash_{\mathsf{ELL}} \Rightarrow \Diamond([T]^{\vee 0}), \Diamond(\mathscr{L}^{\vee 0}), \Box q^-_{\mathsf{root}}.$$

Recall that ELL-provability is reducible to IEL-provability (Theorem 3.1.7). Reachability in ordinary ABVASSes is undecidable. More precisely:

Fact 3.3.14 (Lazić and Schmitz [78], Sections 3.2, 3.3, and 3.4)**.** *The reachability problem for ordinary ABVASSes is* RE*-complete.*

Since it is clear that the set of provable terms (or sequents) in IEL is in RE, i.e., recursively enumerable, we conclude:

Corollary 3.3.15. *The provability problems for* ELL *and* IEL *are* RE*-complete.*

Chapter 4

Nonelementary lower bounds for various substructural logics

> We do not choose our friends because they embody all the pleasant qualities of humanity, but because they are the people that they are. And so in mathematics; a property common to too many objects can hardly be very exciting, and mathematical ideas also become dim unless they have plenty of individuality.
>
> Godfrey H. Hardy
> *A Mathematician's Apology*, 1940

In the previous chapter, we have obtained a tight complexity estimate for deducibility in some substructural logics. However, the logical systems that we have investigated so far are very few in number. It thus is now time to consider if we may provide a more general characterization of substructural logics in terms of computational complexity; that is, we have a question:

> *Is it possible to provide in one shot a complexity bound for a large range of substructural logics?*

The main aim of this chapter is to offer a partial answer to this question by establishing the following three claims.

1. Every substructural logic weaker than $\mathsf{FL}_{\mathsf{ew}}$ is TOWER-hard with respect to deducibility (Corollary 4.1.5).
2. Every substructural logic weaker than $\mathsf{FL}_{\mathsf{eco}}$ is ACK-hard with respect to deducibility (Corollary 4.2.5).

3. Every substructural logic weaker than $\mathsf{FL_w}$ is HACK-hard with respect to deducibility (Corollary 4.1.8).

It follows from Claims (1)–(3) that various decision problems, algebraically shown by Blok and van Alten [15], van Alten [124], and Horčík [62] to be decidable, are not elementary recursive. Below are a few examples of nonelementary deducibility problems that fit within the scope of Claims (1)–(3).

- The deducibility problems for $\mathsf{FL_i}$ and $\mathsf{FL_w}$: TOWER-hard (Corollary 4.1.6).[1]
- The deducibility problem for $\mathsf{FL_e}{}^n_m$ ($m > n \geq 0$): TOWER-hard (Corollary 4.1.6).
- The deducibility problem for $\mathsf{FL_e}{}^n_m$ ($n > m \geq 1$): ACK-hard (Corollary 4.2.6).[2]
- The deducibility problem for FL^1_m ($m \geq 2$): HACK-hard (Corollary 4.1.9).

Corollaries 4.1.6 and 4.1.9 improve, for instance, the existing polynomial space lower bounds for the deducibility problems for FL^1_m (where $m \geq 2$) and $\mathsf{FL_e}{}^n_m$ shown by Horčík and Terui [64].

Organization of this chapter. In Section 4.1, we prove the TOWER-hardness of deducibility in substructural logics weaker than $\mathsf{FL_{ew}}$. To this end, we construct an efficient reduction from deducibility in $\mathsf{FL_{ew}}$ into deducibility in FL. The key insight is that we may simulate the rules of exchange, left-weakening, and right-weakening by using the corresponding nonlogical axioms. The idea is reminiscent of the nice technique of Lincoln, Scedrov, and Shankar [82], used in the undecidability proof of provability in second-order intuitionistic multiplicative linear logic ($\mathsf{IMLL2}$).[3] Since it is difficult to directly prove that our translation is sound and complete, we again make use of intuitionistic affine logic ($\mathsf{ILZ_w}$). However, the Gentzen-style sequent system $\mathsf{ILZ_w}$ that we employ in this chapter differs slightly from

[1]Recall that the tight complexity, as observed by Greati and Ramanayake [51], is by far higher than TOWER, i.e., HACK-complete.

[2]St. John [110] proved a stronger result, Fact 4.2.10.

[3]The main insight behind the proof by Lincoln, Scedrov, and Shankar [82] is that the rules of (left-)weakening and contraction can be simulated by second-order quantified terms consisting only of multiplicative connectives. This leads to a reduction of second-order intuitionistic propositional logic ($\mathsf{LJ2}$) to $\mathsf{IMLL2}$; note that provability in $\mathsf{LJ2}$ is recursively unsolvable (see [84]).

the version defined in Chapter 3. Unlike the latter, the former is based on a noncommutative language (see Chapter 2) and is equipped with the rule of explicit exchange. In Subsection 4.1.1, we thus start with a description of a sequent calculus for $\mathsf{ILZ_w}$ with explicit exchange and summarize some of its important properties, such as cut elimination. In Subsection 4.1.2, we show the first main result of this chapter, i.e., Corollary 4.1.5. In Subsection 4.1.3, we prove that deducibility in a substructural logic is HACK-hard if the logic is between FL and $\mathsf{FL_w}$.

In Section 4.2, we confirm that the argument of Section 4.1 also works for the case of logics weaker than $\mathsf{FL_{eco}}$. In conjunction with the famous achievement by Urquhart [122], the ACK-completeness of provability in (the positive fragment of) $\mathsf{FL_{ec}}$, we can infer the ACK-hardness of deducibility in all these logics. In Subsection 4.2.1, for our purpose in this section, we introduce a cut-free sequent calculus for $\mathsf{FL_{eco}}$. In Subsection 4.2.2, we then prove that there is a straightforward reduction from $\mathsf{FL_{eco}}$-deducibility to FL-deducibility. In fact, it suffices to show that there is a reduction from $\mathsf{FL_{eco}}$-*provability* to FL-deducibility, as provability in $\mathsf{FL_{eco}}$ is already ACK-complete.[4]

In Section 4.3, we mention some immediate consequences that follow from the main theorems shown in Sections 4.1 and 4.2.

4.1 Nonelementary problems in substructural logics weaker than $\mathsf{FL_{ew}}$

4.1.1 Sequent calculus for $\mathsf{ILZ_w}$ with explicit exchange

All the logical systems in this chapter—in contrast to the sequent systems in Chapter 3—are based on noncommutative intuitionistic languages. For starters, we extend the noncommutative intuitionistic language $\mathcal{L}_{\mathsf{FL}}$ by adding to the unary operation symbol $\Box$. For simplicity, we write $\mathcal{L}_{\mathsf{FL}_\Box}$ for the resulting language. The notions of intuitionistic $\mathcal{L}_{\mathsf{FL}_\Box}$-terms and intuitionistic $\mathcal{L}_{\mathsf{FL}_\Box}$-sequents are defined in the obvious manner. We furthermore reformulate intuitionistic affine logic with zero ($\mathsf{ILZ_w}$) as a single-succedent sequent calculus with the rule of explicit exchange. In what follows, the notation $\mathsf{ILZ_w}$ denotes the sequent calculus (over the language $\mathcal{L}_{\mathsf{FL}_\Box}$) obtained from FL by adding the rules of (i), (o), (e), and the following inference rules concerning exponential modality:

[4]This result, Proposition 4.2.8, will be also explicitly proven.

$$\frac{\Gamma, s, \Delta \Rightarrow \sigma}{\Gamma, \Box s, \Delta \Rightarrow \sigma}\ (\Box \Rightarrow) \qquad \frac{\Box\Gamma \Rightarrow s}{\Box\Gamma \Rightarrow \Box s}\ (\Rightarrow \Box) \qquad \frac{\Gamma, \Box s, \Box s, \Delta \Rightarrow \sigma}{\Gamma, \Box s, \Delta \Rightarrow \sigma}\ (\Box \mathsf{c})$$

It is straightforward to show the following propositions:

Proposition 4.1.1. *Cut elimination holds for the sequent calculus for* $\mathsf{ILZ_w}$ *with explicit exchange.*

Proof. The reader may consult Lincoln, Mitchell, Scedrov, and Shankar [81, Appendix A] for a Gentzen-style proof of cut elimination for linear logic. The normalization procedure provided by them can be adapted to $\mathsf{ILZ_w}$ with minor modification. $\Box$

Proposition 4.1.2. *Let* $\Phi = \{s_1, \ldots, s_n\}$ *be a finite set of intuitionistic* $\mathcal{L}_{\mathsf{FL}}$*-terms and* $\Gamma \Rightarrow \sigma$ *be an intuitionistic* $\mathcal{L}_{\mathsf{FL}}$*-sequent.*

$$\vdash_{\mathsf{ILZ_w}} \Box\Phi, \Gamma \Rightarrow \sigma \text{ whenever } \Phi \vdash_{\mathsf{FL_{ew}}} \Gamma \Rightarrow \sigma.$$

Here, $\Box\Phi$ *denotes the sequence* $\Box s_1, \ldots, \Box s_n$*.*

Proof. Almost the same as the proof of Lemma 3.2.10; the proof is by induction on the size of the derivation for $\Phi \vdash_{\mathsf{FL_{ew}}} \Gamma \Rightarrow \sigma$. $\Box$

It is also not very difficult to show that the converse direction of this lemma holds; however, it is not necessary in the argument below.

4.1.2 Tower-hardness of deducibility in substructural logics weaker than $\mathsf{FL_{ew}}$

We define the notion of *simple sequent* that is essentially the same as !-prenex sequents by Terui [118]. A *simple intuitionistic* $\mathcal{L}_{\mathsf{FL}_\Box}$*-term* is an intuitionistic $\mathcal{L}_{\mathsf{FL}}$-term or a term of the form $\Box s$, where s is an intuitionistic $\mathcal{L}_{\mathsf{FL}}$-term. A *simple intuitionistic* $\mathcal{L}_{\mathsf{FL}_\Box}$*-sequent* is an intuitionistic $\mathcal{L}_{\mathsf{FL}_\Box}$-sequent of the form $\Gamma \Rightarrow \sigma$, where Γ is a finite sequence of simple intuitionistic $\mathcal{L}_{\mathsf{FL}_\Box}$-terms and σ is an intuitionistic $\mathcal{L}_{\mathsf{FL}}$-term or empty.

Given a simple intuitionistic $\mathcal{L}_{\mathsf{FL}_\Box}$-term s, we define:

$$\mathsf{Sub}^-(s) := \begin{cases} \mathsf{Sub}(s) & \text{if } s \text{ is an intuitionistic } \mathcal{L}_{\mathsf{FL}}\text{-term,} \\ \mathsf{Sub}(t) & \text{if } s = \Box t, \text{ where } t \text{ is an intuitionistic } \mathcal{L}_{\mathsf{FL}}\text{-term.} \end{cases}$$

For a sequence $\Gamma = s_1, \ldots, s_n$ of simple intuitionistic $\mathcal{L}_{\mathsf{FL}_\Box}$-terms, we write:

- $\mathsf{Sub}^-(\Gamma)$ for the set $\mathsf{Sub}^-(s_1) \cup \cdots \cup \mathsf{Sub}^-(s_n)$,

- $\Gamma^\star$ for the sequence of terms obtained from Γ by removing all the terms of the shape $\Box t$, and
- $\eta^\Box(\Gamma)$ for the set $\{t \mid \Box t \in \eta^*(\Gamma)\}$; here, $\eta^*(\Gamma)$ denotes the support of Γ, i.e., the *set* of terms occurring in Γ.

For instance, if $\Gamma = s, t, \Box u, \Box u, v, \Box v, \Box w, w, w$ (where s, t, u, v, w are assumed to be intuitionistic $\mathcal{L}_{\mathsf{FL}}$-terms),

$$\begin{array}{lcl} \Gamma^\star & = & s, t, v, w, w, \\ \eta^\Box(\Gamma) & = & \{u, v, w\}, \\ \mathsf{Sub}^-(\Gamma) & = & \mathsf{Sub}(s) \cup \mathsf{Sub}(t) \cup \mathsf{Sub}(u) \cup \mathsf{Sub}(v) \cup \mathsf{Sub}(w). \end{array}$$

Given a simple intuitionistic $\mathcal{L}_{\mathsf{FL}_\Box}$-sequent $\Gamma \Rightarrow \sigma$, we define $\mathsf{Sub}^-(\Gamma \Rightarrow \sigma) = \mathsf{Sub}^-(\Gamma) \cup \mathsf{Sub}(\sigma)$, where:

$$\mathsf{Sub}(\sigma) := \begin{cases} \mathsf{Sub}(s) & \text{if } \sigma = s \\ \emptyset & \text{if } \sigma \text{ is empty.} \end{cases}$$

Given a set X of intuitionistic $\mathcal{L}_{\mathsf{FL}}$-terms, we define the set $\Psi_{\mathsf{ew}}(X)$ of intuitionistic $\mathcal{L}_{\mathsf{FL}}$-terms by:

$$\begin{array}{rcl} \Psi_{\mathsf{ew}}(X) & := & \{s \backslash 1 \mid s \in X\} \\ & \cup & \{0 \backslash s \mid s \in X\} \\ & \cup & \{(s \circ t) \backslash (t \circ s) \mid s, t \in X\}. \end{array}$$

The following lemma is the main engine of our proof.

Lemma 4.1.3. *Let $\Gamma \Rightarrow \sigma$ be a simple intuitionistic $\mathcal{L}_{\mathsf{FL}_\Box}$-sequent.*

$$\textit{If } \vdash_{\mathsf{ILZ_w}} \Gamma \Rightarrow \sigma \textit{ then } \Psi_{\mathsf{ew}}(\mathsf{Sub}^-(\Gamma \Rightarrow \sigma)) \cup \eta^\Box(\Gamma) \vdash_{\mathsf{FL}} \Gamma^\star \Rightarrow \sigma.$$

Proof. Thanks to Proposition 4.1.1, the cut elimination theorem for $\mathsf{ILZ_w}$, we can show this by induction on the cut-free derivation in $\mathsf{ILZ_w}$ for $\Gamma \Rightarrow \sigma$. Below we analyze some typical cases; the other cases being similar.

- If the derivation ends with an application of (init), then the derivation is of the form $s \Rightarrow s$, where s must be an intuitionistic $\mathcal{L}_{\mathsf{FL}}$-term. Clearly, $\vdash_{\mathsf{FL}} s \Rightarrow s$ $(= s^\star \Rightarrow s)$; thus $\Psi_{\mathsf{ew}}(\mathsf{Sub}^-(s \Rightarrow s)) \cup \eta^\Box(s) \vdash_{\mathsf{FL}} s \Rightarrow s$.

- In the case where $\Gamma \Rightarrow t \,/\, s$ is obtained from $\Gamma, s \Rightarrow t$ by the rule of $(\Rightarrow /)$, note that s, t and $t \,/\, s$ must be intuitionistic $\mathcal{L}_{\mathsf{FL}}$-terms. By the ind. hyp.,

$$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma, s \Rightarrow t)) \cup \eta^{\Box}(\Gamma, s) \vdash_{\mathsf{FL}} \Gamma^{\star}, s \Rightarrow t.$$

Clearly, $\eta^{\Box}(\Gamma, s) = \eta^{\Box}(\Gamma)$ since s is not of the form $\Box t$. We also have:

$$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma, s \Rightarrow t)) \subseteq \Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma \Rightarrow t \,/\, s)).$$

Thus

$$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma \Rightarrow t \,/\, s)) \cup \eta^{\Box}(\Gamma) \vdash_{\mathsf{FL}} \Gamma^{\star}, s \Rightarrow t.$$

By $(\Rightarrow /)$, we obtain a desired sequent.

- In the case where $\Sigma, t \,/\, s, \Gamma, \Delta \Rightarrow \sigma$ is obtained from $\Gamma \Rightarrow s$ and $\Sigma, t, \Delta \Rightarrow \sigma$ by $(/ \Rightarrow)$, observe first that s, t, $t \,/\, s$ must be intuitionistic $\mathcal{L}_{\mathsf{FL}}$-terms. By the ind. hyp.,

$$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma \Rightarrow s)) \cup \eta^{\Box}(\Gamma) \vdash_{\mathsf{FL}} \Gamma^{\star} \Rightarrow s,$$

and

$$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Sigma, t, \Delta \Rightarrow \sigma)) \cup \eta^{\Box}(\Sigma, t, \Delta) \vdash_{\mathsf{FL}} \Sigma^{\star}, t, \Delta^{\star} \Rightarrow \sigma.$$

Note that the following inclusion relations hold:

1. $\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma \Rightarrow s)) \subseteq \Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Sigma, t \,/\, s, \Gamma, \Delta \Rightarrow \sigma))$,
2. $\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Sigma, t, \Delta \Rightarrow \sigma)) \subseteq \Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Sigma, t \,/\, s, \Gamma, \Delta \Rightarrow \sigma))$,
3. $\eta^{\Box}(\Gamma) \cup \subseteq \eta^{\Box}(\Sigma, t \,/\, s, \Gamma, \Delta)$,
4. $\eta^{\Box}(\Sigma, t, \Delta) \subseteq \eta^{\Box}(\Sigma, t \,/\, s, \Gamma, \Delta)$.

We thus have:

$$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Sigma, t \,/\, s, \Gamma, \Delta \Rightarrow \sigma)) \cup \eta^{\Box}(\Sigma, t \,/\, s, \Gamma, \Delta) \vdash_{\mathsf{FL}} \Gamma^{\star} \Rightarrow s,$$

and

$$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Sigma, t \,/\, s, \Gamma, \Delta \Rightarrow \sigma)) \cup \eta^{\Box}(\Sigma, t \,/\, s, \Gamma, \Delta) \vdash_{\mathsf{FL}} \Sigma^{\star}, t, \Delta^{\star} \Rightarrow \sigma.$$

By applying $(/ \Rightarrow)$, we have the following:

$$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Sigma, t \,/\, s, \Gamma, \Delta \Rightarrow \sigma)) \cup \eta^{\Box}(\Sigma, t \,/\, s, \Gamma, \Delta) \vdash_{\mathsf{FL}} \Sigma^{\star}, t \,/\, s, \Gamma^{\star}, \Delta^{\star} \Rightarrow \sigma.$$

- If $\Gamma, \Box s, \Delta \Rightarrow \sigma$ is deduced from $\Gamma, s, \Delta \Rightarrow \sigma$ by the rule of $(\Box \Rightarrow)$, by the ind. hyp.:

$$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma, s, \Delta \Rightarrow \sigma)) \cup \eta^{\Box}(\Gamma, s, \Delta) \vdash_{\mathsf{FL}} \Gamma^{\star}, s, \Delta^{\star} \Rightarrow \sigma.$$

The following two inclusion relations hold:

$$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma, s, \Delta \Rightarrow \sigma)) \subseteq \Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma, \Box s, \Delta \Rightarrow \sigma)),$$

and

$$\eta^{\Box}(\Gamma, s, \Delta) \subseteq \eta^{\Box}(\Gamma, \Box s, \Delta).$$

Hence

$$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma, \Box s, \Delta \Rightarrow \sigma)) \cup \eta^{\Box}(\Gamma, \Box s, \Delta) \vdash_{\mathsf{FL}} \Gamma^{\star}, s, \Delta^{\star} \Rightarrow \sigma.$$

Clearly, s is a member of $\eta^{\Box}(\Gamma, \Box s, \Delta)$; thus,

$$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma, \Box s, \Delta \Rightarrow \sigma)) \cup \eta^{\Box}(\Gamma, \Box s, \Delta) \vdash_{\mathsf{FL}} \Rightarrow s.$$

By applying the cut rule, we have the following as desired:

$$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma, \Box s, \Delta \Rightarrow \sigma)) \cup \eta^{\Box}(\Gamma, \Box s, \Delta) \vdash_{\mathsf{FL}} \Gamma^{\star}, \Delta^{\star} \Rightarrow \sigma.$$

- If $\Gamma, t, s, \Delta \Rightarrow \sigma$ is obtained from $\Gamma, s, t, \Delta \Rightarrow \sigma$ by applying the exchange rule, then by the ind. hyp., we have:

$$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma, s, t, \Delta \Rightarrow \sigma)) \cup \eta^{\Box}(\Gamma, s, t, \Delta) \vdash_{\mathsf{FL}} (\Gamma, s, t, \Delta)^{\star} \Rightarrow \sigma.$$

We need to analyze some subcases here. For instance, let us consider the case where s and t are both intuitionistic $\{\backslash, /, \circ, \wedge, \vee, 1, 0\}$-terms; one may check the other cases more easily. Obviously,

$$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma, s, t, \Delta \Rightarrow \sigma)) \subseteq \Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma, t, s, \Delta \Rightarrow \sigma)),$$

and

$$\eta^{\Box}(\Gamma, s, t, \Delta) \subseteq \eta^{\Box}(\Gamma, t, s, \Delta).$$

Thus

$$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma, t, s, \Delta \Rightarrow \sigma)) \cup \eta^{\Box}(\Gamma, t, s, \Delta) \vdash_{\mathsf{FL}} \Gamma^{\star}, s, t, \Delta^{\star} \Rightarrow \sigma.$$

By $(\circ \Rightarrow)$,

$$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma, t, s, \Delta \Rightarrow \sigma)) \cup \eta^{\Box}(\Gamma, t, s, \Delta) \vdash_{\mathsf{FL}} \Gamma^{\star}, s \circ t, \Delta^{\star} \Rightarrow \sigma.$$

Note that $(t \circ s) \backslash (s \circ t) \in \Psi_{\mathsf{ew}}(\Gamma, t, s, \Delta \Rightarrow \sigma)$. Since $(\Rightarrow \backslash)$ and $(\circ \Rightarrow)$ are both invertible:

$$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma, t, s, \Delta \Rightarrow \sigma)) \cup \eta^{\Box}(\Gamma, t, s, \Delta) \vdash_{\mathsf{FL}} t, s \Rightarrow s \circ t.$$

By using (cut), we have:

$$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma, t, s, \Delta \Rightarrow \sigma)) \cup \eta^{\Box}(\Gamma, t, s, \Delta) \vdash_{\mathsf{FL}} \Gamma^{\star}, t, s, \Delta^{\star} \Rightarrow \sigma.$$

- If $\Gamma \Rightarrow s$ is obtained from $\Gamma \Rightarrow$ by an application of (o), by the ind. hyp., we have:

  $$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma \Rightarrow)) \cup \eta^{\Box}(\Gamma) \vdash_{\mathsf{FL}} \Gamma^{\star} \Rightarrow .$$

  Since $\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma \Rightarrow)) \cup \eta^{\Box}(\Gamma) \subseteq \Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma \Rightarrow s)) \cup \eta^{\Box}(\Gamma)$, the following holds:

  $$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma \Rightarrow s)) \cup \eta^{\Box}(\Gamma) \vdash_{\mathsf{FL}} \Gamma^{\star} \Rightarrow .$$

  By applying $(\Rightarrow 0)$,

  $$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma \Rightarrow s)) \cup \eta^{\Box}(\Gamma) \vdash_{\mathsf{FL}} \Gamma^{\star} \Rightarrow 0.$$

  Due to the fact that $0 \backslash s$ is in $\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma \Rightarrow s))$ and the invertibility of $(\Rightarrow \backslash)$, we have:

  $$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma \Rightarrow s)) \cup \eta^{\Box}(\Gamma) \vdash_{\mathsf{FL}} 0 \Rightarrow s.$$

  By using cut,

  $$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma \Rightarrow s)) \cup \eta^{\Box}(\Gamma) \vdash_{\mathsf{FL}} \Gamma^{\star} \Rightarrow s.$$

- In the case where $\Gamma, s, \Delta \Rightarrow \sigma$ is obtained from $\Gamma, \Delta \Rightarrow \sigma$ by the left-weakening (i), we consider here the case where s is a $\{\backslash, /, \circ, \wedge, \vee, 1, 0\}$-term; the other case where s is of the form $\Box t$ is easier. By the ind. hyp., the following holds:

  $$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma, \Delta \Rightarrow \sigma)) \cup \eta^{\Box}(\Gamma, \Delta) \vdash_{\mathsf{FL}} \Gamma^{\star}, \Delta^{\star} \Rightarrow \sigma.$$

  Clearly,

  $$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma, s, \Delta \Rightarrow \sigma)) \cup \eta^{\Box}(\Gamma, s, \Delta) \vdash_{\mathsf{FL}} \Gamma^{\star}, \Delta^{\star} \Rightarrow \sigma.$$

  We can easily check that:

  $$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma, s, \Delta \Rightarrow \sigma)) \cup \eta^{\Box}(\Gamma, s, \Delta) \vdash_{\mathsf{FL}} s \Rightarrow 1.$$

  By $(1 \Rightarrow)$ and (cut), we have:

  $$\Psi_{\mathsf{ew}}(\mathsf{Sub}^{-}(\Gamma, s, \Delta \Rightarrow \sigma)) \cup \eta^{\Box}(\Gamma, s, \Delta) \vdash_{\mathsf{FL}} \Gamma^{\star}, s, \Delta^{\star} \Rightarrow \sigma.$$

□

Corollary 4.1.4. *Let* L *be a substructural logic weaker than* $\mathsf{FL_{ew}}$*. For any finite set* $\Phi \cup \{s\}$ *of intuitionistic* $\mathcal{L}_{\mathsf{FL}}$*-terms,*

$$\Phi \vdash_{\mathsf{FL_{ew}}} s \text{ if and only if } \Phi \cup \Psi_{\mathsf{ew}}(\mathsf{Sub}(\Phi) \cup \mathsf{Sub}(s)) \vdash_{\mathsf{L}} s.$$

Proof. Left-to-right: suppose that $\Phi \vdash_{\mathsf{FL_{ew}}} s$. By Proposition 4.1.2 $\vdash_{\mathsf{ILZ_w}} \Box\Phi \Rightarrow s$, where $\Box\Phi \Rightarrow s$ is a simple intuitionistic $\mathcal{L}_{\mathsf{FL}_\Box}$-sequent. Hence, by Lemma 4.1.3:

$$\Psi_{\mathsf{ew}}(\mathsf{Sub}^-(\Box\Phi \Rightarrow s)) \cup \eta^{\Box}(\Box\Phi) \vdash_{\mathsf{FL}} \Rightarrow s.$$

Clearly, $\Psi_{\mathsf{ew}}(\mathsf{Sub}^-(\Box\Phi \Rightarrow s)) = \Psi_{\mathsf{ew}}(\mathsf{Sub}(\Phi) \cup \mathsf{Sub}(s))$ and $\eta^{\Box}(\Box\Phi) = \Phi$. Thus, $\Phi \cup \Psi_{\mathsf{ew}}(\mathsf{Sub}(\Phi) \cup \mathsf{Sub}(s)) \vdash_{\mathsf{L}} s$ for each substructural logic L weaker than $\mathsf{FL_{ew}}$, as desired.

Right-to-left: suppose that $\Phi \cup \Psi_{\mathsf{ew}}(\mathsf{Sub}(\Phi) \cup \mathsf{Sub}(s)) \vdash_{\mathsf{L}} s$. By assumption, $\Phi \cup \Psi_{\mathsf{ew}}(\mathsf{Sub}(\Phi) \cup \mathsf{Sub}(s)) \vdash_{\mathsf{FL_{ew}}} s$. Clearly, all the terms in $\Psi_{\mathsf{ew}}(X)$ for a given set X of intuitionistic $\mathcal{L}_{\mathsf{FL}}$-terms are already provable in $\mathsf{FL_{ew}}$ without any set of nonlogical axioms; thus $\Phi \vdash_{\mathsf{FL_{ew}}} s$. □

Clearly, Corollary 4.1.4 serves as a polynomial time reduction from $\mathsf{FL_{ew}}$-deducibility to L-deducibility, where L is a substructural logic weaker than $\mathsf{FL_{ew}}$. By the TOWER-hardness of deducibility in $\mathsf{FL_{ew}}$ (Theorem 3.3.6), we obtain:

Corollary 4.1.5. *The deducibility problem for any substructural logic weaker than* $\mathsf{FL_{ew}}$ *is* TOWER*-hard.*

As an immediate consequence, we obtain new TOWER lower bounds for deducibility in various substructural logics weaker than $\mathsf{FL_{ew}}$.

Corollary 4.1.6. *The following are* TOWER*-hard problems:*

- *the deducibility problem for* $\mathsf{FL_i}$*,*
- *the deducibility problem for* $\mathsf{FL_w}$*,*
- *the deducibility problem for* FL_m^1 *(where* $m \geq 2$*),*
- *the deducibility problem for* $\mathsf{FL}_{\mathsf{e}m}^n$ *(where* $m > n \geq 0$*).*

4.1.3 HACK-hardness of deducibility in substructural logics weaker than $\mathsf{FL_w}$

We have shown in the previous subsection that deducibility in a substructural logic L is hard for TOWER whenever L is weaker than $\mathsf{FL_{ew}}$. Hence, for instance, deducibility in $\mathsf{FL_w}$ is at least TOWER-hard. However, the problem, as observed by Greati and Ramanayake [51], requires nonmultiply recursive time (and space). Using the result, we show that deducibility in a substructural logic L is HACK-hard if L is weaker than $\mathsf{FL_w}$.

Proposition 4.1.7. *Let* L *be a substructural logic weaker than* $\mathsf{FL_w}$*. The deducibility problem for* $\mathsf{FL_w}$ *is many-one reducible to the* L*-deducibility problem in polynomial time.*

Proof. The proof proceeds similarly to that of Corollary 4.1.4. Let $\mathsf{ILZ_w^-}$ denote the sequent calculus (over the language $\mathcal{L}_{\mathsf{FL}_\Box}$) that consists of:

- the inference rules of $\mathsf{FL_w}$;
- the following rules concerning the modality:

$$\frac{\Gamma, s, \Delta \Rightarrow \sigma}{\Gamma, \Box s, \Delta \Rightarrow \sigma}\ (\Box \Rightarrow) \qquad \frac{\Box\Gamma \Rightarrow s}{\Box\Gamma \Rightarrow \Box s}\ (\Rightarrow \Box)$$

$$\frac{\Gamma, \Box s, t, \Delta \Rightarrow \sigma}{\Gamma, t, \Box s, \Delta \Rightarrow \sigma}\ (\Box\mathsf{e}_1) \qquad \frac{\Gamma, s, \Box t, \Delta \Rightarrow \sigma}{\Gamma, \Box t, s, \Delta \Rightarrow \sigma}\ (\Box\mathsf{e}_2)$$

$$\frac{\Gamma, \Box s, \Box s, \Delta \Rightarrow \sigma}{\Gamma, \Box s, \Delta \Rightarrow \sigma}\ (\Box\mathsf{c})$$

The system $\mathsf{ILZ_w^-}$ is just an affine version of the *multiplicative-additive-exponential Lambek calculus* (see, e.g., [68]). As is the case with the cut elimination theorem and the deduction theorem for $\mathsf{ILZ_w}$, it is not difficult to confirm that the following (1) and (2) hold:

1. $\mathsf{ILZ_w^-}$ admits cut elimination.
2. For any finite set Φ of intuitionistic $\mathcal{L}_{\mathsf{FL}}$-terms and any intuitionistic $\mathcal{L}_{\mathsf{FL}}$-sequent $\Gamma \Rightarrow \sigma$, if $\Phi \vdash_{\mathsf{FL_w}} \Gamma \Rightarrow \sigma$ then $\vdash_{\mathsf{ILZ_w^-}} \Box\Phi, \Gamma \Rightarrow \sigma$.

Let X be a finite set of intuitionistic $\mathcal{L}_{\mathsf{FL}}$-terms. Define:

$$\Psi_{\mathsf{w}}(X) := \{s \setminus 1 \mid s \in X\} \cup \{0 \setminus s \mid s \in X\}.$$

The claim below follows by induction on the cut-free derivations for simple intuitionistic $\mathcal{L}_{\mathsf{FL}_\Box}$-sequents:

Claim. *For any simple intuitionistic $\mathcal{L}_{\mathsf{FL}_\Box}$-sequent $\Gamma \Rightarrow \sigma$:*

$$\textit{If } \vdash_{\mathsf{ILZ}_\mathsf{w}^-} \Gamma \Rightarrow \sigma \textit{ then } \Psi_\mathsf{w}(\mathsf{Sub}^-(\Gamma \Rightarrow \sigma)) \cup \eta^\Box(\Gamma) \vdash_{\mathsf{FL}} \Gamma^\star \Rightarrow \sigma.$$

Now consider an instance (Φ, s) of the FL_w-deducibility problem. Using the above claim, we can show the following in nearly the same way as Corollary 4.1.4:

$$\Phi \vdash_{\mathsf{FL}_\mathsf{w}} s \text{ if and only if } \Psi_\mathsf{w}(\mathsf{Sub}(\Phi) \cup \mathsf{Sub}(s)) \cup \Phi \vdash_{\mathsf{L}} s.$$

Constructing the corresponding instance $(\Psi_\mathsf{w}(\mathsf{Sub}(\Phi) \cup \mathsf{Sub}(s)) \cup \Phi, s)$ may be done in polynomial time. $\Box$

By appealing to Fact 2.5.3, we have:

Corollary 4.1.8. *Let L be a substructural logic. The deducibility problem for L is* HACK*-hard whenever L is weaker than FL_w.*

For instance, the complexity lower bound for FL^1_m-deducibility is therefore drastically improved for each $m \geq 2$.

Corollary 4.1.9. *The deducibility problem for FL^1_m is* HACK*-hard for each $m \geq 2$.*

4.2 Nonprimitive recursive problems in substructural logics weaker than $\mathsf{FL}_{\mathsf{eco}}$

4.2.1 Cut-free sequent calculus for $\mathsf{FL}_{\mathsf{eco}}$

We first recall a sequent calculus for $\mathsf{FL}_{\mathsf{eco}}$ with the explicit exchange. In Chapter 2, we have defined it as the extension of FL by the rules of (e), (c) and (o). However, the system has a somewhat problematic feature, i.e., lacks the cut elimination property.[5]

Proposition 4.2.1. *The sequent calculus $\mathsf{FL}_{\mathsf{eco}}$ does not enjoy the cut elimination property.*

[5]Theorem 4.1 of [39, p. 211] asserts that every extension of FL by $R \subseteq \{\mathsf{e}, \mathsf{c}, \mathsf{i}, \mathsf{o}\}$ admits cut elimination, except for the case where $R = \{\mathsf{c}\}$. Proposition 4.2.1 however refutes this claim. (A similar confusion can be found in [64, p. 4004].) Likewise, one can easily check by an argument similar to the proof of Proposition 4.2.1 that the sequent calculi FL_o, $\mathsf{FL}_{\mathsf{eo}}$ and $\mathsf{FL}_{\mathsf{co}}$ also fail to enjoy cut elimination. Nevertheless, this is not very fatal; indeed, the algorithm developed by Ciabattoni, Galatos, and Terui [24] automatically generates cut-free sequent calculi for all the basic substructural logics.

Proof. Consider a sequent $p, 0 \Rightarrow$, where p is a variable. It suffices to show that this sequent is provable in $\mathsf{FL_{eco}}$ but not provable in $\mathsf{FL_{eco}}$ without cut. We may construct a derivation for $p, 0 \Rightarrow$ in $\mathsf{FL_{eco}}$ without using any nonlogical axioms as follows:

$$\dfrac{\dfrac{0 \Rightarrow}{0 \Rightarrow p \setminus 0}\,(\mathsf{o}) \qquad \dfrac{p \Rightarrow p \qquad 0 \Rightarrow}{p, p \setminus 0 \Rightarrow}}{p, 0 \Rightarrow}\,(\mathrm{cut})$$

Suppose that there is a cut-free derivation P for $p, 0 \Rightarrow$. Inspecting all the inference rules of $\mathsf{FL_{eco}}$, we see that P must end with an application of (c) or (e). This means that P is one of the following:

$$\dfrac{\dfrac{\vdots}{p, 0, 0 \Rightarrow}}{p, 0 \Rightarrow}\,(\mathsf{c}) \qquad \dfrac{\dfrac{\vdots}{p, p, 0 \Rightarrow}}{p, 0 \Rightarrow}\,(\mathsf{c}) \qquad \dfrac{\dfrac{\vdots}{0, p \Rightarrow}}{p, 0 \Rightarrow}\,(\mathsf{e})$$

Checking again the rules of $\mathsf{FL_{eco}}$, we see that the immediate subderivation must end with an application of either (c) or (e). Even if we repeat the same argument, we never find an initial sequent in P—it is a contradiction. $\square$

Nevertheless, one can find a cut-free sequent calculus equivalent to $\mathsf{FL_{eco}}$. Roughly speaking, such a cut-free sulculus is obtained from $\mathsf{FL_{eco}}$ by transforming the three structural rules into *analytic* ones; see [24, Definition 4.6] for the precise definition of *analytic structural rules*. The transformation procedure, which is due to Ciabattoni, Galatos, and Terui [22], does not unconditionally work for all structural rules; see [22, Section 6] and [24, Section 4] for details. Henceforth, the notation $\mathsf{FL^*_{eco}}$ stands for the extension of FL by the following *analytic* versions of (e), (c), (o):[6]

$$\dfrac{\Gamma, \Sigma, \Xi, \Delta \Rightarrow \sigma}{\Gamma, \Xi, \Sigma, \Delta \Rightarrow \sigma}\,(\mathsf{e}^*) \quad \dfrac{\Gamma, \Sigma, \Sigma, \Delta \Rightarrow \sigma}{\Gamma, \Sigma, \Delta \Rightarrow \sigma}\,(\mathsf{c}^*) \quad \dfrac{\Gamma \Rightarrow}{\Delta, \Gamma, \Sigma \Rightarrow \sigma}\,(\mathsf{o}^*)$$

It is quite easy to check that a sequent $\Gamma \Rightarrow \sigma$ is provable $\mathsf{FL^*_{eco}}$ if and only if it is provable in $\mathsf{FL_{eco}}$. Surprisingly enough, Ciabattoni, Galatos, and Terui [24] also showed that every extension of FL by analytic structural rules enjoys a strong form of cut elimination, i.e., the strong analyticity property. The following is just a corollary of uniform cut elimination for a wide range of substructural logics:

[6]In [115], Surarso and Ono firstly investigated the rule of (c^*). They referred to the rule as *global contraction* and showed that the extension of FL by the rule of global contraction enjoys cut elimination.

Fact 4.2.2 (Ciabattoni, Galatos, and Terui [24], Sections 4 and 5)**.** *The Gentzen-style sequent calculus* $\mathsf{FL}^*_{\mathsf{eco}}$ *is a cut-free sequent calculus such that for any intuitionistic* $\mathcal{L}_{\mathsf{FL}}$*-sequent* $\Gamma \Rightarrow \sigma$*,*

$$\vdash_{\mathsf{FL}_{\mathsf{eco}}} \Gamma \Rightarrow \sigma \textit{ if and only if } \vdash_{\mathsf{FL}^*_{\mathsf{eco}}} \Gamma \Rightarrow \sigma.$$

4.2.2 Ack-hardness of deducibility in substructural logics weaker than $\mathsf{FL}_{\mathsf{eco}}$

In [122], Urquhart proved that the positive fragment $\mathsf{FL}^+_{\mathsf{ec}}$ of $\mathsf{FL}_{\mathsf{ec}}$ is already Ack-complete; see Fact 2.5.2.[7] Using this, we can easily show the following:

Proposition 4.2.3. *Provability in* $\mathsf{FL}_{\mathsf{eco}}$ *is* Ack*-hard.*

Proof. By Facts 4.2.2 and 2.5.2, it suffices to show that: for any intuitionistic $\mathcal{L}_{\mathsf{FL}^+}$-sequent $\Gamma \Rightarrow s$,

$$\vdash_{\mathsf{FL}^+_{\mathsf{ec}}} \Gamma \Rightarrow s \text{ if and only if } \vdash_{\mathsf{FL}^*_{\mathsf{eco}}} \Gamma \Rightarrow s.$$

Left-to-right: trivial.

Right-to-left: by induction on the length of the *cut-free* derivation of $\Gamma \Rightarrow s$. The only nontrivial case seems to be the case where $\Delta, \Gamma, \Sigma \Rightarrow s$ is obtained from $\Gamma \Rightarrow$ by the rule of (o^*). However, it is impossible that the last applied rule in the derivation is (o^*). This is due to the following claim:

> (†): *If* P *is a cut-free derivation in* ${\mathsf{FL}_{\mathsf{eco}}}^*$ *that contains a sequent with the empty succedent, then the constant* 0 *is a subterm of some term in the endsequent of* P.

This claim follows by induction on the length of P. Now observe that the cut-free derivation for $\Delta, \Gamma, \Sigma \Rightarrow s$ contains $\Gamma \Rightarrow$. By Claim (†), it follows that 0 is a subterm of some term t in $\Delta, \Gamma, \Sigma \Rightarrow s$; however, this violates the fact that $\Delta, \Gamma, \Sigma \Rightarrow s$ is an intuitionistic $\mathcal{L}_{\mathsf{FL}^+}$-sequent. □

Let X be a finite set of intuitionistic $\mathcal{L}_{\mathsf{FL}}$-terms. Define a set $\Psi_{\mathsf{eco}}(X)$ of

[7] The positive fragment $\mathsf{FL}^+_{\mathsf{ec}}$ of $\mathsf{FL}_{\mathsf{ec}}$ is called LR+ in [122].

intuitionistic $\mathcal{L}_{\mathsf{FL}}$-terms as follows:

$$\begin{array}{rcl}\Psi_{\mathsf{eco}}(X) & := & \{s \backslash s^2 \mid s \in X\} \\ & \cup & \{0 \backslash s \mid s \in X\} \\ & \cup & \{0 \backslash {\sim} s \mid s \in X\} \\ & \cup & \{0 \backslash -s \mid s \in X\} \\ & \cup & \{(s \circ t) \backslash (t \circ s) \mid s, t \in X\}.\end{array}$$

Proposition 4.2.4. *Let* L *be a substructural logic weaker than* $\mathsf{FL}_{\mathsf{eco}}$ *and* s *be an intuitionistic* $\mathcal{L}_{\mathsf{FL}}$*-term.*

$$\vdash_{\mathsf{FL}_{\mathsf{eco}}} s \textit{ if and only if } \Psi_{\mathsf{eco}}(\mathsf{Sub}(s)) \vdash_{\mathsf{L}} s.$$

Proof. Left-to-right: by Fact 4.2.2, it suffices to prove the claim below:

(✠): *For any intuitionistic* $\mathcal{L}_{\mathsf{FL}}$*-sequent* $\Gamma \Rightarrow \sigma$,

$$\textit{if } \vdash_{\mathsf{FL}^*_{\mathsf{eco}}} \Gamma \Rightarrow \sigma \textit{ then } \Psi_{\mathsf{eco}}(\mathsf{Sub}(\Gamma \Rightarrow \sigma)) \vdash_{\mathsf{FL}} \Gamma \Rightarrow \sigma.$$

Clearly, every cut-free derivation P in $\mathsf{FL}^*_{\mathsf{eco}}$ can be transformed into a cut-free derivation P' satisfying the following conditions:

- ▶ In every application of (e^*), each of the exchanged sequences is a sequence consisting only of exactly one occurrence of a term.
- ▶ In every application of (c^*), the contracted sequence is a sequence consisting only of exactly one occurrence of a term.

Claim (✠) follows by induction on the cut-free derivation in $\mathsf{FL}^*_{\mathsf{eco}}$ for $\Gamma \Rightarrow \sigma$ satisfying these conditions. We only consider some critical cases.

- ▶ If $\Delta, \Gamma, \Sigma \Rightarrow \sigma$, where $\Delta = u_1, \ldots, u_n$ and $\Sigma = v_1, \ldots, v_m$, is deduced from $\Gamma \Rightarrow$ by the rule of (o^*), by the ind. hyp.,

$$\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Gamma \Rightarrow)) \vdash_{\mathsf{FL}} \Gamma \Rightarrow .$$

Since it holds that $\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Gamma \Rightarrow)) \subseteq \Psi_{\mathsf{eco}}(\mathsf{Sub}(\Delta, \Gamma, \Sigma \Rightarrow \sigma))$,

$$\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Delta, \Gamma, \Sigma \Rightarrow \sigma)) \vdash_{\mathsf{FL}} \Gamma \Rightarrow .$$

Note that:

$$\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Delta, \Gamma, \Sigma \Rightarrow \sigma)) \vdash_{\mathsf{FL}} 0 \Rightarrow {\sim} u_i$$

for each u_i $(1 \leq i \leq n)$. Applying $(\Rightarrow 0)$ and (cut), we have:

$$\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Delta, \Gamma, \Sigma \Rightarrow \sigma)) \vdash_{\mathsf{FL}} \Gamma \Rightarrow {\sim} u_n.$$

By $(0 \Rightarrow)$ and $(/ \Rightarrow)$,

$$\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Delta, \Gamma, \Sigma \Rightarrow \sigma)) \vdash_{\mathsf{FL}} -{\sim} u_n, \Gamma \Rightarrow .$$

Repeating the same argument for $u_{n-1}, \ldots, u_1$, we have:

$$\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Delta, \Gamma, \Sigma \Rightarrow \sigma)) \vdash_{\mathsf{FL}} -{\sim} u_1, \ldots, -{\sim} u_n, \Gamma \Rightarrow .$$

On the other hand,

$$\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Delta, \Gamma, \Sigma \Rightarrow \sigma)) \vdash_{\mathsf{FL}} 0 \Rightarrow -v_j,$$

for every v_j $(1 \leq j \leq m)$. Thus we have:

$$\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Delta, \Gamma, \Sigma \Rightarrow \sigma)) \vdash_{\mathsf{FL}} -{\sim} u_1, \ldots, -{\sim} u_n, \Gamma \Rightarrow -v_1.$$

By $(0 \Rightarrow)$ and $(\backslash \Rightarrow)$,

$$\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Delta, \Gamma, \Sigma \Rightarrow \sigma)) \vdash_{\mathsf{FL}} -{\sim} u_1, \ldots, -{\sim} u_n, \Gamma, {\sim}{-} v_1 \Rightarrow .$$

Consequently,

$$\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Delta, \Gamma, \Sigma \Rightarrow \sigma)) \vdash_{\mathsf{FL}} -{\sim} u_1, \ldots, -{\sim} u_n, \Gamma, {\sim}{-} v_1, \ldots, {\sim}{-} v_m \Rightarrow .$$

Note that $\vdash_{\mathsf{FL}} t \Rightarrow -{\sim} t$ and $\vdash_{\mathsf{FL}} t \Rightarrow {\sim}{-} t$ for every term t. Hence by applying (cut),

$$\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Delta, \Gamma, \Sigma \Rightarrow \sigma)) \vdash_{\mathsf{FL}} \Delta, \Gamma, \Sigma \Rightarrow .$$

If σ is empty, we have nothing further to do. If $\sigma = s$, since it is clear that $\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Delta, \Gamma, \Sigma \Rightarrow s)) \vdash_{\mathsf{FL}} 0 \Rightarrow s$, by applying $(\Rightarrow 0)$ and (cut) we have:

$$\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Delta, \Gamma, \Sigma \Rightarrow s)) \vdash_{\mathsf{FL}} \Delta, \Gamma, \Sigma \Rightarrow s.$$

► If $\Gamma, \Sigma, \Delta \Rightarrow \sigma$ is obtained from $\Gamma, \Sigma, \Sigma, \Delta \Rightarrow \sigma$ by the rule of (c^*), we can assume without loss of generality that $\Sigma = s$ for some term s. By the ind. hyp.,

$$\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Gamma, s, s, \Delta \Rightarrow \sigma)) \vdash_{\mathsf{FL}} \Gamma, s, s, \Delta \Rightarrow \sigma.$$

Since it is clear that $\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Gamma, s, s, \Delta \Rightarrow \sigma)) = \Psi_{\mathsf{eco}}(\mathsf{Sub}(\Gamma, s, \Delta \Rightarrow \sigma))$,

$$\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Gamma, s, \Delta \Rightarrow \sigma)) \vdash_{\mathsf{FL}} \Gamma, s, s, \Delta \Rightarrow \sigma.$$

By applying $(\circ \Rightarrow)$,

$$\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Gamma, s, \Delta \Rightarrow \sigma)) \vdash_{\mathsf{FL}} \Gamma, s^2, \Delta \Rightarrow \sigma.$$

By the construction of $\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Gamma, s, s, \Delta \Rightarrow \sigma))$, we obviously have:

$$\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Gamma, s, \Delta \Rightarrow \sigma)) \vdash_{\mathsf{FL}} s \Rightarrow s^2.$$

By applying the cut rule, we have the following:

$$\Psi_{\mathsf{eco}}(\mathsf{Sub}(\Gamma, s, \Delta \Rightarrow \sigma)) \vdash_{\mathsf{FL}} \Gamma, s, \Delta \Rightarrow \sigma.$$

Right-to-left: this direction is obvious, as every term in $\Psi_{\mathsf{eco}}(\mathsf{Sub}(s))$ is already provable in $\mathsf{FL}_{\mathsf{eco}}$ without any additional assumptions. □

By Proposition 4.2.4, if a substructural logic L is weaker than $\mathsf{FL}_{\mathsf{eco}}$, $\mathsf{FL}_{\mathsf{eco}}$-provability is polynomial time reducible to L-deducibility. In view of Proposition 4.2.3, we hence obtain:

Corollary 4.2.5. *The deducibility problem for any substructural logic weaker than* $\mathsf{FL}_{\mathsf{eco}}$ *is* ACK*-hard.*

Since $\mathsf{FL}_{\mathsf{e}_m^n}$ is clearly a logic between FL and $\mathsf{FL}_{\mathsf{eco}}$ for $n > m \geq 1$, we immediately obtain:

Corollary 4.2.6. *The deducibility problem for* $\mathsf{FL}_{\mathsf{e}_m^n}$ *is* ACK*-hard for* $n > m \geq 1$.

St. John [110] already proved a stronger result than Corollary 4.2.6; see Fact 4.2.10. The following is a specific instance of Proposition 4.2.4:

$$\vdash_{\mathsf{FL}_{\mathsf{eco}}} s \text{ if and only if } \Psi_{\mathsf{eco}}(\mathsf{Sub}(s)) \vdash_{\mathsf{FL}_{\mathsf{ec}}} s.$$

Since deducibility in $\mathsf{FL}_{\mathsf{ec}}$ is decidable in ACK (see [122]), so is provability in $\mathsf{FL}_{\mathsf{eco}}$ by Fact 2.4.4.

Let Φ be a finite set of intuitionistic terms. We write $\Phi^{\wedge 1}$ for the set $\{t \wedge 1 \mid t \in \Phi\}$ and define

$$\prod \Phi := \begin{cases} 1 & \text{if } \Phi \text{ is empty,} \\ s_1 \circ \cdots \circ s_n & \text{if } \Phi = \{s_1, \ldots, s_n\} \ (n \geq 1). \end{cases}$$

For our purpose here, we make use of the following deduction theorem:

Lemma 4.2.7 (Galatos, Jipsen, Kowalski, and Ono [39], Corollary 2.15)**.** *Let* L *be a substructural logic over* $\mathsf{FL_{ec}}$ *and* $\Phi \cup \{s\}$ *be a finite set of intuitionistic* $\mathcal{L}_{\mathsf{FL}}$*-terms.*

$$\Phi \vdash_{\mathsf{L}} s \textit{ if and only if } \vdash_{\mathsf{L}} (\prod \Phi^{\wedge 1}) \setminus s.$$

In particular, deducibility in $\mathsf{FL_{eco}}$ is reduced to provability in $\mathsf{FL_{eco}}$. We thus obtain:

Corollary 4.2.8. *The provability problem for* $\mathsf{FL_{eco}}$ *is* ACK*-complete. So is the deducibility problem for* $\mathsf{FL_{eco}}$*.*

Now let us focus on axiomatic extensions of $\mathsf{FL}_{\mathsf{e}_m^n}$ for $n > m \geq 1$. The following is a variant of the deduction theorem for n-contractive (fuzzy) logics; see [63, Theorem 3.3].

Proposition 4.2.9. *Let* L *be a substructural logic over* $\mathsf{FL}_{\mathsf{e}_m^n}$ *(where* $n > m \geq 1$*). For any finite set* $\Phi \cup \{s\}$ *of intuitionistic* $\mathcal{L}_{\mathsf{FL}}$*-terms,*

$$\Phi \vdash_{\mathsf{L}} s \textit{ if and only if } \vdash_{\mathsf{L}} (\prod \Phi^{\wedge 1})^m \setminus s.$$

Proof. Clearly, it suffices to show that the following claim.

> *Let* L *be a substructural logic over* $\mathsf{FL}_{\mathsf{e}_m^n}$ *(where* $n > m \geq 1$*). For any finite set* Φ *of intuitionistic* $\mathcal{L}_{\mathsf{FL}}$*-terms and for any intuitionistic* $\mathcal{L}_{\mathsf{FL}}$*-sequent* $\Gamma \Rightarrow \sigma$*,*
>
> $$\Phi \vdash_{\mathsf{L}} \Gamma \Rightarrow \sigma \textit{ if and only if } \vdash_{\mathsf{L}} (\Phi^{\wedge 1})^{(m)}, \Gamma \Rightarrow \sigma.$$

Right-to-left: this direction is trivial due to the fact that $\Phi \vdash_{\mathsf{FL}_{\mathsf{e}_m^n}} t \wedge 1$ for every term t in Φ.

Left-to-right: by induction on the length of the derivation for $\Phi \vdash_{\mathsf{L}} \Gamma \Rightarrow \sigma$. For the induction step, we consider, for instance, the case where $\Sigma, \Gamma, \Delta \Rightarrow \sigma$ is deduced from $\Gamma \Rightarrow s$ and $\Sigma, s, \Delta \Rightarrow \sigma$ by an application of (cut). By the ind. hyp.,

$$\vdash_{\mathsf{L}} (\Phi^{\wedge 1})^{(m)}, \Gamma \Rightarrow s,$$

and

$$\vdash_{\mathsf{L}} (\Phi^{\wedge 1})^{(m)}, \Sigma, s, \Delta \Rightarrow \sigma.$$

By applying (cut),

$$\vdash_{\mathsf{L}} (\Phi^{\wedge 1})^{(m)}, \Sigma, (\Phi^{\wedge 1})^{(m)}, \Gamma, \Delta \Rightarrow \sigma.$$

Applying the exchange rule several times, we obtain:

$$\vdash_{\mathsf{L}} (\Phi^{\wedge 1})^{(2m)}, \Sigma, \Gamma, \Delta \Rightarrow \sigma.$$

- If $2m \leq n$, by applying $(1 \Rightarrow)$ and $(\wedge \Rightarrow)$ several times,

  $$\vdash_{\mathsf{L}} (\Phi^{\wedge 1})^{(n)}, \Sigma, \Gamma, \Delta \Rightarrow \sigma.$$

  Applying the (m, n)-knotted rules (and the exchange rule) several times, we obtain:

  $$\vdash_{\mathsf{L}} (\Phi^{\wedge 1})^{(m)}, \Sigma, \Gamma, \Delta \Rightarrow \sigma.$$

- Otherwise, by applying the exchange rule and the (m, n)-knotted rule repeatedly, we may obtain a derivation for:

  $$\vdash_{\mathsf{L}} (\Phi^{\wedge 1})^{(k)}, \Sigma, \Gamma, \Delta \Rightarrow \sigma,$$

  for some k such that $k \leq n$. By applying the rules for $(1 \Rightarrow)$ and $(\wedge \Rightarrow)$ several times, we obtain a derivation for:

  $$\vdash_{\mathsf{L}} (\Phi^{\wedge 1})^{(n)}, \Sigma, \Gamma, \Delta \Rightarrow \sigma.$$

  By using the (m, n)-knotted rule (and the exchange rule), we obtain:

  $$\vdash_{\mathsf{L}} (\Phi^{\wedge 1})^{(m)}, \Sigma, \Gamma, \Delta \Rightarrow \sigma.$$

The rest of the proof proceeds similarly. □

By Proposition 4.2.9 and Corollary 4.2.6, we have obtained an alternative proof of the following fact, essentially shown by St. John [110].

Fact 4.2.10 (St. John [110]; see Corollary 4.4.2 and Section 4.4.3)**.** *Provability in $\mathsf{FL}_{\mathsf{e}m}^{n}$ is* ACK*-hard for $n > m \geq 1$.*

Remark 4.2.11. Also, St. John essentially proved the ACK-hardness for the deducibility problem for the axiomatic extension of FL_{e} by a *weakly expansive axiom*, i.e., a single-variable axiom of the form $x^m \backslash (x^{n_1} \vee \cdots \vee x^{n_k})$, where $m > 0$ and $m < n_i$ for some $1 \leq i \leq k$; see [110, Section 4.4.3] for details. Note that Corollary 4.2.5 subsumes the result.

4.3 Further results

We close this chapter by summarizing some further consequences, which we have not mentioned earlier.

4.3.1 Case study on knotted extensions of FL

Regarding knotted extensions of $\mathsf{FL_e}$, we have essentially shown the following:

Corollary 4.3.1. *Let (m, n) be a knotted pair. The deducibility problem for $\mathsf{FL_e}{}_m^n$ is not elementary recursive. To be precise, the following statements hold:*

1. *The deducibility problem for $\mathsf{FL_e}{}_m^n$ is* TOWER*-hard for $m > n \geq 0$.*
2. *The provability problem for $\mathsf{FL_e}{}_m^n$ is* ACK*-hard for $n > m \geq 1$.*

Proof. Immediate from Corollary 4.1.6 and Fact 4.2.10. $\square$

While Horčík [61] proved that deducibility in FL_m^n is undecidable for almost every knotted pair (see Table 1.2 in Chapter 1 for details), Cardona and Galatos [19] proved that all of those problems become decidable when the axiom of *weak commutativity*:

$$(xyx \setminus x^2y) \wedge (x^2y \setminus xyx) \tag{e'}$$

is added to FL_m^n. To be precise, Cardona and Galatos proved in the paper that the variety of residuated lattices (or FL-algebras) axiomatized by the equations $xyx = x^2y$ and $x^m \leq x^n$ has the finite embeddability property for each knotted pair (m, n); see [19, Theorem 2.6] and [62, Section 7.2] for details. Hence the universal theory of each of those varieties is decidable. Via the *algebraization theorem for substructural logics* (see [40, Theorem 3.4] and [39, Theorem 2.29]), the result is rephrased as the decidability of deducibility in FL_m^n with the axiom of (e'). Clearly, FL_m^n with the axiom of (e') is between FL and $\mathsf{FL_{ew}}$ if $m > n \geq 0$ and is between FL and $\mathsf{FL_{eco}}$ if $n > m \geq 1$. As a by-product of Corollaries 4.1.5 and 4.2.5, we thus have new nonelementary bounds for these logics.

Corollary 4.3.2. *Deducibility in FL_m^n with the axiom of (e') is not elementary recursive for each knotted pair (m, n). More precisely, the following hold.*

1. *The deducibility problem for FL_m^n with the axiom of (e') is* TOWER*-hard for $m > n \geq 0$.*
2. *The deducibility problem for FL_m^n with the axiom of (e') is* ACK*-hard for $n > m \geq 1$.*

4.3.2 PSPACE-hardness of deducibility in logics weaker than intuitionistic propositional logic

Given the pattern of our reductions thus far, it would be no surprise if provability in the logic $\mathsf{FL_{ecw}}$, a cut-free calculus equivalent to intuitionistic propositional logic (i.e., LJ), is polytime reducible to deducibility in any substructural logic weaker than $\mathsf{FL_{ecw}}$. We now show:

Proposition 4.3.3. *The provability problem for $\mathsf{FL_{ecw}}$ is polynomial-time reducible to the deducibility problem for any substructural logic weaker than $\mathsf{FL_{ecw}}$.*

Proof. We only present a proof sketch. For a finite set X of intuitionistic $\mathcal{L}_{\mathsf{FL}}$-terms, we define:

$$\begin{aligned}
\Psi_{\mathsf{ecw}}(X) \quad &:= \quad \{s \backslash 1 \mid s \in X\} \\
&\cup \quad \{0 \backslash s \mid s \in X\} \\
&\cup \quad \{(s \circ t) \backslash (t \circ s) \mid s, t \in X\} \\
&\cup \quad \{s \backslash s^2 \mid s \in X\}.
\end{aligned}$$

Despite the fact that none of the rules of (e), (c), (i), and (o) are analytic in the sense of [24], we can show that $\mathsf{FL_{ecw}}$ admits cut elimination.[8] We can furthermore show the following claim, thus obtaining the desired polynomial time reduction:

Claim. *Let L be a substructural logic weaker than $\mathsf{FL_{ecw}}$ and s be an intuitionistic $\mathcal{L}_{\mathsf{FL}}$-term.*

$$\vdash_{\mathsf{FL_{ecw}}} s \textit{ if and only if } \Psi_{\mathsf{ecw}}(\mathsf{Sub}(s)) \vdash_{\mathsf{L}} s.$$

The left-to-right direction follows by induction on the cut-free derivations in $\mathsf{FL_{ecw}}$. The converse direction is trivial. □

[8]The verification is easy. Observe that the following structural rules are analytic in the sense of [24, Definition 4.6] and correspond to (e), (i), (o), and (c), respectively:

$$\frac{\Gamma, \Sigma, \Xi, \Delta \Rightarrow \sigma}{\Gamma, \Xi, \Sigma, \Delta \Rightarrow \sigma}\,(\mathsf{e}^*) \quad \frac{\Gamma, \Delta \Rightarrow \sigma}{\Gamma, \Sigma, \Delta \Rightarrow \sigma}\,(\mathsf{i}^*) \quad \frac{\Gamma \Rightarrow}{\Delta, \Gamma, \Sigma \Rightarrow \sigma}\,(\mathsf{o}^*) \quad \frac{\Gamma, \Sigma, \Sigma, \Delta \Rightarrow \sigma}{\Gamma, \Sigma, \Delta \Rightarrow \sigma}\,(\mathsf{c}^*)$$

The rule extension of FL by these analytic rules is equivalent to $\mathsf{FL_{ecw}}$ and is strongly analytic by [24, Theorem 5.22]; hence, this analytic extension *a fortiori* admits cut elimination. To show the cut elimination for $\mathsf{FL_{ecw}}$, it thus suffices to confirm that these analytic structural rules are all derivable in the cut-free variant of $\mathsf{FL_{ecw}}$.

Combined with the PSPACE-completeness of $\mathsf{FL}_{\mathsf{ecw}}$-provability shown by Statman [111] (see also [64]), the above proposition entails:

Proposition 4.3.4. *For every substructural logic* L*, the deducibility problem for* L *is* PSPACE*-hard if* L *is weaker than* $\mathsf{FL}_{\mathsf{ecw}}$*.*

It is instructive to compare Proposition 4.3.4 with the result of Horčík and Terui [64], i.e., the PSPACE-hardness of provability in consistent substructural logics with the disjunction property. Our condition does not encompass, e.g., the extension of FL by the axiom(s) of cancellativity (see [64, Figure 4]), whereas the logic possesses the disjunction property; see Remark 4.3.5. On the other hand, the axiomatic extension of FL with the axiom $(x \setminus 1) \vee (x \setminus x^2)$ is an example of substructural logics that match the condition of Proposition 4.3.4 but lack the disjunction property; see Remark 4.3.6 for details. Deducibility in the logic is thus PSPACE-hard; more to the point, it is HACK-hard by Corollary 4.1.8 and might even be undecidable.

Remark 4.3.5. The axioms of cancellativity:

$$x \,/\, (xy \,/\, y) \qquad\qquad (x \setminus xy) \setminus y$$

are $\mathcal{M}_2$-axioms in the sense of [64, Section 3.3]. Since any axiomatic extension of FL by $\mathcal{M}_2$-axioms has the disjunction property (see [64, Corollary 3.14]), so do the extensions of FL with these axioms. The corresponding equations:

$$x = xy \,/\, y \qquad\qquad x \setminus xy = y$$

may be refuted in, e.g., the two-element Boolean algebra $\mathbf{2}$; thus, the extensions of FL by the cancellative axioms are not weaker than classical propositional logic and, a priori, are not weaker than intuitionistic propositional logic.

Remark 4.3.6. It is clear that the extension of FL by the axiom $(x \setminus 1) \vee (x \setminus x^2)$ is weaker than $\mathsf{FL}_{\mathsf{ecw}}$. In order to check that this substructural logic fails to have the disjunction property, it suffices by the algebraization theorem of Galatos and Ono [40, Theorem 3.4] to find:

- an FL-algebra, in which the equation $1 \leq (x \setminus 1) \vee (x \setminus x^2)$ is valid, but in which the equation $1 \leq x \setminus 1$ is not valid, and
- an FL-algebra, in which the equation $1 \leq (x \setminus 1) \vee (x \setminus x^2)$ is valid, but in which the equation $1 \leq x \setminus x^2$ is not valid.

To this end, the reader may use, e.g., the five-element FL-algebras **A** and **B** of van Alten with the same universe $\{a, b, c, d, 1\}$; see [123, Examples 6.1 and 6.2] for the precise definition of these algebras. (Caveat lector: his algebras are actually residuated lattices; however, every residuated lattice is regarded as an FL-algebra in a trivial way.) Clearly, the algebra **A** validates the equation $1 \leq (x \setminus 1) \vee (x \setminus x^2)$ since **A** is integral, that is to say, the unit for the monoid reduct of **A** is the greatest element of **A** with respect to the lattice-order. The equation $1 \leq x \setminus x^2$ fails to hold for **A** since $c \setminus c^2 = c \setminus d = b < 1$. On the other hand, the algebra **B** validates the equation $1 \leq (x \setminus 1) \vee (x \setminus x^2)$ since the monoid operation is obviously idempotent; see the Cayley-table of [123, Figure 2]. Finally, the equation $1 \leq x \setminus 1$ is refuted in **B** since, for instance, $a \setminus 1 = d < 1$.

Chapter 5

Conclusion

> A new problem, especially when it comes from the world of outer experience, is like a young twig, which thrives and bears fruit only when it is grafted carefully and in accordance with strict horticultural rules upon the old stem, the established achievements of our mathematical science.
>
> David Hilbert
> *Mathematical Problems* (*Lecture delivered before the International Congress of Mathematicians at Paris in 1900*), 1902

Since Schmitz's proposal of the fast-growing complexity hierarchy [105], there have been many efforts across several fields of theoretical computer science to find problems complete for certain levels of this hierarchy. This trend has been particularly pronounced, for example, in the context of *well-structured transition systems* (WSTSes, for short), such as VASSes [77, 31, 80, 79, 32]; see, e.g., [107] for general background on the complexity issues concerning WSTSes. However, it seems that many substructural logicians—except for a few logicians, like Urquhart [122], who published path-breaking work in the 1990s on the nonprimitive recursive complexity of relevance logics—have paid less attention to logical decision problems of enormous complexity.[1] With hindsight, this may be because we do not often encounter decidable logics of nonelementary complexity when considering only provability (cf. Table 1.1 in Chapter 1).

[1] With that being said, the research trend emerging from the WSTS literature is gradually gaining traction in the field of substructural logic, as is evident from some recent papers [8, 117, 51].

However, exploring the computational complexity of deducibility, a natural generalization of provability, we have seen that there are many substructural logics requiring an astronomical amount of time and/or space resources. In Chapter 3, we have provided the tight complexity bounds for deducibility in (commutative) contraction-free logics, which had remained uncharted until recently. In particular, it has turned out that the deducibility problems for $\mathsf{FL_{ei}}$ and $\mathsf{FL_{ew}}$ are decidable in primitive recursive time but require "barely nonelementary" time. More precisely, we have shown the following:

- The TOWER-completeness of deducibility in $\mathsf{FL_{ew}}$ and its variants (Corollary 3.3.6).

To show this, we have used various ideas that come from the literature of linear logic rather than that of substructural logic, e.g., Avron's deduction theorem for linear logic [6, 81], Terui's !-prenex sequents [118], Laurent's negative translation for linear logic [76]. Among them, we have made heavy use of the nice reduction techniques developed by Lazić and Schmitz [78].

As a by-product of our proof idea, we have also shown the following.

- The TOWER-completeness of deducibility in $\mathsf{ELL_w}$ and its variants (Corollary 3.3.9 and Theorem 3.3.10).

However, the realm that we have explored in Chapter 3 is quite confined in terms of the whole "zoo" of substructural logics. In Chapter 4, we therefore have attempted to find handy criteria for classifying a plethora of substructural logics in terms of the fast-growing complexity hierarchy. The main outcomes are as follows.

- The TOWER-hardness of deducibility in substructrural logics weaker than $\mathsf{FL_{ew}}$ (Corollary 4.1.5).
- The ACK-hardness of deducibility in substructrural logics weaker than $\mathsf{FL_{eco}}$ (Corollary 4.2.5).
- The HACK-hardness of deducibility in substructrural logics weaker than $\mathsf{FL_{w}}$ (Corollary 4.1.8).

Of importance to our proofs was the observation that each basic structural rule can be implemented by the corresponding non-logical axiom. The idea has also yielded the PSPACE-hardness for deducibility in substructural logics weaker than intuitionistic propositional logic (Proposition 4.3.4). The result is in contrast to the coNP-hardness of deducibility (or, more precisely, provability) in nontrivial substructural logics (cf. [64]).

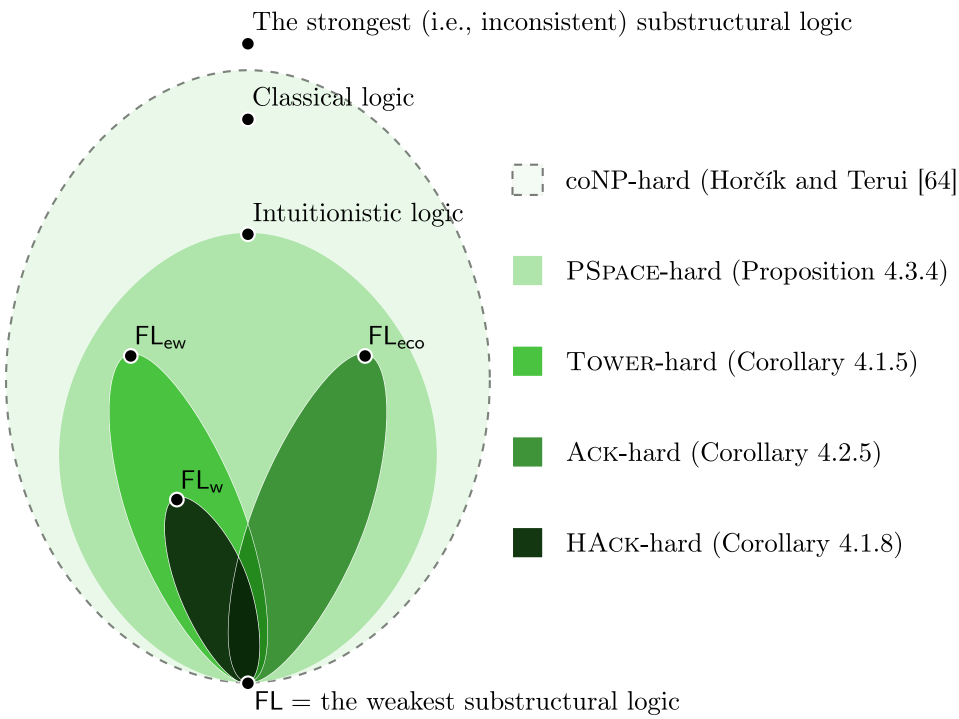

Figure 5.1: Complexity map of deducibility in substructural logics.

Figure 5.1 is a graphical representation of some "hardness conditions." The lightest-colored region enclosed by the dashed line represents the universe of consistent substructural logics. The second-lightest (resp. third-, fourth-, fifth-lightest) subregion represents the collection of logics between FL and intuitionistic propositional logic (resp. $\mathsf{FL_{ew}}$, $\mathsf{FL_{eco}}$, $\mathsf{FL_{w}}$).

From these hardness conditions, we may derive many corollaries. Below is a non-exhaustive list of such corollaries:

- The TOWER-hardness of deducibility in $\mathsf{FL_{e}}{}^{n}_{m}$ for $m > n \geq 0$.
- The TOWER-hardness of deducibility in FL^{n}_{m} with the axiom of (e') for $m > n \geq 0$.
- The ACK-hardness of deducibility in $\mathsf{FL_{e}}{}^{n}_{m}$ for $n > m \geq 1$.
- The ACK-hardness of deducibility in FL^{n}_{m} with the axiom of (e') for $n > m \geq 1$.

- The HACK-hardness of deducibility in FL^1_m for $m \geq 2$.

Open questions

There are a number of open questions involved in our work. Among them, we describe some selected topics below.

A grand challenge: towards a uniform understanding of substructural logic. In view of a series of the existing results culminating in the work of Greati and Ramanayake [51], we may conclude that the deducibility problems for most basic substructural logics are extremely intractable in the sense of Section 1.3. Consequently, we can no longer keep any hope of finding elementary decision procedures for deducibility in basic substructural logics except for intuitionistic propositional logic and $\mathsf{FL}_{\mathsf{ci}}$. In this respect, the complexity results so far offer a little bit pessimistic view of most basic substructural logics.

However, the above is just a small observation derived from the main results. Our purpose is not to individually examine whether each substructural logic is practical or not. Our primary aim—or rather a grand program of substructural logic—is to organize in a unified way the seemingly chaotic universe of substructural logics. In other words, we are seeking a *good* classification of substructural logics.

To this end, *substructural hierarchy* [22, 23, 24, 25], which we have not mentioned in the main chapters, should be of great importance. This is due to the historical fact that so far this concept has been useful in obtaining a unified view of various logical properties, such as cut elimination [22, 24, 25], standard completeness [10, 9, 11], and the computational complexity of proof-search [100, 8, 26]. In this respect, however, it is somewhat unfortunate that the conditions given in Chapter 4, unlike the uniform decidability/complexity statements provided in [100, 8, 26], have little direct connection to substructural hierarchy. It would thus be interesting if we could present a novel taxonomy of nonclassical logics by exploring the interplay between substructural hierarchy and Schmitz's hierarchy.

Climbing up the fast-growing complexity hierarchy. In Chapter 4 we have found various nonelementary problems, some of which are even nonprimitive recursive or nonmultiply recursive. However, it is also likely that some decidable substructural logics require more computational resources than logics with HACK-complete deducibility such as $\mathsf{FL_w}$. We thus pose the following question:

Question. *Does there exist a substructural logic complete for a fast-growing complexity class larger than* HACK *with respect to some type of decision problem?*

Rule admissibility problems. The deducibility problem for a substructural logic L is equivalent to the *rule derivability problem* for L, which determines whether a given (Hilbert-)rule is derivable in the Hilbert-style system for L. As is well known, there is another important property concerning Hilbert-rules, i.e., the notion of *admissibility*; see, e.g., Rybakov's book [104]. In view of this, we can consider another type of decision problem not addressed in the thesis. It is called the *rule admissibility problem* (cf. [104]), i.e., the problem of asking whether a given (Hilbert-)rule is admissible in the logic under consideration. For instance, Rybakov [103] showed that rule admissibility in intuitionistic propositional logic is decidable. After a while, Jeřábek [67] proved that it is coNEXPTIME-complete.[2] However, to the best of our knowledge, it remains unclear whether rule admissibility is decidable for most basic substructural logics.

Question. *Let* L *be a basic substructural logic (other than intuitionistic propositional logic). Is the rule admissibility problem for* L *decidable? If so, what is the precise complexity of the problem?*

Problems spanning across logic and automata theory. In Chapter 3, we have shown that deducibility in BCK is TOWER-complete. However, we do not know whether the deducibility problem for BCI (i.e., BCK without the rule of left-weakening, or equivalently, the implicational fragment of intuitionistic linear logic) is decidable.[3]

Question. *Is the deducibility problem for* BCI *decidable?*

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[2]As a side remark, admissibility in classical propositional logic is coNP-complete since, as is well-known, it is *structurally complete*; that is, every admissible rule is derivable. See, e.g., [20, Section 1.4].

[3]In Table 5.1, we summarize the complexity status of implicational fragments of basic commutative substructural logics. The notation is as follows:

- BCIW (see, e.g., [89]) = the implicational fragment of $\mathsf{FL_{ec}}$ (i.e., BCI with the rule of contraction),
- $\mathsf{LJ}_{\to}$ = the implicational fragment of LJ (i.e., BCI with the rules of contraction and weakening).

	Provability	Deducibility
BCI	NP-complete [18]	? (in RE, but ACK-hard, cf. [117, Sec. 7])
BCIW	2-EXPTIME-complete [106]	2-EXPTIME-complete[4]
BCK	NP-complete (Cor. 3.1.9)	TOWER-complete (Cor. 3.3.6)
$\mathsf{LJ}_{\to}$	PSPACE-complete [111]	PSPACE-complete [111]

Table 5.1: Complexity status of some implicational substructural logics.

Although the question is seemingly minor, it is deeply intertwined in two open questions from linear logic and vector addition systems, i.e., provability in MELL and reachability in BVASSes. The (un)decidability of provability in MELL is a major open problem in the community of linear logic for more than 30 years; see, e.g., [114, Section 1] for historical background on this problem. In [35], de Groote, Guillaume and Salvati were the first to discover a direct correspondence between BVASS-reachability and MELL-provability; that is, they proved that BVASS-reachability is inter-reducible to MELL-provability. Furthermore, it is not difficult to confirm that BCI-deducibility is inter-reducible to MELL-provability and to BVASS-reachability.[5] There-

[4]The 2-EXPTIME-hardness is a direct consequence of [106, Theorem 5.8]. Below we give a sketch of the proof of the 2-EXPTIME upper bound for BCIW-deducibility. Schmitz also proved that provability in the *multiplicative exponential fragment of intuitionistic contractive linear logic* (IMELLC), which is equivalent to the $\{\to, \circ, \Box, 1\}$-fragment of ILZ with the rule of (c), is 2-EXPTIME-complete; see [106, Section 6.2]. For our purpose, it thus suffices to show the following:

> *For any intuitionistic $\{\to\}$-sequent $\Gamma \Rightarrow s$ and any finite set Φ of intuitionistic $\{\to\}$-terms,*
>
> $$\Phi \vdash_{\mathsf{BCIW}} \Gamma \Rightarrow s \text{ if and only if } \vdash_{\mathsf{IMELLC}} \Box\Phi, \Gamma \Rightarrow s.$$

The only-if-part is shown by tedious induction on the length of derivation for $\Phi \vdash_{\mathsf{BCIW}} \Gamma \Rightarrow s$. Notice that the strategy we employed in the proof of Lemma 3.2.10 does not work for the if-part here. The if-part is however just a consequence of the following claim:

> *For any $\Box$-prenex $\{\to\}$-sequent $\Box\Gamma, \Delta \Rightarrow s$,*
>
> $$\vdash_{\mathsf{IMELLC}} \Box\Gamma, \Delta \Rightarrow s \text{ only if } \eta(\Gamma) \vdash_{\mathsf{BCIW}} \Delta \Rightarrow s.$$

The claim follows by induction on the cut-free derivation for $\Box\Gamma, \Delta \Rightarrow s$. (Note that IMELLC enjoys cut elimination; see, e.g., [96].)

[5]See, e.g., [117, Section 7] for a proof of a reduction from (ordinary) BVASS-reachability into BCI-deducibility.

fore, showing that any one of these three decision problems is (un)decidable amounts to establishing the (un)decidability of the other two.

To our knowledge, until around 2010, the best-known lower bound for BVASS-reachability was an exponential space shown by Lipton [83]. However, some major improvements appeared one after another over the past 15 years. In 2010, Lazić [77] proved that this problem is at least 2-ExpSpace-hard. In 2015, Lazić and Schmitz [78] proved that it is Tower-hard. In 2019, Czerwiński, Lasota, Lazić, Leroux, and Mazowiecki [31] improved the work of Lazić and Schmitz by proving that the reachability problem for VASSes, which *was* one of the most important open questions in this area, is already Tower-hard. Shortly afterward, Leroux [79], Czerwiński and Orlikowski [32] delivered the final blow to the precise complexity of VASS-reachability by showing that it is Ack-complete.[6]

In light of the latest breakthrough by Czerwiński, Orlikowski, and Leroux, we know only that BVASS-reachability, MELL-provability, and BCI-deducibility are not primitive recursive. At all events, the (un)decidability of these problems is still wrapped in mystery.

[6]In 2019, Leroux and Schmitz [80] already showed that the problem is decidable in Ack.

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