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On Gentzen's Three Consistency Proofs for Arithmetic

by

Yuta TAKAHASHI

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Contents

Chapter 1

Introduction

Gerhard Gentzen was a German logician in the early 20th century. He provided three proofs for the consistency of elementary number theory, which showed that no contradictory results are derivable in the theory. Gentzen inherited the research on the consistency of elementary number theory from David Hilbert. Hilbert was a German mathematician and significantly contributed to the foundational issues of mathematics between the late 19th and early 20th centuries. Members of his school, including Gentzen, also investigated these issues, one of them being the proof of the consistency of elementary number theory and analysis.

This thesis discusses Gentzen's three consistency proofs for elementary number theory. In this introduction, we explain the background of the thesis.

1.1 The Axiomatic Method and Hilbert's Program

In the paper "Mathematische Probleme" published in 1900, David Hilbert proposed the *axiomatic method* (die axiomatische Methode), from which the Hilbert School's consistency proofs, including Gentzen's proofs, are derived. The axiomatic method involves a procedure of foundational investigation: One begins with a presentation of axioms to formulate a mathematical theory and then shows the consistency and the completeness of the axioms. Hilbert explained the notions of consistency and completeness as follows. An axiomatic system of a mathematical theory is complete if no statement within the theory is held to be correct unless it can be derived from the axioms of the system by means of a finite number of logical steps. Furthermore, an axiomatic system is consistent if a finite number of logical steps based on the system's axioms never lead to contradictory results.[1]

On the significance of the problems of consistency, Hilbert made the following observation. A proof for the consistency of an axiomatic system is at the same time a proof of the existence of the mathematical structure this system describes.[2] This idea about the existence of a mathematical structure came from Hilbert's conception of a mathematical structure: It is a system of things such that mutual relations among these things are governed by the axioms for the structure.[3] For Hilbert, it *suffices* to show that, for example, the axioms of real numbers or the axioms of analysis are consistent to prove the existence of the complete system of real numbers.

At the International Congress of Mathematicians in 1900, Hilbert posed the problem of the consistency of analysis as the second of 23 mathematical problems. This problem was to prove the consistency of analysis *directly*; that is, to prove its consistency without reducing the consistency of the axioms of analysis to the consistency of another branch of mathematics.

Henri Poincaré, in his paper [Poi06] published in 1906, raised an objection against Hilbert's consistency proofs. Poincaré claimed that Hilbert's

[1]The following passage from "Mathematische Probleme" includes these explanations:

> When we are engaged in investigating the foundations of a science, we must set up a system of axioms which contains an exact and complete description of the relations subsisting between the elementary ideas of that science. The axiom so set up are at the same time the definitions of those elementary ideas; and no statement within the realm of the science whose foundation we are testing is held to be correct unless it can be derived from those axioms by means of a finite number of logical steps.
>
> [...]
>
> But above all I wish to designate the following as the most important among the numerous questions which can be asked with regard to the axioms: *To prove that they are not contradictory, that is, that a finite number of logical steps based on them can never lead to contradictory results.* ([Ewa96, p.1104], italics original)

[2]The following passage includes this observation:

> If contradictory attributes be assigned to a concept, I say, that *mathematically the concept does not exist.* So, for example, a real number whose square is -1 does not exist mathematically. But if it can be proved that the attributes assigned to the concept can never lead to a contradiction by the application of a finite number of logical inferences, I say that the mathematical existence of the concept (for example, of a number or a function which satisfies certain conditions) is thereby proved. ([Ewa96, p.1105], italics original)

[3]Cf. [Ewa96, p.1105].

argument in [Hil05] for the consistency of a certain arithmetical theory used the principle of complete induction, i.e., the induction principle on the length of a given derivation for an equation.[4] Here, Poincaré suggested that consistency proofs for an arithmetical theory with the induction axiom must include circular reasoning.

During the 1920s, Hilbert developed his mature method for the problem of the consistency of analysis and responded to Poincaré's objection above. Let us explain Hilbert's mature method first. To prove the consistency of analysis, he proposed the *finitary standpoint* (der finite Standpunkt), which he considered the foundation of not only mathematics but all sciences. Roughly speaking, the finitary standpoint admits only inferences and principles about the definite manipulation of concrete symbols.[5] This standpoint identifies numerals with natural numbers and considers the operation of concatenating the symbol $'$ to the right-hand side of a given numeral to be the successor function for natural numbers. Addition and multiplication are defined by primitive recursion, which is also admissible from the finitary standpoint:

$$\begin{aligned} n+0 &= n, \\ n+m' &= (n+m)', \end{aligned}$$

$$\begin{aligned} n\cdot 0 &= 0, \\ n\cdot m' &= (n\cdot m)+n. \end{aligned}$$

Hilbert's strategy for solving the problem of the consistency of analysis was to prove the following from the finitary standpoint: the consistency of the axiomatic system of analysis that is formulated as a *formal system*, namely, a meaningless symbolic system. This strategy is now called *Hilbert's Program*. Through this program, Hilbert aimed to justify the practice of classical mathematics: He aimed to justify both the introduction of ideal elements into mathematics and the application of classical reasoning to those elements.[6]

Hilbert's strategy above responded to Poincaré's objection against consistency proofs. In this strategy, Hilbert distinguished the induction principle used in the finitary standpoint from the one included in a formal system of analysis. The former is different from the latter in the respect that the application of the former is considerably restricted. In applying it, an inductive

[4][Poi06, Section XX].

[5][Hil26, pp.170-171]. For lack of space, the formal theory to which the finitary standpoint corresponds cannot be discussed here. There is a fairly general agreement that it includes Primitive Recursive Arithmetic. See [Tai02], [Zac03] and [SR05].

[6][Hil26, pp.178-179], [Hil28, p.74].

predicate $A(x)$ must be decidable; that is, it must be possible to decide in finitely many steps whether $A(n)$ holds for a given numeral n. For example, the commutative law for addition

$$n + m = m + n$$

is provable by means of complete induction in the finitary standpoint. Hilbert's Program aimed to prove the consistency of analysis with this restricted induction principle.

Despite Hilbert's efforts, Kurt Gödel published his proofs of the incompleteness theorems in 1931, which showed that Hilbert's Program at least needed some modification. Roughly speaking, the incompleteness theorems showed that if second-order arithmetic, which is a formal system of analysis, is consistent, its consistency cannot be proved from the finitary standpoint in the original form. The incompleteness theorems also apply to first-order arithmetic, which is a formal system of elementary number theory: If first-order arithmetic is consistent, its consistency cannot be proved from the finitary standpoint unless it is extended.

Gentzen's consistency proofs for first-order arithmetic were completed in the period following Gödel's incompleteness theorems. Gentzen proved its consistency by extending the finitary standpoint.

1.2 Gentzen's Works on the Consistency of Arithmetic

Gentzen, who was born in Greifswald on November 24, 1909, studied mathematics at the University of Greifswald for one year and then spent the academic year 1929–30 at the University of Göttingen.[7] Gentzen attended Hilbert's lectures on set theory there. After this academic year, he spent one semester at Munich, a further semester at Berlin, and then returned to Göttingen. He was granted a doctorate by the University of Göttingen in 1933 and became Hilbert's assistant in 1934. As is well known, Hilbert's works on foundations of mathematics had a great influence on Gentzen's research.

Modifying Hilbert's finitary standpoint, Gentzen gave three proofs for the consistency of first-order arithmetic.[8] He explicitly used the induction

[7] We owe the brief bibliographical explanation in this paragraph to [Gen69, p.vii] and [Sie12, pp.108-109]. For a detailed bibliographical description, see [Men07].

[8] Note that Gentzen intended his proofs to comprehend not only first-order Peano arithmetic but also any system that may have as an axiom any finitistically valid (quantifier-free) formula. See [Gen36, §6.2] and [Gen38b, §1.4].

principle up to ε_0 for quantifier-free arithmetical formulas in his second and third consistency proofs.

The first proof was included in a paper submitted in 1935 and posthumously published in 1974 ([Gen74]), since Gentzen withdrew the paper because of Paul Bernays' criticism.[9] The second and third proofs were published in 1936 ([Gen36]) and 1938 ([Gen38b]), respectively. Note that the method of the third proof, i.e., the cut elimination method became a standard technique of proof theory of first-order arithmetic.[10] In this thesis, we discuss the first and second proofs rather than the third proof. Hereafter, we call the first, second, and third proofs the *1935 proof*, the *1936 proof*, and the *1938 proof*, respectively.[11]

This thesis focuses on Gentzen's interpretation of first-order arithmetical formulas, i.e., his assignment of a sense or meaning to every first-order arithmetical formula. He formulated this interpretation in the 1935 and 1936 proofs. We state it in the following form:

> A formula A is correct
>
> if and only if
>
> a reduction procedure is statable for the sequent $\rightarrow A$.

We will explain the details of this interpretation in the next chapter. Here, it suffices to say that Gentzen, in this interpretation, assigned a sense to each arithmetical formula A by explaining when A is correct.

Gentzen considered the interpretation above *finitist* (finit). He wrote,

[9] Cf. [Ber70, p.409]. An English translation of (the main part of) this paper was published in 1969 as an appendix to [Gen69, ch.4]. For the historical description of the submission of the first proof, see [Men07, pp.57-62].

[10] For example, see [Take87, §12].

[11] There have been many papers with expository sections for the 1935 proof or the 1936 proof. The first reconstruction of Gentzen's 1935 proof was made by [Ber70], and [Neg80] refined this reconstruction. [Kre71] claimed that the bar rule is used in the 1935 proof. [Sun83] explained the relation between the devices of the 1935 proof and infinitary derivations. [Coq95], [Tai05], and [vPl09b] independently revealed the game theoretic aspect of the 1935 proof. [Tai05] also pointed out the relation between the 1935 proof and the no-counterexample interpretation. [Aki10] reconstructed the 1935 proof, using Mints-Buchholz's method of finite notations for infinitary derivations. [Sie12] investigated the historical relation between Hilbert's proof theory and the 1935 proof by scrutinizing Gentzen's unpublished manuscripts. [Tai15] argued that Gentzen did not employ the bar theorem in the 1935 proof.

[Yas80] presented a reconstruction of the 1936 proof and some applications of it. [SP95] remarked the relation between the 1936 proof and the no-counterexample interpretation. [Buc15] reconstructed the 1936 proof, using Mints-Buchholz's method of finite notations for infinitary derivations.

> Having rejected the actualist interpretation (an-sich-Auffasung) of transfinite propositions, we are still left with the possibility of *ascribing a 'finitist' sense* (finiter Sinn) *to such propositions, i.e., of interpreting* (zu deuten) them in each case as expressions for definite *finitely* (endlich) characterizable states of affairs. ([Gen36, p.525], [Gen69, pp.162-163], italics original)[12]

The "actualist interpretation of transfinite propositions" means the interpretation of quantified arithmetical formulas that treats the infinite sequence of all natural numbers as closed, i.e., finished. This interpretation admits, for example, the inference from $\neg\neg\exists x A(x)$ to $\exists x A(x)$ for an arbitrary arithmetical predicate $A(x)$. Gentzen wanted to replace such an interpretation with his interpretation of arithmetical formulas. For our purpose, the following is important: Gentzen aimed to interpret them as expressing finitely characterizable states of affairs.[13] As we will see in the next chapter, he interpreted an arithmetical formula to express that a certain process of rewriting syntactical objects always terminates. This is Gentzen's example of finitely characterizable states of affairs and he considered his interpretation of arithmetical formulas finitist in this respect. Although one might wonder about the relation between finiteness according to Gentzen and Hilbert, we cannot discuss it in this thesis for lack of space.

1.3 Objectives of the Thesis

Let us give an outline of the chapters of this thesis. In Chapter 2, we discuss the relation of Gentzen's interpretation of arithmetical formulas with the debates between the Hilbert School and the Brouwer School on the significance of consistency proofs. The Hilbert School was a major force in the foundations of mathematics during the early 20th century. The Brouwer School, whose founder was Luitzen Egbertus Jan Brouwer, was also a formidable player then.

Brouwer held *intuitionism* and maintained that classical mathematics, whose consistency Hilbert aimed to prove, was based on incorrect methods

[12]In this thesis, translations of quotations from Gentzen are by Szabo ([Gen69]), with a few exceptions.

[13]A comment on our usage. By "a finitist sense of a proposition or formula," we mean a *meaning* of a proposition or formula that is admissible from Gentzen's finitist standpoint. In contrast, by "a finitist interpretation of a proposition or formula," we mean an *assignment* of a meaning to a proposition or formula such that the assigned meaning is admissible from Gentzen's finitist standpoint.

and must be revised using intuitionist methods. Moreover, he raised objections against the significance of consistency proofs for classical mathematics. According to Brouwer, consistency proofs for classical mathematics hold no significance, because the theorems of classical mathematics have no sense regardless of whether classical mathematics is consistent, as it is based on incorrect methods. Classical mathematics is nothing but a game of manipulating meaningless symbols and only theorems proved in an intuitionist way have sense.

Gentzen responded to this objection in [Gen74, Gen36] using his interpretation of arithmetical formulas. Gentzen worked to convince intuitionists that his 1935 and 1936 proofs, which include Gentzen's interpretation of arithmetical formulas, not only showed the consistency of first-order classical arithmetic but also gave a finitist sense to each of its theorems.

The aim of Chapter 2 is to explore this feature of Gentzen's 1935 proof. First, we show why Gentzen took the above objection seriously and responded to it, scrutinizing not only Gentzen's papers but also Hilbert's and Brouwer's. Next, we prove the main lemma of the 1935 proof, which is crucial to Gentzen's response, using intuitionist methods, and show that it is possible for intuitionists to admit a sense that Gentzen gave to each theorem of first-order classical arithmetic.

Recently, Detlefsen also focused on the above feature of the 1935 proof, that is, the intention to convince intuitionists that Gentzen's interpretation of arithmetical formulas not only showed the consistency of first-order classical arithmetic but also gave a finitist sense to each of its theorems. Detlefsen compared Gentzen's conception of consistency proofs comprehensively with Hilbert's ([Det15]). In addition, Tait formulated Gentzen's interpretation of arithmetical formulas using intuitionist methods ([Tai15]), which are essentially the same as the intuitionist methods that we have used in Chapter 2. The key contribution of Chapter 2 consists in our argument showing that it is possible for intuitionists to admit a sense that Gentzen's interpretation of arithmetical formulas gave to each theorem of first-order classical arithmetic.

In Chapter 3, we discuss Gentzen's interpretation of arithmetical formulas in light of the problem of interpreting the implication formulas $A \supset B$. This problem is a recurrent topic in the foundations of mathematics. For example, Hilbert and Bernays investigated which implication formulas are interpretable from the finitary standpoint. They argued that the range of implication formulas interpretable from the intuitionist viewpoint is wider than the range of implication formulas interpretable from the finitary standpoint.[14] Gentzen focused on the problem of implications for interpretation as

[14] Cf. [HB1934, p.43], [Ber83, pp.265-267].

well. In his papers for the 1935 and 1936 proofs, he showed the circularity of a certain interpretation of implication, stating that one of the main objectives of his 1935 and 1936 proofs was to offer an interpretation of implication that avoids this circularity.[15] However, he offered only an indirect interpretation of implication via the translation of $A \supset B$ into $\neg(A \wedge \neg B)$. Moreover, he did not argue for the claim that this interpretation avoids circularity.

In this chapter, we first formulate a direct Gentzen-style interpretation of implication by adapting Tait's formulation of the 1935 proof to the case of implication.[16] Next, we argue that this interpretation avoids the circularity Gentzen urged against.

Okada, in [Oka88], has proposed a way out from this circularity, but it is independent of Gentzen's consistency proofs. The main contribution in this chapter is that we explain how the 1935 proof avoids this circularity. As a consequence of this explanation, we will point out the following significance of the 1935 proof: It gave a non-circular interpretation to implication formulas of first-order arithmetic from Gentzen's standpoint.

Finally, in Chapter 4, we discuss Gentzen's interpretation of arithmetical formulas in relation to the distinction between *contentual correctness* proofs (inhaltliche Richtigkeitsbeweise) and *formal correctness* proofs (formale Richtigkeitsbeweise). As Sieg explained in [Sie12], the terms "formal correctness proofs" and "contentual correctness proofs" were used by Gentzen in his unpublished manuscripts about consistency proofs. According to Sieg, contentual correctness proofs show the consistency of a theory by verifying that its axioms and theorems are all correct. In contrast, formal correctness proofs show the consistency of a theory by assigning a normal derivation to each numeric equation that is derivable in the theory. Its consistency follows from the fact that no normal derivation for a numeric equation has an incorrect conclusion. Sieg observed that Gentzen eventually considered his 1936 proof as intermediate between these two kinds. This observation induces the following questions: Is the 1936 proof both a contentual correctness proof and a formal correctness proof? How do the 1936 proof's contentual aspects, especially Gentzen's interpretation of arithmetical formulas, relate to its formal aspects?

In Chapter 4, we answer these two questions. First, we argue that the 1935 proof is a contentual correctness proof and that the 1938 proof is a formal correctness proof. Second, we show that the 1936 proof is both contentual and formal because the main lemma of the 1936 implies both the

[15] [Gen36, p.530], [Gen69, p.168].

[16] In contrast, we will not use Tait's formulation in Chapter 2, but, as we have said, the intuitionist methods we use in Chapter 2 are essentially the same ones used by Tait.

main lemmas of the 1935 proof and of the 1938 proof. To show this in a uniform way, we formulate the 1936 proof in the framework of *normalization trees*, which was introduced in [Aki10, AT13] by means of finite notations for infinitary derivations. Third, we explain how the 1936 proof's contentual aspects relate to its formal aspects. Our argument for the claim that the 1936 proof is both contentual and formal enables us to see the following: The correctness of a derivable formula of first-order arithmetic is in fact shown by means of syntactic transformation of a given derivation of the formula. In other words, Gentzen's interpretation of arithmetical formulas can be given by the 1936 proof's formal aspects.

Our major contribution in Chapter 4 is that we show the 1936 proof to be both contentual and formal in the sense of Sieg's explanation. Furthermore, our argument also comprehensively explains the relation between its contentual aspects and its formal aspects.

The entire argument of this thesis will show a connection between some aims of Gentzen's research in 1935 and a result that he reached in 1938, i.e., the cut elimination method employed for the 1938 proof. We show it by examining Gentzen's research on the consistency of first-order arithmetic in terms of his three consistency proofs.

Chapter 3 is an English translation of the following paper with minor modifications: Yuta Takahashi. "A Philosophical Significance of Gentzen's 1935 Consistency Proof for First-Order Arithmetic: on a Circularity of Implication." (in Japanese) *Kagaku Tetsugaku* (*PHILOSOPHY OF SCIENCE*), 49 (2016): 49-66. Chapter 4 is a pre-copyedited version (with minor modifications) of the following contribution: Ryota Akiyoshi and Yuta Takahashi. "Contentual and Formal Aspects of Gentzen's Consistency Proofs." In *Philosophical Logic: Current Trends in Asia – Proceedings of AWPL-TPLC 2016*, edited by Syraya Chin-Mu Yang, Kok Yong Lee and Hiroakira Ono.

Chapter 2

Philosophical Background of Gentzen's Interpretation for Arithmetic

2.1 Introduction to Chapter 2

Gentzen's three consistency proofs have a common aim that originates from Hilbert's Program. In this program, as seen in Chapter 1, Hilbert aimed to justify the practice of classical mathematics. In other words, he aimed to justify both the introduction of ideal elements into mathematics and the application of classical reasoning to these elements, by proving the following from his finitary standpoint: No contradiction can be derived in formal systems codifying ideal parts of mathematics.[1] Gentzen also had these aims. At the outset of his attempt to achieve them, Gentzen aimed to justify with these three consistency proofs the application of classical reasoning to quantified formulas of first-order arithmetic.

In addition to his three consistency proofs' common aim, Gentzen gave a *"finitist" interpretation* to every first-order arithmetical formula by means of his first two consistency proofs, i.e., the 1935 and 1936 proofs.[2] Even though Hilbert began to investigate interpretations of mathematical propositions for foundational purposes after Gödel's incompleteness theorems,[3] Hilbert's 1920s proof theory included no such interpretation. In this respect, Gentzen's

[1][Hil26, pp.178-179], [Hil28, p.74].

[2]This reading of the 1935 and 1936 proofs has been adopted by [Kre71], [Neg80], [Coq95], [Tai05, Tai15], [Sie12] and [AT13]. Especially, in [Tai15, p.215], Tait extracted from the 1935 and 1936 proofs the same interpretation as the interpretation (GI) that we state below.

[3]Cf. [Sie12, §5.5].

proof theory transcended Hilbert's proof theory of the 1920s.

Several studies have provided some clues to the investigation into the relationship of Gentzen's interpretation of arithmetical formulas with intuitionism. Bernays remarked in [Ber70] that reduction procedures (Reduziervorschrift), a key concept for that interpretation, "involves universal quantification over free choice sequences."[4] In [Kre71], Kreisel claimed, "In fact, not the fan theorem, but rather the *bar theorem* is involved in Gentzen's [first] proof; or, to be precise, [...] Gentzen uses the corresponding *rule*."[5]

Here, we investigate the relationship of Gentzen's interpretation of arithmetical formulas with intuitionism, in terms of the debate between the Hilbert School and the Brouwer School over consistency proofs' significance. First, we argue that this interpretation functioned as a response to a Brouwer-style objection against consistency proofs' significance. Brouwer had raised such an objection on the basis of his claim about the interpretation of mathematical propositions; his claim was very close to Hilbert's claim about the finitist interpretation of existential propositions. Gentzen had accepted Hilbert's claim; hence, Gentzen took this objection seriously and responded to it. Second, we propose a way to understand the 1935 proof's response to the Brouwer-style objection from an intuitionist perspective. We formulate Gentzen's interpretation of arithmetical formulas by means of spreads, which are infinite trees in intuitionistic mathematics, and then prove the key lemma to his response by monotone bar induction.[6] Note that we do not claim Gentzen himself used monotone bar induction to prove the lemma.

This chapter is structured as follows. In Section 2.2, we explain both Hilbert's claim and the Brouwer-style objection mentioned above. In Section 2.3, we argue that Gentzen's interpretation of arithmetical formulas served as a response to the Brouwer-style objection. In Section 2.4, we provide an intuitionist formulation of reduction procedures–key to this interpretation–by using the notion of spreads. Section 2.5 gives a proof for the key lemma of Gentzen's response, and Section 2.6 concludes the chapter.

[4] [Ber70, p.417].

[5] [Kre71, p.262].

[6] As to monotone bar induction, see [TD88, ch.4, §8]. In [Tai15, p.223], Tait has given a proof of this key lemma by means of the principle that corresponds to decidable bar induction.

2.2 Hilbert and Brouwer on Mathematical Propositions

In this section, we first explain Hilbert's claim that set the scene for Gentzen's interpretation of arithmetical formulas. Next, we argue that Brouwer raised an objection to consistency proofs' significance and that one of its premises is very close to Hilbert's claim.

In papers published during the 1920s, Hilbert established the *finitary standpoint* to provide a firm foundation for the introduction of ideal elements into mathematics, such as irrational numbers and complex numbers. The finitary standpoint admits only facts and concepts about the definite manipulation of concrete symbols.[7] Hilbert aimed to justify the introduction of ideal elements and the application of classical reasoning to these elements by proving the following from the finitary standpoint: No contradiction can be derived in formal systems codifying ideal parts of mathematics.[8] This is the main aim of *Hilbert's Program*.

In developing this program, Hilbert made several claims about the finitist interpretation of quantified propositions. Most important here is the finitist interpretation of existential propositions. Hilbert wrote,[9]

> In general, from the finitist point of view an existential proposition of the form "There exists a number having this or that property" has a sense (Sinn) only as a *partial proposition* (Partialaussage), that is, as part of a proposition that is more precisely determined but whose exact content is unessential for many applications. ([Hil26, p.173])

Here, Hilbert claimed that from the finitary standpoint an existential proposition has sense only as a "partial proposition." He provide an example of partial propositions in the following passage:

> This proposition ["Between $\mathfrak{p}+1$ and $\mathfrak{p}!+1$ there certainly exists a new prime number"] itself, moreover, is completely in conformity with our finitist attitude. For "there exists" here serves merely to abbreviate the proposition:
>
>> Certainly $\mathfrak{p} + 1$ or $\mathfrak{p} + 2$ or $\mathfrak{p} + 3$ or $\ldots$ or $\mathfrak{p}! + 1$ is a prime number.

[7] [Hil26, pp.170-171].
[8] [Hil26, pp.178-179], [Hil28, p.74].
[9] The next two translations of [Hil26] are taken from [vHe67, pp.377-378].

> But let us go on. Obviously, to say
>
> > There exists a prime number that (1) is $> \mathfrak{p}$ and (2) is at the same time $\leq \mathfrak{p}! + 1$ [(E)]
>
> would amount to the same thing, and this leads us to formulate a proposition that expresses only a *part* of Euclid's assertion, namely: [(P)] there exists a prime number that is $> \mathfrak{p}$. ([Hil26, p.172], italics added.)

In this passage, the example of partial propositions is the proposition (P), which is part of the proposition (E), whose content is more exactly determined.

On the basis of the last quotation, we propose the following characterization of partial propositions:

> A partial proposition is an existential proposition $\exists x A(x)$ such that it alludes to some effective way of yielding its witness in some definite and finite totality.

Note that the word "definite" means "not hazy." If we adopt this characterization, we can ascribe the following claim to Hilbert: From the finitary standpoint, an existential proposition $\exists x A(x)$ means that one possesses some effective way of yielding a witness for $\exists x A(x)$ in some definite and finite totality.

This is the content of Hilbert's claim that an existential proposition has sense only as a partial proposition. Of course, one can admit that an existential proposition $\exists x A(x)$ has an ordinary sense, according to which $\exists x A(x)$ means that there is an object a satisfying $A(x)$ *somewhere*. Here, one does not need to possess a way of yielding an object satisfying $A(x)$. Let us call this interpretation of existential propositions the *classical* interpretation. The above quotations' point is that the classical interpretation of existential propositions is not admissible from the finitary standpoint. An existential proposition must be treated as a partial proposition.

According to the reading of Hilbert as an instrumentalist, it is evident he considered that existential propositions without witnesses have no sense from *any* standpoint and that they are merely useful instruments to prove propositions that have senses.[10] In this thesis, we are not committed to the question of whether Hilbert really held with instrumentalism or not. As stated in the last paragraph, what we want to emphasize is the following:

[10] This reading is, for example, presented in [Det90, pp.346-347]. As to the reading of Hilbert as a non-instrumentalist, see [Hal90] and [AR01, §4].

Hilbert claimed that existential propositions without witnesses have no sense from the *finitary* standpoint.

About the interpretation of existential propositions, Brouwer took a position very close to Hilbert's: The classical interpretation of existential propositions is not admissible in Brouwer's intuitionism. We ascribe this position to Brouwer on the basis of the following two passages from his writings. First, in his 1933 paper "Willen, Weten, Spreken," Brouwer wrote,

> [...]: [S]uppose that human beings with unlimited power of memory recorded their constructions in shortened form in a suitable language, surveyed the strings of their affirmations in this language and then would be able to see the occurrence of the linguistic figures of logical principles in all their mathematical modifications. Careful rational reflection would then show that as far as the principles of identity, contradiction and syllogism are concerned such an occurrence could be expected; as to the linguistic figure of the principle of the excluded middle, this would only occur if one restricted oneself to affirmations concerning parts of a definite, once and for all given, finite mathematical system. Wider applications of the latter principle would never occur since such applications to pure-mathematical affirmations usually lead to verbal complexes *devoid of any mathematical sense and therefore of any sense*. ([vSt90, pp.427-428], italics added.)

For simplicity's sake, consider an instance $\exists x A(x) \vee \neg \exists x A(x)$ of the Principle of the Excluded Middle (PEM), where $A(x)$ is a unary predicate. According to Brouwer, the use of this instance leads to a correct conclusion, namely, one accompanied by some mental constructions if the variable x runs over a finite collection and the predicate $A(x)$ is decidable. However, if it runs over an *infinite* collection, its use produces a proposition not accompanied by any construction. For it is not always the case that one possesses a way of yielding either a witness for $\exists x A(x)$ or a function f such that $\neg A(f(a))$ holds for every a in the range of x. That is to say, one does not always have some mental constructions witnessing either of the disjuncts.[11] In his 1922 paper "Intuitionistische Mengenlehre" ([Bro22]), Brouwer made the same remark as in the last quotation:

> The Principle of the Excluded Middle has only scholastic and heuristic value, so that theorems that in their proof cannot avoid

[11] As to Brouwer's conception of the relationship between the content of a mathematical proposition and mental constructions, see [Kan06].

> the use of this principle lack all mathematical content (Inhalt). ([Bro22, pp.949-950].)[12]

If one admits the classical interpretation of existential propositions, it is inevitable to admit the validity of PEM of the form $\exists x A(x) \vee \neg \exists x A(x)$ such that the predicate $A(x)$ is not decidable. However, as we have seen, Brouwer did not admit the validity of such instances, so the classical interpretation of existential propositions is not admissible by Brouwer's intuitionism.

In our reading, Brouwer's position above–that the classical interpretation of existential propositions is not admissible–is a premise of his objection to the consistency proofs, for which the Hilbert School asked. Remember that the Hilbert School aimed to show the consistency of *classical* mathematics, in which the use of PEM or a similar principle is not restricted. Brouwer's objection can be summarized as follows.

1. The classical interpretation of existential propositions is not admissible.
2. In the intuitionist interpretation, the alternative to the classical interpretation, the existential propositions of classical mathematics proved with substantial use of PEM are incorrect or at least not proved yet.
3. Thus, classical mathematics is incorrect.
4. Moreover, classical mathematics remains incorrect even if its consistency is proved by using the Hilbert School's methods, since such a proof ascribes no sense to those existential propositions.
5. Accordingly, the consistency proofs that the Hilbert School requested are of no significance.

Let us explain our reading above in detail. First of all, we want to call attention to the following passage in Brouwer's 1923 paper "Über die Bedeutung des Satzes vom ausgeschlossenen Dritten in der Mathematik, insbesondere in der Funktionentheorie."

> The contradictions that, as a result, one repeatedly encountered gave rise to the *formalistic critique*, a critique which in essence comes to this: the *language accompanying the mathematical mental activity* is subjected to a mathematical examination. To such an examination the laws of theoretical logic present themselves as operators acting on primitive formulas or axioms, and one sets himself the goal of transforming these axioms in such a way that

[12] This English translation is from [Man98, p.23].

> the linguistic effect of the operators mentioned (which are themselves retained unchanged) can no longer be disturbed by the appearance of the linguistic figure of a contradiction. We need by no means despair of reaching this goal, but nothing of mathematical value will thus be gained: an incorrect theory, even if it cannot be inhibited by any contradiction that would refute it, is none the less incorrect, just as a criminal policy is none the less criminal even if it cannot be inhibited by any court that would curb it. ([Bro23, pp.2-3].)[13]

Here, Brouwer argued that consistency proofs for "an incorrect theory" are of no significance because such theories are *incorrect* regardless of their consistency. The following passage, which occurs just after the previous quotation, indicates that Brouwer considered classical mathematics an example of such theories:

> The following two fundamental properties, which follow from the principle of excluded middle, have been of basic significance for this incorrect "logical" mathematics of infinity ("logical" because it makes use of the principle of excluded middle), especially for the *theory of real functions* (developed mainly by the Paris school):
>
> 1. *The points of the continuum form an ordered point species;*
> 2. *Every mathematical species is either finite or infinite.*
>
> ([Bro23, p.3].)

Details of the "two fundamental properties" are not relevant to our purpose. Rather, what is important is that Brouwer obviously referred to classical mathematics by "this incorrect 'logical' mathematics of infinity." In sum, Brouwer argued in these quotations that consistency proofs for classical mathematics are of no significance because classical mathematics is *incorrect* regardless of its consistency.

Moreover, we can understand Brouwer's word "incorrect" as follows. For Brouwer, classical mathematics is incorrect because its existential propositions proved with substantial use of PEM either are incorrect or are at least not proved yet. Then, we can read from the passages cited above Brouwer's objection to consistency proofs requested by the Hilbert School.

[13]This and the next translations of [Bro23] are from [vHe67, pp.336-337].

2.3 Gentzen's Response to the Brouwer-style Objection

In this section, we argue that Gentzen's interpretation of first-order arithmetical formulas served as a response to the Brouwer-style objection as explained in the last section. First of all, we can say that Gentzen clearly recognized the objection and he took it seriously. In the following passage, which appeared in both [Gen74] and [Gen36] including the 1935 and 1936 proofs, respectively, Gentzen mentioned the objection and then responded to it:

> On the part of the *intuitionists*, the following *objection* is raised against the significance of consistency proofs*: even if it had been demonstrated that the disputable forms of inference cannot lead to mutually contradictory results, these results would nevertheless be propositions *without sense* (sinnlos) and their investigation therefore an idle pastime; real *knowledge* (wirkliche Erkenntnisse) could be gained only by means of indisputable intuitionist (or finitist, as the case may be) forms of inference.
>
> Let us, for example, consider the existential proposition cited at 10.6, for which the *statement* of a number whose existence is asserted is not possible. According to the intuitionist view, this proposition is therefore *without sense* (sinnlos); an existential proposition can after all be significantly asserted only if a numerical example is available.
>
> What can we say to this?
>
> [. . .]
>
> [. . .] The major part of my consistency proof, however, consists precisely in *ascribing* (beilegen) *a finitist sense* to actualist propositions (an-sich-Aussagen), viz.: for every arbitrary proposition, as long as it is provable, a *reduction procedure* according to 13.6 *can be stated*, and this fact represents the finitist sense of the proposition concerned and this sense is gained precisely through the consistency proof. ([Footnote]*: For example, cf.: L. E. J. Brouwer, Intuitionistische Betrachtungen über den Formalismus, Sitzungsber. d. Preuß. Akad. d. Wiss., phys.-math. KI. (1928), S.48–52.)
> ([Gen74, pp.117-118], [Gen36, pp.563-564], [Gen69, pp.200-201],

italics original)[14]

In this quotation's latter part, Gentzen claimed that the major part of his consistency proof consists in ascribing "finitist" *senses* to all theorems of first-order classical arithmetic, which include the theorems proved with PEM. This claim is obviously Gentzen's response to the Brouwer-style objection. Moreover, Gentzen said that he responded to the objection with "the major part" of his consistency proofs. This indicates that Gentzen took the objection seriously.

Indeed, Gentzen took the Brouwer-style objection seriously because he had followed Hilbert's claim about the finitist interpretation of existential propositions. Gentzen wrote,

> What sense should we concede to a proposition of the form $\exists\mathfrak{x}\mathfrak{F}(\mathfrak{x})$? The *actualist* interpretation that *somewhere* in the infinite number sequence there exists a number with the property $\mathfrak{F}$ is for us without sense. If, on the other hand, the proposition $\mathfrak{F}(\mathfrak{n})$ has been recognized as significant and valid for a definite number $\mathfrak{n}$, we wish to be able to conclude ($\exists$-introduction): $\exists\mathfrak{x}\mathfrak{F}(\mathfrak{x})$. There are no objections to this; the proposition $\exists\mathfrak{x}\mathfrak{F}(\mathfrak{x})$ now constitutes only a *weakening* of the proposition $\mathfrak{F}(\mathfrak{n})$ ('Partialaussage' for Hilbert, 'Urteilsabstrakt' for Weyl) in that it now attests merely that we have found a number $\mathfrak{n}$ with property $\mathfrak{F}$, although this number itself is no longer mentioned. Thus, $\exists\mathfrak{x}\mathfrak{F}(\mathfrak{x})$ acquires in this way a finitary sense. ([Gen36, p.527], [Gen69, pp.164-165], italics original)

Gentzen followed Hilbert's claim that an existential proposition has sense only as a partial proposition. Thus, he aimed at responding to the Brouwer-style objection, since a very close claim to Hilbert's is used as a premise in the objection. Gentzen was concerned that the theorems proved with PEM or a similar principle also have no sense from his standpoint for consistency proofs. The 1935 and 1936 proofs have dealt with not only the problem of the consistency of first-order classical arithmetic, but also this concern.

In the rest of this section, we explain how Gentzen responded to the Brouwer-style objection. Our key claim is as follows: Gentzen aimed to

[14] As said in Chapter 1, translations of quotations from Gentzen are by Szabo ([Gen69]), with a few exceptions. The translation of "eine Reduziervorschrift" is one of such exceptions: We translate "eine Reduziervorschrift" as "a reduction procedure" (we owe this translation to [Sie12]), whereas Szabo translates it as "a reduction rule." For the other exception, see Footnote 16.

Moreover, note that the version of this passage in [Gen36] also includes the reference to Heyting's work ([Hey34]).

ascribe a finitist sense to each theorem of first-order classical arithmetic by giving his own interpretation, according to which all theorems of first-order classical arithmetic are correct.

In papers for the 1935 and 1936 proofs, Gentzen manifestly pointed out the possibility of a finitist interpretation of arithmetical formulas, after having rejected the standard interpretation ("the actualist interpretation," in his term) because of its inadmissibility from his standpoint.[15] He wrote,

> Having rejected the actualist interpretation of transfinite propositions, we are still left with the possibility of *ascribing a 'finitist' sense* (ein 'finiter' Sinn) *to such propositions, i.e., of interpreting* (zu deuten) them in each case as expressions for definite *finitely* (endlich) characterizable states of affairs. ([Gen36, p.525], [Gen69, pp.162-163], italics original)

In the following passage, Gentzen suggested how to give such an interpretation, although he did not present it explicitly.

> The concept of the 'statability of a *reduction procedure*' (die Angebbarkeit einer Reduziervorschrift) for a sequent, to be defined below, will serve as the formal replacement (formaler Ersatz) of the contentual *concept of correctness* (der inhaltliche Richtigkeitsbegriff); it provides us with a *special finitist interpretation* (finite Deutung) of propositions and takes the place of their *actualist interpretation* [...]. ([Gen74, p.100], [Gen36, p.536], [Gen69, p.173], italics original)[16]

According to Gentzen, his interpretation of arithmetical formulas is given by the *statability of a reduction procedure*. The statability of a reduction procedure serves as an alternative concept of *correctness* and gives this interpretation by explaining the correctness of arithmetical formulas.

We extract from the last quotation the following interpretation, which we saw briefly in Chapter 1. Let $\Gamma \to A$ be a sequent of first-order arithmetic, where Γ is a finite set of formulas.[17] Then,

(GI) $\Gamma \to A$ is correct

if and only if

a reduction procedure is statable for $\Gamma \to A$.

[15] [Gen36, p.524], [Gen69, p.162].

[16] Following [Sie12], we translate "der inhaltliche Richtigkeitsbegriff" as "the contentual concept of correctness." In [Gen69] Szabo translates it as "the informal concept of truth."

[17] Note that the formal system used in the 1935 and 1936 proofs is a natural deduction system in sequent-style.

Definition 2.4.4 in Section 2.4 provides an (equivalent) version of Gentzen's notion of a reduction procedure. Moreover, in Section 2.5, we explain when a reduction procedure is statable. Here, note that the statability of a reduction procedure requires the availability of not only this procedure but also a proof for its termination, as Tait pointed out in [Tai15, Footnote 4].[18] If we stipulate that a reduction procedure for a formula A means one for $\rightarrow A$, we obtain the following from (GI):

> A is correct
>
> if and only if
>
> a reduction procedure is statable for A.

As stated above, Gentzen's response to the Brouwer-style objection was to define the sense of an arithmetical formula, or more generally of a sequent to be that a reduction procedure is statable for it and to show that classically derivable sequents are all correct in that sense. The main lemma of the 1935 proof, which was crucial not only to the proof of the consistency of first-order classical arithmetic Z but also to Gentzen's response, is as follows:[19]

Main Lemma. *For every sequent $\Gamma \rightarrow A$ of Z, if $\Gamma \rightarrow A$ is derivable in Z, then there is a reduction procedure for $\Gamma \rightarrow A$.*

In Section 2.5, we see the following: This lemma shows the correctness of each Z-derivable sequent $\Gamma \rightarrow A$ in the sense of (GI) because the proof of the lemma gives not only a reduction procedure for $\Gamma \rightarrow A$, but also a proof for its termination. Therefore, all classical theorems of Z are also correct in the sense of (GI).

Let us summarize our arguments in Section 2.2 and 2.3. First, we have seen the following claim made by Hilbert: An existential proposition without witnesses has no sense from the finitary standpoint. Next, we have argued that Brouwer made a very close claim to Hilbert's and raised an objection to the significance of consistency proofs by using his own claim as a premise of the objection. Finally, we have maintained that Gentzen, who followed

[18] For Gentzen, the statability of a reduction procedure for $\Gamma \rightarrow A$ gives a sense to $\Gamma \rightarrow A$ from his finitist standpoint. It is not easy to estimate the exact strength of Gentzen's finitist standpoint. As far as we know, his standpoint should be constructive in the following sense. First, all infinite totalities must be generated by some finitary rules ([Gen36, pp.524-525], [Gen69, p.162]). For example, the totality of all natural numbers is generated from 0 by the successor rule. Second, one must avoid the use of the principle of the excluded middle for non-decidable predicates ([Gen36, pp.527-528], [Gen69, pp.164-165]).

[19] [Gen74, p.103], [Gen36, p.539], [Gen69, p.177].

Hilbert's claim, responded to the Brouwer-style objection in the following way: He formulated the interpretation (GI) of arithmetical formulas and ascribed a sense to every theorem of first-order classical arithmetic by means of it.[20] The interpretation (GI), according to which all theorems of first-order classical arithmetic are correct, had the role of responding to the Brouwer-style objection.

2.4 Formulation of Reduction Procedures with Spreads

In this section, we give our definition of reduction procedures by means of spreads, which are infinite trees in intuitionistic mathematics.

First, we introduce the proof system $\mathcal{Z}$ of first-order classical arithmetic. The system $\mathcal{Z}$ is the same as the proof system of the 1935 proof, except for the following two minor points. First, the connectives $\exists, \vee$ and $\supset$ are excluded from the language of $\mathcal{Z}$.[21] Second, a sequent of $\mathcal{Z}$ includes not finite sequences of formulas but finite sets of them, so the structural rules are omitted.

We assume a language $\mathcal{L}$ of first-order arithmetic with the following vocabulary: the constant 0 (*zero*), the unary function symbol S (*successor*), some predicate symbols for primitive recursive relations and the logical connectives $\wedge, \neg, \forall$. *Terms* and *formulas* are defined in a usual way. *Atomic formulas* are formulas of the form $p(t_1, \ldots, t_n)$, where p is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms. We use the following syntactic variables possibly with suffixes: k, m, n for numerals, A, B, C, D for formulas and Γ, Δ for finite sets of formulas. We denote the union $\Gamma \cup \Delta$ of Γ and Δ by Γ, Δ and the union $\{A\} \cup \Gamma$ by A, Γ or Γ, A. *Sequents* are expressions of the form $\Gamma \to A$. We call a formula in Γ an *antecedent formula* and A the *succedent*

[20] One might wonder why Gentzen did not respond to the objection by means of Gödel-Gentzen's double-negation translation. He could simply argue that theorems of first-order classical arithmetic have senses also for intuitionists, because the translation enables them to interpret these theorems as theorems of first-order Heyting arithmetic. The reason he did not argue in this manner is that intuitionist implication and negation were not admissible for him. He thought that there is circularity in intuitionist implication, which can be found also in intuitionist negation because the latter is a special case of the former. We discuss this circularity in Chapter 3. As to Gentzen's argument for "circularity," see [Gen36, §11]. A reconstruction of Gentzen's argument is given in [Oka88, pp.200-201] and [Oka08, pp.3-4].

[21] Gentzen did not exclude $\exists, \vee$ and $\supset$ from the language of the proof system of the 1935 proof, but he translated $\exists x A(x)$, $A \vee B$ and $A \supset B$ as $\neg\forall x \neg A(x)$, $\neg(\neg A \wedge \neg B)$ and $\neg(A \wedge \neg B)$, respectively. See [Gen74, §12].

formula. Sequents are denoted by S possibly with suffixes. If θ is a term, a formula, a finite set of formulas or a sequent, then $FV(\theta)$ denotes the set of all free variables in θ and we say θ is *closed* whenever $FV(\theta) = \emptyset$.

For readers' convenience, we define some notions and notations that are not found in Gentzen's original presentation. The set $\mathcal{TRUE}$ (resp. $\mathcal{FALSE}$) consists of all closed atomic formulas that are true (resp. false) in the usual sense. We use $\perp$ as a variable for formulas in $\mathcal{FALSE}$. When $FV(S) \subseteq \{x_0, \ldots, x_k\}$, the substitution instance $S[x_0 := n_0, \ldots, x_k := n_k]$ is the sequent obtained by substituting n_i for x_i in S for $i = 0, \ldots, k$ and we abbreviate it as $S[\vec{x} := \vec{n}]$. The set $\mathcal{CSEQ}$ is the set of all closed sequents.

The axioms and inference rules of $\mathcal{Z}$ are as follows:

Logical axioms:

$$\Gamma, A \to A.$$

Non-Logical axioms: Let us assume a primitive recursive set $\mathcal{AX}$ of some arithmetical axioms. Here, we do not need to specify this set.[22] We require that $\mathcal{AX}$ includes the defining axioms for each predicate symbol p and that every sequent in $\mathcal{AX}$ may have an arbitrary set of formulas as auxiliary antecedent formulas. For example, the following sequents may be included in $\mathcal{AX}$:

$$\Gamma \to t = t, \quad \Gamma, S(s) = S(t) \to s = t.$$

Logical rules:

$$\frac{\Gamma \to A_0 \quad \Gamma \to A_1}{\Gamma \to A_0 \wedge A_1}\ (\wedge\mathrm{I}) \qquad \frac{\Gamma \to A_0 \wedge A_1}{\Gamma \to A_i}\ (\wedge\mathrm{E}) \text{ with } i \in \{0, 1\}$$

$$\frac{\Gamma \to A(y)}{\Gamma \to \forall x A(x)}\ (\forall\mathrm{I}) \text{ with } y \notin FV(\Gamma) \qquad \frac{\Gamma \to \forall x A(x)}{\Gamma \to A(n)}\ (\forall\mathrm{E})$$

$$\frac{\Gamma \to \neg\neg A}{\Gamma \to A}\ (\mathrm{DNE}) \qquad \frac{A, \Gamma \to B \quad A, \Gamma \to \neg B}{\Gamma \to \neg A}\ (\mathrm{RED})$$

Mathematical Induction:

$$\frac{\Gamma \to A(0) \quad \Gamma, A(y) \to A(S(y))}{\Gamma \to A(n)}\ (\mathrm{IND}) \text{ with } y \notin FV(\Gamma)$$

[22] Gentzen also did not specify which arithmetical axioms are included in the proof system of the 1935 proof. See [Gen36, §6.2].

We can show in a standard way that the left weakening rule is admissible in $\mathcal{Z}$.

Next, we define *reduction steps* (Reduktionsschritte) of the 1935 proof in our manner.[23] Our definition is an application of the method for representing infinitary derivations as functions, which is found in [Min78] and [Buc91]. We use the notation of [Buc91] with some modifications.[24] This method enables us to define reduction procedures by means of spreads. Preliminary to our definition of reduction steps, the set $\mathcal{STEP}$ of the symbols for reduction steps is defined as follows:

$$\begin{aligned}
\mathcal{STEP} \;:=\; & \{\mathrm{Ax}\} \\
\cup\; & \{(\forall, A) | A \text{ is of the form } \forall x B(x)\} \\
\cup\; & \{(\forall, k, A) | A \text{ is of the form } \forall x B(x), k \in \mathbb{N}\} \\
\cup\; & \{(\wedge, A) | A \text{ is of the form } B_0 \wedge B_1\} \\
\cup\; & \{(\wedge, i, A) | A \text{ is of the form } B_0 \wedge B_1, i \in \{0,1\}\} \\
\cup\; & \{(\neg, i, A) | A \text{ is of the form } \neg B, i \in \{r, l\}\}
\end{aligned}$$

We use R as a variable for elements of $\mathcal{STEP}$.

Definition 2.4.1 (Reduction Steps)**.** For every $\langle R, \Gamma \to A\rangle \in \mathcal{STEP} \times \mathcal{CSEQ}$ and every infinite sequence $\langle \Delta_n \to B_n\rangle_{n\in\mathbb{N}}$ of closed sequents, $\mathcal{RED}(\langle R, \Gamma \to A\rangle, \langle \Delta_n \to B_n\rangle_{n\in\mathbb{N}})$ holds if and only if all of the following statements hold:

(i) If $R = \mathrm{Ax}$,

then either $A \in \mathcal{TRUE}$ holds or both $A \in \mathcal{FALSE}$ and $\Gamma \cap \mathcal{FALSE} \neq \emptyset$ hold,

(ii) if $R = (\forall, \forall x C(x))$,

then $A = \forall x C(x)$ holds and for every $n \in \mathbb{N}$, $\Delta_n = \Gamma$ and $B_n = C(n)$ hold,

(iii) if $R = (\forall, k, \forall x C(x))$,

then $\forall x C(x) \in \Gamma$, $\Delta_0 = \Gamma \cup \{C(k)\}$ and $B_0 = A \in \mathcal{FALSE}$ hold,

(iv) if $R = (\wedge, C_0 \wedge C_1)$,

then $A = C_0 \wedge C_1$ holds and for every $i \in \{0,1\}$, $\Delta_i = \Gamma$ and $B_i = C_i$ hold,

[23] For Gentzen's definition of reduction steps, see [Gen74, pp.100-102], [Gen36, pp.536-537], [Gen69, pp.173-175].

[24] [Sun83] also gave a reconstruction of reduction procedures in the 1935 proof by using the method of [Min78].

(v) if $R = (\wedge, i, C_0 \wedge C_1)$,
then $C_0 \wedge C_1 \in \Gamma$, $\Delta_0 = \Gamma \cup \{C_i\}$ and $B_0 = A \in \mathcal{FALSE}$ hold,

(vi) if $R = (\neg, r, \neg C)$,
then $A = \neg C$, $\Delta_0 = \Gamma \cup \{C\}$ and $B_0 \in \mathcal{FALSE}$ hold,

(vii) if $R = (\neg, l, \neg C)$,
then $\neg C \in \Gamma$, $A \in \mathcal{FALSE}$, $\Delta_0 = \Gamma$ and $B_0 = C$ hold.

Example 2.4.1. It holds that $\mathcal{RED}(\langle(\forall, \forall x A(x)), \Gamma \to \forall x A(x)\rangle, \langle\Gamma \to A(n)\rangle_{n\in\mathbb{N}})$. This corresponds to the following reduction step:[25]

$$\Gamma \to \forall x A(x) \ \rhd \ \Gamma \to A(n)$$

for an arbitrarily chosen numeral n.

Let us turn to the definition of spreads. The set $\mathbb{N}^{<\omega}$ is the primitive recursive set of (the codes of) all finite sequences of natural numbers. The elements of $\mathbb{N}^{<\omega}$ are denoted by $\vec{u}$, $\vec{v}$ and $\vec{w}$. We also denote the code of the empty sequence by $\langle\rangle$, infinite sequences of natural number by α and the concatenation function for $\vec{u}$ and $\vec{v}$ by $\vec{u} * \vec{v}$. The ordering $\leq$ on $\mathbb{N}^{<\omega}$ is defined by

$\vec{u} \leq \vec{v}$ if and only if $\vec{u} * \vec{w} = \vec{v}$ holds for some $\vec{w}$.

$\vec{u} < \vec{v}$ if and only if $\vec{u} \leq \vec{v}$ and $\vec{u} \neq \vec{v}$ hold.

For an arbitrary α, the *initial segment* $\bar{\alpha}(n)$ of length n is defined by

$$\bar{\alpha}(n) := \langle\alpha(0), \ldots, \alpha(n-1)\rangle.$$

Hereafter, we denote effectively calculable total functions from $\mathbb{N}^{<\omega}$ to $\{0, 1\}$ by s. We owe the following definition of spreads to [Dum00, pp.47-48].

Definition 2.4.2 (Spreads)**.** We define *speads*, *correlation laws* for spreads and *dressed spreads* as follows.

(i) s is a *spread* if and only if all of the following statements hold:

- $s(\langle\rangle) = 0$,
- for every $\vec{u}$,
  if $s(\vec{u}) = 0$, then $s(\vec{u} * \langle k\rangle) = 0$ for some $k \in \mathbb{N}$,

[25] Cf. [Gen74, §13.21].

- for every $\vec{u}$ and $\vec{v}$,
  if $\vec{u} \leq \vec{v}$ and $s(\vec{v}) = 0$, then $s(\vec{u}) = 0$.

(ii) Let X be a decidable set and c be an effectively calculable partial function from $\mathbb{N}^{<\omega}$ to X, then c is a *correlation law* for s with respect to X if and only if for every $\vec{u}$,

$$\text{if } s(\vec{u}) = 0, \text{ then } c(\vec{u}) \text{ is defined and } c(\vec{u}) \in A.$$

(iii) We call the pair $\langle s, c\rangle$ of a spread s and a correlation law c for s with respect to some X a *dressed spread*.

Let s^* be the function from $\mathbb{N}^{<\omega}$ to $\{0, 1\}$ defined by $s^*(\vec{u}) := 0$ for all $\vec{u}$. It is obvious that s^* is a spread. We call it the *universal spread*. We use φ, ψ as variables for effectively calculable total functions from $\mathbb{N}^{<\omega}$ to $\mathcal{STEP} \times \mathcal{CSEQ}$. Since each φ is a correlation law for the universal spread s^* with respect to $\mathcal{STEP} \times \mathcal{CSEQ}$, the pair $\langle s^*, \varphi\rangle$ is a dressed spread.

Definition 2.4.3. For each φ, we define the following notions:

(i) If $\varphi(\vec{u}) = \langle R, \Gamma \to A\rangle$, we set $\varphi^0(\vec{u}) := R$, $\varphi^1(\vec{u}) := \Gamma \to A$,
$\mathit{Rule}(\varphi) := \varphi^0(\langle\,\rangle)$ and $\mathit{End}(\varphi) := \varphi^1(\langle\,\rangle)$.

(ii) φ is *monotone* if and only if for every $\vec{u}$ and n,
if $\varphi^0(\vec{u}) = \mathrm{Ax}$, then $\varphi(\vec{u} * \langle n\rangle) = \varphi(\vec{u})$ holds.

(iii) φ is *well-founded* if and only if
for every α there exists n such that $\varphi^0(\bar{\alpha}(n)) = \mathrm{Ax}$ holds.

(iv) φ is *locally correct* if and only if φ is monotone and for every $\vec{u}$,
$\mathcal{RED}(\varphi(\vec{u}), \langle\varphi^1(\vec{u} * \langle n\rangle)\rangle_{n\in\mathbb{N}})$ holds.

Example 2.4.2. There is a function φ such that φ is locally correct but not well-founded. A typical example of such a function φ can be represented as the following tree, where $A(x)$ is of the form $x = x$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\langle(\forall, 3, \forall x A(x)),\ A(0), A(1), A(2), \forall x A(x) \to 0 = 1\rangle}}{\langle(\forall, 2, \forall x A(x)),\ A(0), A(1), \forall x A(x) \to 0 = 1\rangle}}{\langle(\forall, 1, \forall x A(x)),\ A(0), \forall x A(x) \to 0 = 1\rangle}}{\langle(\forall, 0, \forall x A(x)),\ \forall x A(x) \to 0 = 1\rangle}$$

Now, we define reduction procedures as a kind of dressed spreads.

Definition 2.4.4 (Reduction Procedures)**.** For every φ and every $\Gamma \to A \in \mathcal{CSEQ}$, $\langle s^*, \varphi\rangle$ is a *reduction procedure* for $\Gamma \to A$ if and only if $\varphi^0(\langle\rangle) = \Gamma \to A$, φ is well-founded and locally correct.

2.5 Proof of Main Lemma with Monotone Bar Induction

In this section, we prove the main lemma of the 1935 proof. This lemma is a key for Gentzen's response to the Brouwer-style objection, as seen in Section 2.3. The proof below is given from an intuitionist viewpoint: We use monotone bar induction, which is a induction principle on a well-founded tree in intuitionistic mathematics.[26] Then, we propose a way to understand Gentzen's response to the Brouwer-style objection from an intuitionist perspective.

The *rank* $rk(A)$ of a formula A is defined by

$rk(A) := 0$, if A is an atomic formula,

$rk(A \wedge B) := \max(rk(A), rk(B)) + 1$, $rk(\neg A) := rk(A) + 1$,

$rk(\forall x A(x)) := rk(A(x)) + 1$.

Hereafter, we abbreviate "there is a reduction procedure for $\Gamma \to A$" as "$\Gamma \to A$ is reducible." In addition, for simplicity's sake, we consider closed sequents only, unless indicated otherwise.

Lemma 2.5.1. *The following statements hold:*

1. $\Gamma \to A(n)$ *is reducible for every* $n \in \mathbb{N}$ *if and only if* $\Gamma \to \forall x A(x)$ *is reducible,*
2. $\Gamma \to A_0$ *and* $\Gamma \to A_1$ *are reducible if and only if* $\Gamma \to A_0 \wedge A_1$ *is reducible,*
3. *If* $\neg A, \Gamma \to A$ *is reducible, then* $\neg A, \Gamma \to \bot$ *is reducible,*
4. *If* $A, \Gamma \to \bot$ *is reducible, then* $\Gamma \to \neg A$ *is reducible,*
5. *If* $A(n), \forall x A(x), \Gamma \to B$ *is reducible, then* $\forall x A(x), \Gamma \to B$ *is reducible,*

[26] As we have said in the introduction to this chapter, we do not claim that Gentzen himself used monotone bar induction.

6. *If* $A_i, A_0 \wedge A_1, \Gamma \to B$ *is reducible* ($i \in \{0, 1\}$), *then* $A_0 \wedge A_1, \Gamma \to B$ *is reducible.*

Proof. It is obvious by the definition of reduction procedures that the statements 1, 2, 3 and 4 hold. The statements 5 and 6 are proved by induction on $rk(B)$. □

Lemma 2.5.2. $\Gamma, A \to A$ *is reducible for every formula* A.

Proof. By induction on $rk(A)$. Apply Lemma 2.5.1 in the case of the induction steps. □

Lemma 2.5.3. *The following statements hold:*

1. *If* $A, \Gamma \to B$ *is reducible, then* $\neg\neg A, \Gamma \to B$ *is reducible,*

2. $\Gamma, \neg\neg A \to A$ *is reducible.*

Proof. (1) By induction on $rk(B)$.

(2) The sequent $A \to A$ is reducible by Lemma 2.5.2, then apply (1). □

Lemma 2.5.4 (The Soundness of the Weakening Rule)**.** *If* $\Gamma \to A$ *is reducible, then* $\Delta, \Gamma \to A$ *is reducible.*

Lemma 2.5.5 (Main Lemma of the 1935 Proof)**.** *If* $\Gamma \to A$ *is derivable in* $\mathcal{Z}$*, then* $\Gamma \to A$ *is reducible.*

Proof. By induction on the length of the $\mathcal{Z}$-derivation d of $\Gamma \to A$. For simplicity's sake, we focus on the interesting cases.

(1) d is a logical axiom $\Gamma, B \to B$: Apply Lemma 2.5.2.

(2) The last rule of d is:

$$\frac{\Gamma \to A(y)}{\Gamma \to \forall x A(x)}\ (\forall\mathrm{I}).$$

Apply Lemma 2.5.1.(2).

(3) The last rule of d is:

$$\frac{\Gamma \to \neg\neg A}{\Gamma \to A}\ (\mathrm{DNE}).$$

By IH, the sequent $\Gamma \to \neg\neg A$ is reducible. On the other hand, by Lemma 2.5.3.(2), the sequent $\Gamma, \neg\neg A \to A$ is reducible. We assume that the following lemma is proved:

(*) For every Γ, A and B,

if $\Gamma \to A$ and $A, \Gamma \to B$ are reducible, then $\Gamma \to B$ is reducible.

Therefore, $\Gamma \to A$ is reducible.

(4) The last rule of d is:

$$\frac{\Gamma \to A(0) \quad \Gamma, A(y) \to A(S(y))}{\Gamma \to A(n)} \text{ (IND).}$$

Assume that $\Gamma \to A(0)$ and $\Gamma, A(y) \to A(S(y))$ are reducible (IH). Then, $\Gamma, A(m) \to A(S(m))$ is reducible for every m such that $m < n$. By (*), $\Gamma \to A(n)$ is reducible.

To finish the proof of Lemma 2.5.5, it suffices to show that (*) holds. Preliminary to a proof of (*), we formulate monotone bar induction, following [TD88, ch.4, §8].

Monotone Bar Induction. Let P and Q be predicates on $\mathbb{N}^{<\omega}$. If the following four conditions

1. for every infinite sequence α of natural numbers, there is a natual number n such that $P(\bar{\alpha}(n))$ holds,
2. for every $\vec{u}$ and $\vec{v}$, if $P(\vec{u})$ holds then $P(\vec{u} * \vec{v})$ holds,
3. for every $\vec{u}$, if $P(\vec{u})$ holds then $Q(\vec{u})$ holds,
4. for every $\vec{u}$, if $Q(\vec{u} * \langle n \rangle)$ holds for all n then $Q(\vec{u})$ holds and

hold, then $Q(\langle\rangle)$ holds.

Lemma 2.5.6 (The Soundness of the Cut Rule). *If* $\Gamma \to A$ *and* $A, \Gamma \to B$ *are reducible, then* $\Gamma \to B$ *is reducible.*

Proof. Assume that $\Gamma \to A$ and $A, \Gamma \to B$ are reducible. First, we use the induction principle on $rk(A)$. For simplicity's sake, we focus on the case that $A = \forall x C(x)$. Let ψ be a given reduction procedure for $A, \Gamma \to B$.

In the induction steps, we apply monotone bar induction (MBI), setting

$P(\vec{u})$ if and only if $\psi^0(\vec{u}) = \text{Ax}$,

$Q(\vec{u})$ if and only if for every closed Δ_0 and B_0,
if $\psi^1(\vec{u}) = \forall x C(x), \Delta_0 \to B_0$ holds, then $\Gamma, \Delta_0 \to B_0$ is reducible.

If we show that all premises of MBI hold, then we can conclude that $Q(\langle\rangle)$ holds. Since $\psi^1(\langle\rangle) = \forall x C(x), \Gamma \to B$ holds, it follows that $\Gamma \to B$ is reducible.

By the well-foundedness and local correctness of ψ, it is obvious that Premises 1 and 2 of MBI hold.

First, we show that Premise 3 of MBI holds. Assume that $\psi^0(\vec{u}) = \mathrm{Ax}$ and $\psi^1(\vec{u}) = \forall x C(x), \Delta_0 \to B_0$ hold. Then, $\psi(\vec{u}) = \langle \mathrm{Ax}, \forall x C(x), \Delta_0 \to B_0\rangle$ holds. By the local correctness of ψ, $B_0 \in \mathcal{TRUE}$ or $\Delta_0 \cap \mathcal{FALSE} \neq \emptyset$ holds. Define φ as

$$\varphi(\vec{u}) := \langle \mathrm{Ax}, \Gamma, \Delta_0 \to B_0\rangle \text{ for every } \vec{u},$$

then it is obvious that $\langle s^*, \varphi\rangle$ is a reduction procedure for $\Gamma, \Delta_0 \to B_0$, so $\Gamma, \Delta_0 \to B_0$ is reducible.

Next, we show that Premise 4 of MBI holds. Assume that for every n and every closed Δ_0, B_0,

> if $\psi^1(\vec{u} * \langle n\rangle) = \forall x C(x), \Delta_0 \to B_0$ holds, then $\Gamma, \Delta_0 \to B_0$ is reducible (IH of MBI).

To show that $Q(\vec{u})$ holds, assume that $\psi^1(\vec{u}) = \forall x C(x), \Delta_1 \to B_1$ holds for arbitrary closed Δ_1, B_1. We have to distinguish the cases according to the value of $\psi^0(\vec{u})$ and consider the most crucial case only.

Suppose that $\psi^0(\vec{u}) = (\forall, k, \forall x C(x))$ and $C(k) \in \Delta_1$. By the local correctness of ψ, $\psi^1(\vec{u} * \langle 0\rangle) = \forall x C(x), \Delta_1 \to B_1$ holds. Hence, by IH of MBI, the sequent $\Gamma, \Delta_1 \to B_1$ is reducible. Next, consider the case that $C(k) \notin \Delta_1$ holds. By the local correctness of ψ and IH of MBI, the sequent $C(k), \Gamma, \Delta_1 \to B_1$ is reducible. From the assumption that $\Gamma \to \forall x C(x)$ is reducible, it follows that $\Gamma \to C(k)$ is reducible, so $\Gamma, \Delta_1 \to C(k)$ is reducible by Lemma 2.5.4. Therefore, $\Gamma, \Delta_1 \to B_1$ is reducible by IH of the induction on $rk(A)$. □

As said in Section 2.3, the statability of a reduction procedure requires the availability of both a reduction procedure $\langle s^*, \varphi\rangle$ and a proof for the well-foundedness of φ: For every reduction procedure $\langle s^*, \varphi\rangle$ for a closed sequent S, $\langle s^*, \varphi\rangle$ is *statable* for S if and only if both $\langle s^*, \varphi\rangle$ and a proof for the well-foundedness of φ are obtained. For every sequent S with $FV(S) \neq \emptyset$, a reduction procedure is *statable* for S if and only if for every substitution instance S' of S, a reduction procedure is statable for S'.

Then, through the above proof of Lemma 2.5.5, we can show that every $\mathcal{Z}$-derivable sequent S is correct in the sense of (GI). Consider a substitution instance S' of S. Then, the above proof of Lemma 2.5.5 gave a reduction procedure $\langle s^*, \varphi\rangle$ for S' with a proof for the well-foundedness of φ. Therefore, a reduction procedure is statable for S and S is correct in the sense of (GI).

This means that the main lemma of the 1935 proof provides each theorem of first-order classical arithmetic with a sense being admissible to intuitionists. Remember that reduction procedures were defined as dressed spreads, which are infinite trees in intuitionistic mathematics. Moreover, in the proof of the main lemma, we avoided the principle of the excluded middle for non-decidable predicates and used monotone bar induction as an induction principle that was needed to complete the proof. We showed that every theorem of first-order arithmetic is correct, in a manner being admissible to intuitionists.

2.6 Conclusion of Chapter 2

The use of spreads and monotone bar induction in consistency proofs, as in Sections 2.4 and 2.5, results in the use of intuitionistic mathematics in consistency proofs. Brouwer had suggested such use of intuitionistic mathematics. In the paper "Intuitionistische Betrachtungen über den Formalismus" ([Bro28]), to which Gentzen referred in [Gen74, p.117] and [Gen36, p.563], Brouwer discussed the relationship between formalism, namely the Hilbert School, and intuitionism as follows:

> **First Insight.** The distinction in the Formalist practice between the construction of "a stock of mathematical formulae" (the Formalist description of mathematics) and the intuitive (contentual) theory of laws of this construction, as well as *the recognition that for the latter theory the intuitionist mathematics of the set of natural numbers is indispensable.* ([Bro28, p.375], italics added)[27]

Here, Brouwer claimed that "the intuitionist mathematics of the set (Menge) of natural numbers," in the original German text "die intuitionistische Mathematik der Menge der natürlichen Zahlen," is indispensable for Hilbert's finitary standpoint. Note that Brouwer called spreads *Mengen* in his German writings.[28] Moreover, monotone bar induction is an induction principle on a spread being well-founded, so "die intuitionistische Mathematik der Menge der natürlichen Zahlen" includes this induction principle. Thus, Brouwer actually stated that the branch of intuitionistic mathematics, in which spreads and bar induction are used, is indispensable for Hilbert's finitary standpoint.

In the present chapter, we have argued first that Gentzen's interpretation (GI) of arithmetical formulas took the role of responding to the Brouwer-style objection, which opposes the significance of consistency proofs. Specifically,

[27] This English translation is from [Man98, p.41].
[28] For example, see [Bro25, pp.244-245].

to respond to the objection, Gentzen ascribed a sense to each theorem of first-order classical arithmetic from his finitist standpoint: Gentzen proved that such a theorem is still correct in the sense of (GI). Second, we have proposed a way to understand Gentzen's response from an intuitionist viewpoint. On the basis of our definition for reduction procedures by means of spreads, we have proved the key lemma to Gentzen's response by using monotone bar induction. The present chapter's entire argument showed the following: The role of responding to the Brouwer-style objection was given to Gentzen's interpretation of arithmetical formulas by the dialogue between Gentzen and intuitionists, and the response is admissible to an intuitionist viewpoint.

Chapter 3

Gentzen's Interpretation for Arithmetic and Circularity of Implication

3.1 Introduction to Chapter 3

As we saw in the previous chapter, in the early period of the foundations of mathematics, the Hilbert School influenced the Brouwer School and vice versa. However, this of course does not mean that the two schools held the same opinions about the foundations of mathematics.[1] For example, intuitionism admitted that if propositions A, B and C are meaningful from the intuitionist viewpoint, then a nested implication $(A \supset B) \supset C$ is so as well. On the other hand, the finitary standpoint did not admit that a nested implication $(A \supset B) \supset C$ is meaningful, even if A, B and C are meaningful from the finitary standpoint. Hilbert and Bernays wrote,

[1]According to Bernays, it was in the 1930s that the Hilbert School realized some differences between itself and the Brouwer School. Bernays wrote,

> Arend Heyting, in two papers of 1930, set up a formal system of intuitionistic number theory. And, as Gödel and Gerhard Gentzen independently observed, there is a relatively simple method of showing that any contradiction derivable in the formal system of classical number theory would entail a contradiction in Heyting's system. [...]
>
> In this way it appeared that intuitionistic reasoning is not identical with finitist reasoning, *contrary to the prevailing views at that time.* ([Ber67, p.502], italics added)

Note that Gödel's result mentioned in this quotation was published in 1933 with the title "Zur intuitionistischen Arithmetik und Zahlentheorie."

> The methodological standpoint of "intuitionism" underlying Brouwer's approach constitutes a certain *extension of the finitistic attitude* (Erweiterung der finiten Einstellung) insofar as assumptions on the existence of derivations or proofs may be introduced even if their intuitive nature (anschauliche Beschaffenheit) is not determined. For instance, from Brouwer's standpoint, sentences of the following form are allowed: "If the sentence $\mathfrak{B}$ holds under the assumption $\mathfrak{A}$, then $\mathfrak{C}$ holds as well." And also: "The assumption that $\mathfrak{A}$ is refutable leads to a contradiction." Or in Brouwer's way of speaking: "The absurdity of $\mathfrak{A}$ is absurd." ([HB1934, p.43], italics original)[2]

Here, Hilbert and Bernays claimed that intuitionism is an *extension* of the finitary standpoint in the respect that the former admits the propositions of the form $(A \supset B) \supset C$. Though Hilbert and Bernays might consider intuitionism to be still finitist in an extended sense and the finitary standpoint in this sense could admit nested implications to be meaningful, they said that nested implications are not meaningful from the finitary standpoint in the *original* sense.

With this background, Gentzen aimed to give an interpretation for all first-order arithmetical formulas including nested implications, by extending the finitary standpoint in his way. As a problem in achieving this aim, Gentzen pointed out a circularity of implication before the main parts of his papers for the 1935 and 1936 consistency proofs. He said that it was one of the main objectives of his 1935 and 1936 proofs to formulate an interpretation that avoids this circularity.[3]

In spite of his words, Gentzen did not present his interpretation of arithmetical formulas explicitly and did not give an argument for its non-circularity at all. In [Oka88], Okada proposed a method to avoid the circularity urged by Gentzen, which is independent of Gentzen's interpretation of arithmetical formulas. Recently, this interpretation was made explicit by Akiyoshi-Takahashi ([AT13]) and Tait ([Tai15]). The interpretation assigns a sense to each implication formula $A \supset B$ via the translation into the formula $\neg(A \wedge \neg B)$, so it is still desirable to give a direct Gentzen-style interpretation for implication. In [Taka15], such a Gentzen-style interpretation was proposed, but it is not known whether this interpretation avoids circularity.

This chapter aims to show that the Gentzen-style interpretation of first-order arithmetical formulas in [Taka15] avoids the circularity of implication urged by Gentzen himself. In Section 3.2, we recall the circularity in de-

[2] We owe this English translation to [HB2011, p.43].

[3] [Gen36, p.530], [Gen69, p.168].

tail. In Section 3.3, we reformulate this interpretation by means of Tait's method ([Tai15]) of defining reduction procedures. Finally, in Section 3.4, we argue that this interpretation avoids the circularity of implication urged by Gentzen.

3.2 Circular Reasoning Concerning Implication

In this section, we first explain our opinion about why Gentzen, who aimed to prove the consistency of a formal system of first-order arithmetic, was concerned with an interpretation of arithmetical formulas. Next, we recall the circularity of implication urged by Gentzen.

Gentzen wrote that it was of no significance for his attempt at proving the consistency of first-order arithmetic that first-order arithmetic is sound with respect to the standard interpretation of arithmetical formulas.[4] Gentzen also wrote,

> Having rejected the actualist interpretation of transfinite propositions, we are still left with the possibility of *ascribing a 'finitist' sense* (finiter Sinn) *to such propositions, i.e., of interpreting* them in each case as expressions for definite *finitely* (endlich) characterizable states of affairs.
>
> Once this view has been adopted, the relevant *logical forms of inference* must be examined for their compatibility with this interpretation of the *propositions.* ([Gen36, p.525], [Gen69, pp.162-163], italics original)

Here, as we said in Section 1.2, "the actualist interpretation" means the interpretation of quantified arithmetical propositions that treats the infinite sequence of all natural numbers as closed, i.e., finished. From this quotation, we see that Gentzen adopted the following strategy for proving the consistency of first-order arithmetic. He first aimed to give a "finitist" interpretation for arithmetical formulas, then tried to show that first-order arithmetic is sound with respect to this interpretation.

In addition, Gentzen said,

> Such a [consistency] proof would be of little value, however, since it itself would have to *make use* of transfinite propositions and the same associated forms of inference which it is intended to

[4] [Gen36, pp.524-525], [Gen69, p.162].

> '*justify*'. Such a proof would therefore not represent *an appeal to more elementary facts* (Zurückführung), although it would of course *confirm* the finitist character (finiter Charakter) of the formalized rules of inference. ([Gen36, p.529], [Gen69, p.167], italics original)

Gentzen said in this quotation that the significance of consistency proofs in terms of the above strategy consists, not in a *justification* for the inference forms of first-order arithmetic, but in a *confirmation* of their "finitist character." According to our reading, such a confirmation is nothing but to show that the inference rules of first-order arithmetic are understandable from Gentzen's finitist standpoint. He tried to give such a confirmation by explaining that the inference rules in fact preserve correctness from this standpoint.[5] That is why Gentzen was concerned with the interpretation of implication: He aimed to give an interpretation to implication formulas of first-order arithmetic, to show that the inference rules of first-order arithmetic are understandable also from his finitist standpoint.[6]

Next, we proceed to explain the circularity of implication urged by Gentzen.[7] First of all, let us quote a passage from [Gen36]. In the following passage, Gentzen pointed out the circularity and said that one of the main objectives of his consistency proof was to give an interpretation of implication avoiding this circularity.

> This proposition [$\mathfrak{A} \supset \mathfrak{B}$] is merely intended to express the fact that *a proof is available* which permits a proof of the proposition $\mathfrak{B}$ from the proposition $\mathfrak{A}$, once the proposition $\mathfrak{A}$ is proved. [...]
>
> In interpreting $\mathfrak{A} \supset \mathfrak{B}$ in this way, I have presupposed that the available proof of $\mathfrak{B}$ from the assumption $\mathfrak{A}$ contains merely inferences *already recognized as permissible.* On the other hand, such a proof may itself contain other $\supset$-*inferences* and then our interpretation *breaks down.* For, it is *circular* to justify the $\supset$-inferences on the basis of a $\supset$-interpretation which itself already involves the presupposition of the admissibility of *the same* form of inference. The $\supset$-inferences *which occur in the proof* would in that case have to be justified *beforehand*; but this has its difficulties, [$\cdots$].

[5] This reading is adopted by [AT13] as well.

[6] Of course, this does not mean that Gentzen gave no other significance than the above one to consistency proofs.

[7] We owe the following explanation to [Oka88, p.201] and [Oka08, pp.3-4].

> In order to cope with this difficulty, we would really have to formulate a more complicated rule of interpretation. This represents one of the *principal objectives* of the *consistency proof* which follows in section IV. ([Gen36, p.530], [Gen69, pp.167-168], italics original)

Here, Gentzen proposed the following interpretation as a candidate for a finitist interpretation of implication:

(I) $A \supset B$ is correct
    if and only if
    there is a proof d that allows to prove B from any proof of A.

The last quotation appeared in the section titled "The connectives $\supset$ and $\neg$ in transfinite propositions: the intuitionist view (Die Verknüpfungszeichen $\supset$ und $\neg$ in transfiniten Aussagen; die intuitionistische Grenzziehung)." This suggests that Gentzen considered (I) an intuitionist interpretation.

Gentzen claimed that the argument in the quotation above, which was for the soundness of the implication elimination rule on (I), included circular reasoning. This is what we call the circularity of implication urged by Gentzen.

The argument proceeds as follows. To show that an arbitrary instance (†)

$$\frac{A \quad A \supset B}{B}$$

of the implication elimination inference is sound with respect to (I), assume that A and $A \supset B$ are both correct. According to (I), there is a proof d

$$\begin{array}{c} A \\ \vdots\, d \\ B \end{array}$$

that allows us to prove B from any proof of A. Then, the correctness of A is transmitted to B via the intermediate steps in d. Thus, B is correct.

According to Gentzen, this argument includes circular reasoning because it has the following presupposition:

(*) all inference rules in d are sound.

In the argument above, it was claimed that the correctness of A is transmitted to B via the intermediate steps in d. The presupposition (*) is concealed here, and this means that the soundness of the instance (†) is in fact assumed, because (†) itself can occur in d. Thus, the argument above includes trivial circular reasoning of the following form:

(**) A holds. Therefore A holds.

Note that Gentzen did not claim the argument above includes a logical fallacy. The reasoning (**) is logically valid. Gentzen claimed that the argument above includes trivial circular reasoning of the form (**). This is the circularity of implication urged by Gentzen.

As we have seen, Gentzen said that the significance of consistency proofs consists in not a justification of the inference rules of first-order arithmetic, but a confirmation of their finitist character, that is, an explanation that they are understandable from Gentzen's finitist standpoint. The trivial circular reasoning above makes even this confirmation impossible. For we need to give a non-trivial finitist argument for the soundness of the implication elimination rule to *explain* their finitist character. That is why Gentzen did not adopt (I) as a finitist interpretation of implication.

After pointing out the circularity of implication, Gentzen wrote that one of the main objectives of his 1935 and 1936 proofs was to give a finitist interpretation of implication avoiding this circularity. Let us quote the relevant passage again.

> In order to cope with this difficulty, we would really have to formulate a more complicated rule of interpretation. This represents one of the *principal objectives* of the *consistency proof* which follows in section IV. ([Gen36, p.530], [Gen69, p.168], italics original)

However, Gentzen did not present explicitly the interpretation of arithmetical formulas given by his consistency proofs. In [AT13, Tai15] and the previous chapter, this interpretation was made explicit. Consider an arbitrary formula A in a first-order arithmetical language, whose logical constants are only $\wedge, \forall$ and $\neg$. The interpretation, which we call (F) here, is as follows:

(F) A is correct

if and only if

a reduction procedure is statable for A.

Remember that we defined a reduction procedure for A as one for $\rightarrow A$. A rationale for ascribing (F) to Gentzen is given in the following passage, as we explained in the previous chapter.

> The concept of the '*statability of a reduction procedure*' (die Angebbarkeit einer Reduziervorschrift) for a sequent, to be defined below, will serve as the formal replacement (formaler Ersatz) of the

> contentual *concept of correctness* (der inhaltliche Richtigkeitsbegriff); it provides us with a *special finitist interpretation* (finite Deutung) of propositions and takes the place of their *actualist interpretation* [...]. ([Gen74, p.100], [Gen36, p.536], [Gen69, p.173], italics original)

The interpretation above gives a sense to each implication formula $A \supset B$ indirectly, that is, it gives a sense to the formula $\neg(A \wedge \neg B)$ and $A \supset B$ is translated to this formula. In [Taka15], a direct Gentzen-style interpretation of implication was proposed.[8] Consider arbitrary formulas A and B in a first-order arithmetical language, whose logical constants are only $\wedge, \forall$ and $\supset$. The interpretation is as follows:

(IM) $A \supset B$ is correct

if and only if

a reduction procedure is statable for the sequent $A \to B$.

In what follows, we redefine both reduction procedures and the statability of a reduction procedure in terms of Tait's method, in which the notion of pre-reduction procedures is used (Section 3.3). Then, we argue that this chapter's version of (IM) avoids the circularity of implication urged by Gentzen (Section 3.4).

3.3 Gentzen-style Interpretation for Implication

Let $\mathcal{L}^{\supset}$ be the first-order language obtained by replacing $\neg$ with $\supset$ in the language $\mathcal{L}$, which was defined in Section 2.4. In addition, consider the formal system $\mathcal{Z}^{\supset}$ of first-order arithmetic obtained by dropping rules (DNE) and (RED) from system $\mathcal{Z}$ (cf. Section 2.4) and adding the following rules.

$$\frac{\Gamma, A \to B}{\Gamma \to A \supset B}\ (\supset I) \qquad \frac{\Gamma \to A \quad \Gamma \to A \supset B}{\Gamma \to B}\ (\supset E)$$

Definition 3.3.1 (Reduction steps)**.** *Reduction steps* are the following rules that rewrite a closed sequent to another one:

($\forall$) $\Gamma \to \forall x A(x) \ \rhd\ \Gamma \to A(n)$,

($\wedge$) $\Gamma \to A_0 \wedge A_1 \ \rhd\ \Gamma \to A_i$ ($i = 0, 1$),

[8] [Taka15, p.53].

$(\supset, r)$ $\Gamma \to A \supset B \ \triangleright \ A, \Gamma \to B$,

$(\forall, k)$ $\forall x A(x), \Gamma \to \perp \ \triangleright A(k), \forall x A(x), \Gamma \to \perp$,

$(\wedge, i)$ $A_0 \wedge A_1, \Gamma \to \perp \ \triangleright A_i, A_0 \wedge A_1, \Gamma \to \perp$,

$(\supset, l)$ $A \supset B, \Gamma \to \perp \ \triangleright A \supset B, \Gamma \to A$ or $B, A \supset B, \Gamma \to \perp$.

Note that rules $(\forall, k)$, $(\wedge, i)$ and $(\supset, l)$ are applied to a sequent $\Gamma \to A$ only if A is a false sentence and that a choice is included in rules $(\forall)$, $(\wedge)$ and $(\supset, l)$. For example, a choice from numerals is included in $(\forall)$.[9]

A closed sequent $\Gamma \to A$ is of the *end-form* if and only if either $\Gamma \cap \mathcal{FALSE} \neq \emptyset$ or $A \in \mathcal{TRUE}$. Below, we define reduction procedures using pre-reduction procedures, which were introduced by Tait.[10]

Definition 3.3.2 (Pre-reduction Procedures)**.** Let $\Gamma \to A$ be a closed sequent. A *pre-reduction procedure* R for $\Gamma \to A$ is an effective procedure that determines, in the following manner, whether a given finite sequence of sequents is an R-admissible sequence.

1. The sequence $\langle \Gamma \to A \rangle$ is the only R-admissible sequence of length 1.

2. A sequence of closed sequents of length $n+2$ is an R-admissible sequence of length $n+2$ if and only if it is a one-element extension of some R-admissible sequence of length $n+1$. A one-element extension of an R-admissible sequence $\langle \Gamma \to A, \ \dots, \ \Delta \to B \rangle$ of length $n+1$ is defined as follows:

   (a) if $\Delta \to B$ is of the end-form, then there is no one-element extension of $\langle \Gamma \to A, \ \dots, \ \Delta \to B \rangle$,

   (b) if all members of Δ belong to $\mathcal{TRUE}$ and $B \in \mathcal{FALSE}$, then

   $$\langle \Gamma \to A, \ \dots, \ \Delta \to B, \ \Delta \to B \rangle$$

   is the only one-element extension of $\langle \Gamma \to A, \ \dots, \ \Delta \to B \rangle$,

   (c) otherwise, at least one reduction step is applicable to $\Delta \to B$. Then, R determines such a reduction step S and

   $$\langle \Gamma \to A, \ \dots, \ \Delta \to B, \ \Delta_0 \to B_0 \rangle$$

[9] Gentzen did not stipulate the reduction steps for constant $\supset$. For the original definition of the reduction steps given by Gentzen, see [Gen74, pp.101-102], [Gen69, pp.173-175].

[10] [Tai15, pp.217-218]. Tait called this notion pre-reduction *rules*. In case 2.(c) of the following definition, our definition of pre-reduction procedures differs from Tait's, but this difference is not essential.

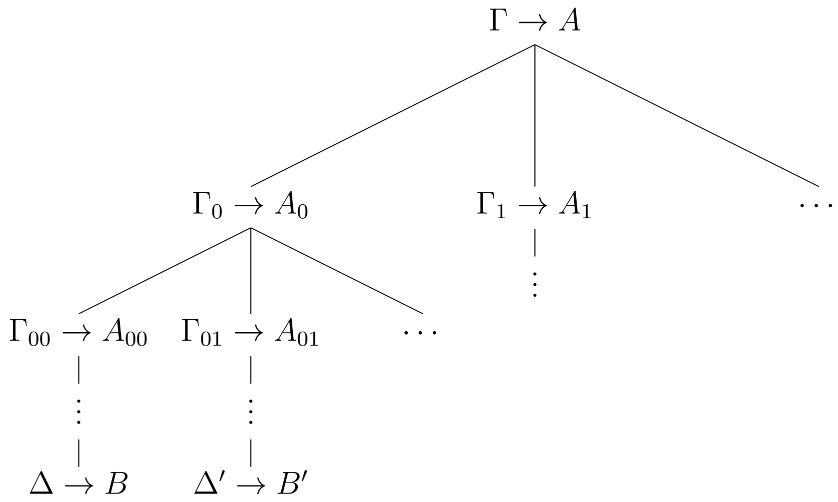

Figure 3.1: (Here, branching corresponds to choice, and all sequents at the bottom are of the end-form.)

is a one-element extension of $\langle \Gamma \to A, \ \ldots, \ \Delta \to B \rangle$ for every $\Delta_0 \to B_0$ that is a result of applying S to $\Delta \to B$.

There is no essential difference between a pre-reduction procedure for $\Gamma \to A$ and one for $\Gamma \to B$ whenever both A and B belong to $\mathcal{FALSE}$, so we stipulate that a pre-reduction procedure for $\Gamma \to A$ is also one for $\Gamma \to B$.

An *initial segment* of a finite or infinite sequence α of closed sequents is a finite sequence

$$\langle \Gamma_1 \to A_1, \ \ldots \ , \Gamma_n \to A_n \rangle$$

such that for every k $(1 \leq k \leq n)$, $\Gamma_k \to A_k$ is equal to the k-th element of α. An initial segment of α is proper if and only if the segment is not equal to α.

Definition 3.3.3 (Reduction Procedures)**.** Let $\Gamma \to A$ be a closed sequent. A pre-reduction procedure R for $\Gamma \to A$ is a reduction procedure for $\Gamma \to A$ if and only if for every infinite sequence α of closed sequents, α includes an initial segment that is not R-admissible.

Informally, reduction procedures are explained as follows. A reduction procedure R for a closed sequent $\Gamma \to A$ is an effective procedure for applying reduction steps to $\Gamma \to A$ repeatedly such that the procedure eventually rewrites $\Gamma \to A$ to a sequent of the end-form, regardless of which choices are made in the applications of steps $(\forall)$, $(\wedge)$ and $(\supset, l)$ (Figure 3.1). In the next section, for the sake of clarity, we often show that there is a reduction procedure for some sequent, by giving such an effective procedure for rewriting it.

Let us reformulate the interpretations (F) and (IM). We define the statability of a reduction procedure as follows: Let R be an arbitrary reduction procedure for a closed sequent S, then R is *statable* for S if and only if both R and a proof for the fact that R is a reduction procedure for S are obtained. For every sequent S such that $FV(S) \neq \emptyset$, we stipulate that a reduction procedure is *statable* for S if and only if for every substitution instance S' of S, a reduction procedure is statable for S'.

With the definitions above, we can reformulate the interpretations (F) and (IM). Let us state them again:

(F) A is correct

if and only if

a reduction procedure is statable for A.

(IM) $A \supset B$ is correct

if and only if

a reduction procedure is statable for the sequent $A \to B$.

Note that, in our setting, (IM) follows from (GI), because the following holds:

A reduction procedure is statable for $\to A \supset B$

if and only if

a reduction procedure is statable for $A \to B$.

3.4 Way Out of Circularity

In this section, we argue that (IM) avoids the circularity of implication urged by Gentzen; that is, the soundness of the implication elimination inference on (IM) can be shown without trivial circular reasoning such as deducing A from A. Consider instance

$$\frac{A \quad A \supset B}{B}$$

of the inference. We show that the instance above preserves the correctness of the premises, without presupposing the soundness of the instance itself.[11]

[11] Of course, this does not mean that we do not use any instance of the implication elimination inference in our argument: We have to use the inference in our meta-theory.

Assume that A and $A \supset B$ are correct; then a reduction procedure is statable for $\to A$ and $A \to B$ by (F) and (IM), respectively. Thus, there are reduction procedures for $\to A$ and $A \to B$. Then, it suffices to explain that the following lemma, which is key to the 1935 proof as we saw in the previous chapter, is provable without trivial circular reasoning. Hereafter, we abbreviate "there is a reduction procedure for $\Gamma \to A$" to "$\Gamma \to A$ is reducible."

Lemma 3.4.1 (The Soundness of the Cut Rule)**.** *If $\Gamma \to A$ and $A, \Gamma \to B$ are reducible, then $\Gamma \to B$ is reducible.*

If this lemma is proved, we obtain both a reduction procedure R for $\to B$ and a proof for the fact that R is a reduction procedure for $\to B$. This means that a reduction procedure is statable for $\to B$.

Below, we explain that the soundness of the cut rule is provable without trivial circular reasoning, especially, without presupposing the soundness of the very instance of the cut rule we are targeting. That is to say, we show that there is a reduction procedure for the conclusion of an instance I of the cut rule, appealing to its premises' reduction procedures and the soundness of the instances of lower "complexity" than I. This is provable by means of the following two induction principles.

1. The induction principle on the length of a formula.

2. The induction principle on a reduction procedure for $\Gamma \to A$.

First of all, let us formulate the second induction principle. Consider an arbitrary pre-reduction procedure R. We say a finite sequence $\vec{S} = \langle \Delta_1 \to B_1, \dots, \Delta_n \to B_n \rangle$ of closed sequents is a *maximal* R-admissible sequence if and only if there is no R-admissible sequence that includes $\vec{S}$ as a proper initial segment.[12] Note that for every maximal R-admissible sequence $\langle \Delta_1 \to B_1, \dots, \Delta_n \to B_n \rangle$, either $\Delta_n \cap \mathcal{FALSE} \neq \emptyset$ or $B_n \in \mathcal{TRUE}$. The second induction principle is an instance of the following induction principle.

Definition 3.4.1 (Induction on a Reduction Procedure)**.** Let P be a property of a finite sequence of closed sequents. If a reduction procedure R for a sequent $\Gamma \to A$ satisfies the following two conditions, then the sequence $\langle \Gamma \to A \rangle$ has P:

- every maximal R-admissible sequence has P,

[12]We owe the notion of maximal R-admissible sequences to Tait ([Tai15, p.223]).

- for every n and every R-admissible sequence $\langle \Delta_1 \to B_1, \ldots, \Delta_n \to B_n \rangle$ of length n, the following holds: If all R-admissible sequences of length $n+1$ including $\langle \Delta_1 \to B_1, \ldots, \Delta_n \to B_n \rangle$ as an initial segment have P, then $\langle \Delta_1 \to B_1, \ldots, \Delta_n \to B_n \rangle$ has P.

Informally, this induction principle says that if every maximal R-admissible sequence has P and P is inherited from each R-admissible sequence to its immediate proper initial segment, then the R-admissible sequence of length 1 has P. This principle can be considered an instance of the principle of decidable bar induction in intuitionistic mathematics.[13]

Now we proceed to our argument that the soundness of the cut rule is provable without circular reasoning.[14] Assume that $\Gamma \to A$ and $A, \Gamma \to B$ are reducible. We show that $\Gamma \to B$ is so as well. For the sake of simplicity, we consider only the case that both $\Gamma \to A$ and $A, \Gamma \to B$ are closed. If $A \in \Gamma$, then $\{A\} \cup \Gamma = \Gamma$, so the assertion already holds. Let A be not in Γ. First, we use induction on the length of the cut formula A.

(i) Base Case

(ia) Consider the case that $A \in \mathcal{TRUE}$. Then, A is irrelevant to a given reduction procedure R for $A, \Gamma \to B$. A reduction procedure for $\Gamma \to B$ is obtained from an effective procedure for rewriting $\Gamma \to B$ in the same way as R rewrites $A, \Gamma \to B$.

(ib) Consider the case that $A \in \mathcal{FALSE}$. We stipulate the following procedure for rewriting $\Gamma \to B$. First, apply $(\forall)$, $(\wedge)$ and $(\supset, r)$ repeatedly until the formula in the right hand of the sequent is decomposed to an atomic formula C. If $C \in \mathcal{TRUE}$, then the resulting sequent is of the end-form, so stop rewriting. If $C \in \mathcal{FALSE}$, the resulting sequent is of the form $\Gamma, \Delta \to C$. Next, we use the following lemma:

Lemma 3.4.2. *For every formula A and every finite set Γ, Δ of formulas, if $\Gamma \to A$ is reducible then $\Delta, \Gamma \to A$ is reducible.*

This lemma immediately follows from the definition of reduction procedures. By this lemma and the assumption that $\Gamma \to A$ is reducible, a

[13] For the principle of decidable bar induction, see [TD88, p.229].

[14] The proof for the soundness of the cut rule in this section differs from the one given by Gentzen in that the former uses the induction principle on a reduction procedure explicitly. For the original proof given by Gentzen, see [Gen74, pp.108-112], [Gen69, pp.207-210]. The proof in this section is essentially the same as the one given by Tait in [Tai15, pp.221-222].

reduction procedure R is given for $\Gamma, \Delta \to A$. Both A and C belong to $\mathcal{FALSE}$, so R is also a reduction procedure for $\Gamma, \Delta \to C$ by the convention in the previous section. Finally, rewrite the sequent $\Gamma, \Delta \to C$ in the same way as R rewrites it, then we always reach to a sequent of the end-form.

Let us summarize the explanation for case (i). In this case, we can show that $\Gamma \to B$ is reducible, by using Lemma 3.4.2 and the given reduction procedures for $\Gamma \to A$ and $A, \Gamma \to B$ only.

(ii) Induction Step The induction hypothesis is as follows: For every formula A_0 such that the length n_0 of A_0 is smaller than the length n of A,

> for every finite set Γ_0 of formulas and every formula B_0, if $\Gamma_0 \to A_0$ and $A_0, \Gamma_0 \to B_0$ are reducible, then $\Gamma_0 \to B_0$ is reducible.

On the assumption that that $\Gamma \to A$ and $A, \Gamma \to B$ are reducible, we show that $\Gamma \to B$ is so. In the case that $A \in \Gamma$, the assertion obviously holds, so let A be not in Γ again. For the sake of simplicity, we consider only the case that A is of the form $\forall x C(x)$. Hereafter, we use the induction principle on a reduction procedure R_1 for $\forall x C(x), \Gamma \to B$. Define the property P on R_1-admissible sequences as follows:

> The R_1-admissible sequence $\langle \Delta_1 \to C_1, \ \ldots, \ \Delta_n \to C_n \rangle$ has P
>
> if and only if
>
> the sequent $\Gamma, \Delta_n \setminus \{\forall x C(x)\} \to C_n$ is reducible.

If we show that the two conditions in the induction principle on R_1 hold, then it follows that the R_1-admissible sequence $\langle \forall x C(x), \Gamma \to B \rangle$ of length 1 has P; that is, $\Gamma \to B$ is reducible. This means that the assertion holds in case (ii) as well. Thus, it suffices to show that those two conditions hold.

(iia) Base Case of the Second Induction Consider an arbitrary maximal R_1-admissible sequence $\langle \Delta_1 \to C_1, \ \ldots, \ \Delta_n \to C_n \rangle$. Then, either $\Delta_n \setminus \{\forall x C(x)\} \cap \mathcal{FALSE} \neq \emptyset$ or $C_n \in \mathcal{TRUE}$, because either $\Delta_n \cap \mathcal{FALSE} \neq \emptyset$ or $C_n \in \mathcal{TRUE}$ by the definition of maximal R_1-admissible sequences. Thus, if we set that $\langle \Gamma, \Delta_n \setminus \{\forall x C(x)\} \to C_n \rangle$ is the only R_2-admissible sequence, R_2 is a reduction procedure for this sequent, so $\langle \Delta_1 \to C_1, \ \ldots, \ \Delta_n \to C_n \rangle$ has P.

(iib) Induction Step of the Second Induction Assume that for every R_1-admissible sequence $\vec{S}$ of length $n+1$ including $\langle \Delta_1 \to C_1, \ \ldots, \ \Delta_n \to C_n \rangle$ as an initial segment, $\vec{S}$ has P. We show that $\langle \Delta_1 \to C_1, \ \ldots, \ \Delta_n \to C_n \rangle$ has P as well. We discuss one of the key cases only, that is, the case that

- $\Delta_n = \Delta' \cup \{\forall x C(x)\}$ for some Δ' with $\forall x C(x) \notin \Delta'$ and
- the one-element extension of $\langle \Delta_1 \to C_1, \ldots, \Delta_n \to C_n \rangle$ is

$$\langle \Delta_1 \to C_1, \ \ldots, \ \Delta' \cup \{\forall x C(x)\} \to C_n, \ \Delta' \cup \{\forall x C(x), C(k)\} \to C_n \rangle.$$

By the induction hypothesis of the second induction, the sequent $\Gamma, \Delta', C(k) \to C_n$ is reducible. If $C(k) \in \Gamma \cup \Delta'$ then we are done, so let $C(k)$ be not in $\Gamma \cup \Delta'$. From the assumption that the sequent $\Gamma \to \forall x C(x)$ is reducible, it can be easily shown that $\Gamma \to C(k)$ is reducible. Thus, $\Gamma, \Delta' \to C(k)$ is reducible by Lemma 3.4.2. The length m of the formula $C(k)$ is smaller than the one of $\forall x C(x)$, so $\Gamma, \Delta' \to C_n$ is reducible by the induction hypothesis of the first induction. This means that $\langle \Delta_1 \to C_1, \ldots, \Delta_n \to C_n \rangle$ has P.

Let us summarize our explanation. First, we saw that no trivial circular reasoning is included in Base Case (i). This enables us to assume in Induction Step (ii) that the soundness of more "simpler" instances of the cut rule than the instance we were targeting is proved without trivial circular reasoning. Then, we show the soundness of this instance by appealing to the soundness of more "simpler" instances. This step-by-step explanation leads to the conclusion that no trivial circular reasoning is included in the whole proof of the soundness of the cut rule.

3.5 Conclusion of Chapter 3

In this chapter, we have argued that the Gentzen-style interpretation (IM) of implication avoids the circularity of implication urged by Gentzen himself. It was the lemma for the soundness of the cut rule in the 1935 proof that was crucial to avoid circularity: It sufficed to confirm that the lemma is proved without trivial circular reasoning. We have confirmed this in the following way. We emphasized two induction principles and explained that due to these principles, the soundness of the cut rule can be shown step-by-step from the base cases of two inductions.

Our whole argument sheds light on the role of the proof-theoretic methods of the Hilbert School. Although there is a gap in Gentzen's own proof for the soundness of the cut rule, Gentzen's interpretation took not only the role of showing the consistency of first-order arithmetic, but also the role of interpreting implication formulas of first-order arithmetic, avoiding the circularity of implication he was concerned with.

Chapter 4

Contentual and Formal Aspects of Gentzen's Interpretation for Arithmetic

4.1 Introduction to Chapter 4

In Chapters 2 and 3, we were mainly concerned with the 1935 proof, i.e., Gentzen's first consistency proof for first-order arithmetic. Those chapters explained Gentzen's two aims that were found in the proof. In this chapter, we examine his second consistency proof for first-order arithmetic, i.e., the 1936 proof. This proof, located between the 1935 and 1938 proofs, was a work during his "transition period 1936-1938."[1] On the one hand, the 1936 proof inherited from the 1935 proof the method of "finitist" interpretation of first-order arithmetical formulas, which we have discussed in Chapters 2 and 3. On the other hand, the 1936 proof was a precursor of the 1938 proof given by the cut elimination method with an ordinal notation system below ε_0.

Because of this intermediate or patchwork feature of the 1936 proof, several researchers have published investigations of its structure. For example, Yasugi ([Yas80]) reformulated the 1936 proof within the framework of the 1938 proof and presented some applications of her reformulation. More recently, Buchholz ([Buc15]) analyzed the structure of the 1936 proof, using the method of finite notations for infinitary derivations.[2] While Yasugi and

[1] [Ara02, p.438]. By examining the 1936 proof, we can see a connection between Gentzen's two aims above and the 1938 proof. We will see this connection in the concluding remarks of this thesis.

[2] The method of finite notations for infinitary derivations originates from the work by Mints ([Min78]) and has been developed further in subsequent papers by Buchholz

Buchholz have focused on the mathematical side of the 1936 proof, we focus not only on its mathematical side, but also on its conceptual side.[3]

Sieg, in [Sie12], explained some background of Gentzen's 1936 proof that enables us to understand the intermediate feature of this 1936 proof from another approach. Sieg claimed, "Hilbert's considerations in ["Beweis des tertium non datur" [Hil31]] were a crucial germ for Gentzen's work on consistency."[4] That is to say, Sieg has claimed that Hilbert's 1931 paper "Beweis des Tertium non datur" had great influence on Gentzen's investigation into the consistency of first-order arithmetic, especially on Gentzen's 1936 proof. In the main argument of [Hil31], Hilbert used a concept of *correctness* (Richtigkeit) of formulas to prove the consistency of a first-order arithmetical theory without any induction axiom schema.[5] He formulated a concept of correctness of formulas and then attempted to show, in this sense, that every derivation of the theory has a correct formula as its conclusion.

According to Sieg, Gentzen scrutinized this argument before completing the 1936 proof. By citing some passages from Gentzen's unpublished manuscripts, Sieg explained background of the 1936 proof as follows.[6] Gentzen called a kind of a consistency proof like Hilbert's argument above a *contentual correctness proof* (inhaltlicher Richtigkeitsbeweis) and contrasted this kind with another kind that he called a *formal correctness proof* (formaler Richtigkeitsbeweis). Formal correctness proofs show a theory's consistency by assigning a normal derivation to every derivation of a numeric equation in the theory. A consistency proof by means of Hilbert's epsilon substitution method and Gentzen's 1938 proof are typical examples of such correctness proofs.[7] Then, Gentzen finished the 1936 proof and considered it *intermediate* between these two kinds of consistency proofs.

([Buc91, Buc97, Buc01]).

[3]The content of this chapter differs from our previous work ([AT13]) written in Japanese with respect to the three main points. First, while the aim of [AT13] was to give only a uniform interpretation of Gentzen's proofs, this chapter considered questions naturally arising from Sieg's paper [Sie12]. Secondly, we use another version of normalization trees, namely, a version of normalization trees reformulated with (possibly) non-well-founded trees. This notion makes it easier to see a connection between the 1936 proof and some notions of intuitionism like spreads and choice sequences. Finally, Section 4.5 of the present chapter gives another proof of Kreisel's no-counterexample interpretation, using normalization trees.

[4][Sie12, p.123].

[5]The theory includes the principle of tertium non datur. Hilbert's aim in this paper was to justify the use of this principle by means of a consistency proof.

[6]In fact, Sieg has explained background of not only the 1936 proof but also the 1935 proof. In the present chapter, we concentrate on an analysis of the 1936 proof in the light of Sieg's explanation.

[7]As to Hilbert's exposition of the epsilon substitution method, see [Hil28].

This explanation by Sieg induces the question whether Gentzen's 1936 proof is both a contentual correctness proof and a formal correctness proof in the sense above, since one might wonder if a consistency proof *in general* can be so. Contentual correctness proofs and formal correctness proofs correspond to semantic consistency proofs and proof-theoretic consistency proofs, respectively, and semantic consistency proofs are usually distinguished from proof-theoretic consistency proofs. (Below we often abbreviate "P is a contentual correctness proof" as "P is contentual." The same abbreviation is used also for formal correctness proofs.) Furthermore, one could ask another question: If the 1936 proof is both contentual and formal, how do its contentual and formal aspects relate to each other?

This chapter aims to answer the two questions above. First, we argue that the 1935 proof is a contentual correctness proof and that the 1938 proof is a formal correctness proof. As seen in Chapters 2 and 3, Gentzen gave the following interpretation for every sequent of first-order arithmetic: Let Γ be a sequent as a finite set of formulas of first-order arithmetic, then

(GI) Γ is correct

if and only if

a reduction procedure is statable for Γ.[8]

The main lemma of the 1935 proof shows that every derivation of first-order arithmetic has the correct endsequent in the sense of (GI); thus the 1935 proof is a contentual correctness proof. After arguing so, we briefly see that the 1938 proof is a formal correctness proof because its main lemma assigns a normal derivation to every derivable numeric equation of first-order arithmetic.

Second, we show that the 1936 proof is both contentual and formal because its main lemma implies both the main lemma of the 1935 proof and the main lemma of the 1938 proof. To show this in a uniform way, we formulate the 1936 proof in the framework of *normalization trees*, which were introduced in [Aki10, AT13] through finite notations for infinitary derivations. Here we use a version of normalization trees reformulated in terms of (possibly) non-well-founded trees, to analyze contentual aspects of the 1936 proof more closely. Then, we derive the main lemma of the 1935 proof from the main lemma of the 1936 proof. Next, to analyze formal aspects of the 1936 proof, we utilize not only normalization trees, but also analyses in Buchholz's two papers [Buc97, Buc15]. We show that the main lemma of the 1938 proof

[8] The reason why we formulate (GI) with one-sided sequents is that we use the one-sided sequent calculus in the arguments of Sections 4.3, 4.4 and 4.5.

is a special case of the main lemma of the 1936 proof. This means that the main lemma of the 1936 proof assigns a normal derivation to every derivable numeric equation as the main lemma of the 1938 proof does.

Third, we explain how contentual and formal aspects of the 1936 proof relate to each other. Our argument for the claim that the 1936 proof is both contentual and formal enables us to see the following: The correctness of a derivable formula of first-order arithmetic in the sense of (GI) is in fact shown by means of syntactic transformation of a given derivation of the formula. This means that contentual aspects of the 1936 proof are formed by its formal aspects.

Finally, we note the following consequence of our arguments: Kreisel's no-counterexample interpretation ([Kre51]) is obtained from our formulation of the 1936 proof.[9]

This chapter is structured as follows. In Section 4.2, we briefly recall Sieg's argument above and then explain that the 1935 proof is a contentual correctness proof and that the 1938 proof is a formal correctness proof. Section 4.3 gives a formulation of finite notations for infinitary derivations in first-order arithmetic, following Buchholz's paper [Buc97]. In Section 4.4, we first define the notion of normalization trees and then answer the foregoing two questions. Finally, in Section 4.5, we show that Kreisel's no-counterexample interpretation is obtained from our formulation of the 1936 proof.

4.2 Contentual and Formal Correctness Proofs

In this section, we first explain the argument by Sieg for the claim that Hilbert's considerations in "Beweis des Tertium non datur" had great influence on Gentzen's 1936 proof. Sieg's argument provides background for the present chapter's contents. Next, preliminarily to our argument after this section, we see that the 1935 proof is a contentual correctness proof and that the 1938 proof is a formal correctness proof.

The following passage from Gentzen's unpublished manuscript *Urdissertation* is key to Sieg's argument.

> The consistency of arithmetic will be proved; in the process, the concept of an infinite sequence of natural numbers will be used, furthermore in one place the principle of the excluded middle.

[9]Several studies have already pointed out the relationship between Gentzen's 1935 and 1936 proofs and the no-counterexample interpretation. Cf. [Kre71, SP95, Tai05]. Furthermore, according to [SP95, Tai01], the idea of the no-counterexample interpretation is found in Gödel's notes for his lecture in 1938.

> The proof is thus not intuitionist. Perhaps the *tertium non datur* can be eliminated. ([Sie12, p.88])

Here, Gentzen made several comments about his tentative proof for the consistency of first-order arithmetic in *Urdissertation*: The concept of an infinite sequence of natural numbers and the principle of tertium non datur were used in that proof. According to Sieg, a connection between Gentzen's investigation into the consistency of first-order arithmetic and Hilbert's considerations in [Hil31] is found here. It was a central feature of the main argument in [Hil31] that both the concept of an infinite number-sequence and the principle of tertium non datur were used in metamathematical investigations.

Hilbert, in [Hil31], formulated a fragment of arithmetic that lacks an induction axiom schema and the axiom (TND) $\forall x A(x) \vee \exists x \neg A(x)$; then he claimed that the theory can be expanded by means of (TND) without deriving contradiction.[10] An outline of Hilbert's main argument is as follows.[11] First, Hilbert stipulated when a formula of the theory is *correct* (richtig). Next, he tried to prove that every correct formula does not imply contradiction. Finally, he argued that every derivation in the expanded theory by means of (TND) can be transformed into a derivation such that the axioms and inference-rules in it are all correct. As seen from this outline, a notion of correctness is crucial to Hilbert's argument. Later in this section, we will see that a notion of correctness played a crucial role in Gentzen's considerations as well.

Let us see briefly how Sieg explained that the concept of an infinite number-sequence was included in Hilbert's considerations. Hilbert stipulated when a universally quantified formula is correct as follows:

> If the statement $\mathfrak{A}(\mathfrak{z})$ is correct as soon as $\mathfrak{z}$ is a numeral, then the statement $(x)\mathfrak{A}(x)$ holds $[\cdots]$. ([Hil31, p.121])[12]

Sieg claimed that this stipulation gave the following rule (HR^*): If $\mathfrak{A}(\mathfrak{z})$ is correct for an arbitrary numeral $\mathfrak{z}$, then the universally quantified formula $(x)\mathfrak{A}(x)$ may be introduced as an *axiom* of theory.[13] Note that the rule (HR^*) differs from standard inference-rules, since standard rules allow us to deduce a conclusion from some premises *within* a formal theory. The rule (HR^*) is for introducing an axiom by means of metatheoretical considerations. According to Sieg, the concept of an infinite sequence of natural numbers is included in

[10] In the notation of [Hil31], (TND) is $(x)\mathfrak{A}(x) \vee (Ex)\bar{\mathfrak{A}}(x)$.
[11] We owe this outline to Sieg. Cf. [Sie12, pp.102-105].
[12] This English translation is by Sieg.
[13] [Sie12, p.92, p.103].

this rule. We omit further details of Sieg's explanation.[14]

In addition to this feature of Hilbert's considerations, Sieg argued that Hilbert made metamathematical use of the principle of tertium non datur to show that his notion of correctness actually works:

> This metamathematical statement [any statement either does or does not lead to a contradiction] is an instance of *tnd*, and Hilbert views it as "necessary" for the founding of mathematics. He then uses *tnd* to show that correctness, falsity, and the generalized negation of statements (see Note 29) harmonize in the appropriate way. ([Sie12, p.105])

Of course, "tnd" denotes the principle of the excluded middle, so Sieg has claimed here that a metamathematical statement used in Hilbert's argument is an instance of the principle of the excluded middle. After quoting the remark from Gentzen's *Urdissertation* (at the beginning of this section), Sieg wrote,

> The connection to Hilbert's considerations in *Beweis des tertium non datur* seems unmistakable, as [Gentzen's] these remarks point exactly to the central features of Hilbert's argument, i.e., the metamathematical use of the rule (HR^*) and *tnd*. ([Sie12, p.107])

This connection between Gentzen's *Urdissertation* and Hilbert's argument in [Hil31] is a rationale for Sieg's claim that Hilbert's consideration in [Hil31] had great influence on Gentzen's 1936 proof.

Let us explain the next rationale. After explaining the connection above between Gentzen's *Urdissertation* and Hilbert's argument in [Hil31], Sieg has described other background of Gentzen's 1936 proof as follows. The manuscript INH, written from October 1932 to October 1934 and titled "Die formale Erfassung des Begriffs der inhaltlichen Richtigkeit in der reinen Zahlentheorie, Beziehungen zum Widerspruchsfreiheitsbeweis," starts with a reflection by Gentzen: how a notion of correctness is defined in a given formal theory.[15] Then, Gentzen contrasted a kind of consistency proof he called a *formal correctness proof* with another kind called a *contentual correctness proof.* The following is a quotation by Sieg from INH, p.2:

[14] Cf. [Sie12, p.107]. Sieg's explanation can be paraphrased as follows: To apply the rule (HR^*), one needs to show that $\mathfrak{A}(\mathfrak{z})$ is correct for an arbitrary numeral $\mathfrak{z}$. Because of Hilbert's finitist attitude, this must be shown by giving an effective method to verify the correctness of $\mathfrak{A}(0)$, $\mathfrak{A}(1)$, $\mathfrak{A}(2)$ and so on. The concept of an infinite sequence of natural numbers is used here.

[15] [Sie12, p.114].

> I seek to clarify the questions: what distinguishes a formal correctness or consistency proof from a contentual one, why is the former for certain inferences not even possible by these same inferences (according to Gödel), is a bridge inference involved then, how secure is that [bridge inference, WS], what are the connections with Gödel's proof, what role do the mathematical axioms play? ([Sie12, p.114])

In the present chapter, what is relevant is the first question, that is, the question of what distinguishes a formal correctness proof from a contentual one.

Citing Gentzen's words from INH, Sieg characterized notions of contentual correctness proofs and formal correctness proofs as follows:

> Calling a proof of a numeric statement a *Normalbeweis* if it contains only numeric statements, Gentzen can now express the difference between (purely) formal and (semi-) contentual correctness proofs by formulating carefully the claim each is to establish. The claim for a *purely formal correctness proof* is, "for every proof of a numeric statement there is a *Normalbeweis* of that statement", and the corresponding claim for the (semi-) contentual correctness proof is, "every proof has a correct result" [where result means endformula, WS]. ([Sie12, p.115])

In sum, a consistency proof is a formal correctness proof if and only if it shows the consistency of a theory by assigning a normal derivation (Normalbeweis) to each derivable numeric equation in the theory. Since no normal derivation of $0 = 1$ exists, the consistency follows. The meaning of "normal derivations" and of "numeric equations" may vary according to which theory is considered. For the former notion, it suffices to require that the truth of the conclusion of a normal derivation is verified in a primitive recursive way. On the other hand, a consistency proof is a contentual correctness proof if and only if it shows the consistency of a theory by verifying that every derivation of the theory has a correct conclusion.[16] Then, it follows that no contradiction is derivable in the theory because no contradiction can be correct. Note that Hilbert's argument in [Hil31] is an example of contentual correctness proofs. Gentzen made a further remark about differences between these two kinds of correctness proofs, but we do not enter into its details. The remark was quoted by Sieg from INH, p.8:

[16] The distinction of *purely* formal correctness proofs from formal correctness proofs does not matter to our present concern; neither does the distinction of *semi*-contentual correctness proofs from contentual correctness proofs. Thus we do not consider these distinctions. In the both cases, the former is subsumed into the latter.

> The semi-contentual proof uses complete induction for a rather complicated statement. This contains Ri erg x [the result of proof x is correct, WS], and this predicate becomes ever more complicated in complicated cases. The formal proof uses complete induction for [the statement] Ey. No y & erg $x =$ erg y [there is a *Normalbeweis* y having the same result as the given proof x, WS]; this is also a statement containing logical signs; it is however of a simple nature, also in more complicated cases. ([Sie12, p.115])

As the last two passages from INH show, Gentzen examined the method of Hilbert's argument in [Hil31], namely, the method of contentual correctness proofs before completing the 1936 consistency proof. This is the second rationale for Sieg's claim that Hilbert's consideration in [Hil31] had great influence on Gentzen's 1936 proof. As we said in the introduction, Sieg eventually concluded that Gentzen gave the 1936 proof in an intermediate way between contentual correctness proofs and formal correctness proofs.[17]

In the rest of this section, we argue that Gentzen's 1935 proof is contentual and that his 1938 proof is formal, preliminarily to our argument in Section 4.4.

As explained in Chapters 2 and 3, Gentzen, in the 1935 proof, gave a notion of correctness called the *statability of a reduction procedure*, and then remarked that it gives a "finitist" interpretation of arithmetical formulas. He wrote,

> The concept of the 'statability of a *reduction procedure*' (die Angebbarkeit einer Reduziervorschrift) for a sequent, to be defined below, will serve as the formal replacement (formaler Ersatz) of the contentual *concept of correctness* (der inhaltliche Richtigkeitsbegriff); it provides us with a *special finitist interpretation* (finite Deutung) of propositions and takes the place of their *actualist interpretation* [...]. ([Gen74, p.100], [Gen36, p.536], [Gen69, p.173], italics original)

From the quotation above, we can extract the following interpretation of every sequent Γ, i.e., every finite set Γ of formulas of first-order arithmetic:

(GI) Γ is correct

if and only if

a reduction procedure is statable for Γ.

[17] [Sie12, p.117].

Later, we will redefine both reduction procedures and the statability of a reduction procedure by the method of this chapter (Section 4.4). But now, it is sufficient to recall that the statability of a reduction procedure is a notion of correctness: A sequent Γ is correct when a reduction procedure is statable for Γ. If we stipulate that a reduction procedure for a formula A as one for $\{A\}$, then we obtain from (GI) the following interpretation of arithmetical formulas:

> A is correct
>
> if and only if
>
> a reduction procedure is statable for A.

The main lemma of the 1935 proof shows that every derivation of a proof-system Z of first-order arithmetic has the correct endsequent in the sense of (GI). The statement of the lemma is as follows.[18]

Main Lemma of the 1935 Proof. For every sequent Γ of Z,

> if Γ is derivable in Z, then there is a reduction procedure for Γ.

Note that, as seen in Chapter 2, this lemma shows the correctness of each Z-derivable sequent Γ in the sense of (GI) because the proof of the lemma gives not only a reduction procedure for Γ, but also a proof for its termination. The 1935 proof shows the consistency of first-order arithmetic in this way. Thus, according to Sieg's characterization above, the 1935 proof is a contentual correctness proof.

On the other hand, the 1938 proof is a formal correctness proof. As we have said previously, Sieg's characterization of formal correctness proofs is not restricted to a particular theory, so the range of numeric equations and normal derivations could vary according to which theory is considered. To see that Gentzen's 1938 proof is a formal correctness proof, consider the proof-system Z of first-order arithmetic that Gentzen formulated in [Gen38b], whose language includes 0 (zero) and S (the successor function) as its all function symbols. Then, the numeric equations we consider are formed by means of these two function symbols and the equality $=$. In addition, we recall Takeuti's formulation of the 1938 proof ([Take87, §12]). A derivation of Z is called *simple* if and only if it includes no free variable and consists only of mathematical initial sequents and structural inferences except cut-inferences with the non-atomic cut formula.[19] Note that a simple derivation

[18] [Gen74, p.103], [Gen36, p.549].

[19] Cf. [Take87, Definition 12.2.].

does not include any logical inference and the induction-rule. Below, by "normal derivation" we mean a simple derivation.

Now, the 1938 proof can be outlined as follows. First, it can be verified in a primitive recursive way that there is no normal derivation of a false numeric equation. Next, the following lemma is proved:

Main Lemma of the 1938 Proof. Let h be a derivation of Z for a numeric equation, then there is a reduction sequence of h that terminates with a normal derivation for the equation.

There is no derivation of Z for $0 = 1$ by the main lemma and the above fact about normal derivations, so Z is consistent. As this outline shows, the 1938 proof is a formal correctness proof. The main lemma of this proof assigns a normal derivation to every derivable numeric equation.

In Section 4.4, we argue that the 1936 proof is both contentual and formal, by showing that the main lemma of the 1936 proof implies the main lemmas of the 1935 and 1938 proofs. In the next section, we explain the method of *finite notations of infinitary derivations.*

4.3 Finite Notations for Infinitary Derivations

In this section, we introduce the finitary system Z^* of first-order arithmetic and the corresponding infinitary system Z^∞, then recall some basic properties of finite notations for infinitary derivations. We define these systems in the style of Tait's calculus. First, a sequent is a finite set of formulas rather than a finite multi-set of them. Second, the contraction, exchange and weakening rules are implicitly assumed. Although we follow Buchholz' paper [Buc97] with some minor modifications, we repeat and explain important definitions and theorems in it for readers' convenience.

We define the basic language L, on which the systems Z^* and Z^∞ are defined. The vocabulary consists of the following symbols:

- Predicate Symbols: p for every primitive recursive relation $\mathbf{P}$,
- Function Symbols: 0, S (successor),
- Variables for natural numbers: $x_0, x_1, x_2, \ldots$,
- Logical connectives: $\forall, \exists, \wedge, \vee, \neg$,
- Auxiliary Symbols: (,).

Terms of L are defined in the standard way: 0 and variables are terms. if t is a term, then $S(t)$ is also a term, and denoted by t, s. The closed terms are called the *numerals*. *Atomic formulas* are formulas of the form $pt_1 \cdots t_n$, where p is a predicate symbol and $t_1, \ldots, t_n$ are terms. We say an expression is a *literal* if it is an atomic formula or of the form $\neg pt_1 \cdots t_n$. Formulas of L are defined from literals by means of $\forall, \exists, \wedge, \vee$. For every formula A, the *negation* $\neg A$ of A is defined via de Morgan's laws. From this definition, it follows that the negation symbol $\neg$ is put only before an atomic formula. For example, $\neg(0 = 0 \wedge 1 = 1)$ denotes the formula $\neg(0 = 0) \vee \neg(1 = 1)$. Finite sets of formulas are called *sequents*.

When θ is a term, a formula or a sequent, we denote the set of the free variables in θ by $FV(\theta)$ and say θ is *closed* if $FV(\theta) = \emptyset$. The expression $\theta(x/t)$ is the result of substituting t for every free occurrence of x in θ after renaming some bound variables in θ if necessary. The set of all closed true literals is denoted by TRUE_0. For example, $0 = 0$ belongs to this set. We use syntactic variables x, y, z for variables, n, m, k, l for terms, A, B, C, D for formulas and Γ, Δ for sequents, possibly with suffixes. We often abbreviate $\Gamma \cup \Delta$ (resp. $\Gamma \cup \{A\}$) as Γ, Δ (resp. Γ, A or A, Γ).

A *proof system* $\mathfrak{S}$ is defined by a set of inference symbols. We denote inference symbols by $\mathcal{I}$, the indices of $\mathcal{I}$ by $|\mathcal{I}|$, the principle formulas of $\mathcal{I}$ by $\Delta(\mathcal{I})$ and the premises of $\mathcal{I}$ by $(\Delta_i(\mathcal{I}))_{i \in |\mathcal{I}|}$. By writing

$$(\mathcal{I})\ \frac{\ldots \Delta_i \ldots (i \in I)}{\Delta}\ ,$$

we mean that $\mathcal{I}$ is an inference symbol such that $|\mathcal{I}| = I$, $\Delta(\mathcal{I}) = \Delta$ and $\Delta_i(\mathcal{I}) = \Delta_i$. When $|\mathcal{I}| = \{0, 1, \ldots, n-1\}$, we write simply

$$(\mathcal{I})\ \frac{\Delta_0\ \Delta_1\ \ldots\ \Delta_{n-1}}{\Delta}\ .$$

For example, in the case of conjunction rule, we write as follows.

$$(\textstyle\bigwedge_{A_0 \wedge A_1})\ \frac{A_0 \quad A_1}{A_0 \wedge A_1}$$

Then, $I = \bigwedge_{A_0 \wedge A_1}$, $\Delta(I) = \{A_0 \wedge A_1\}$, and $\Delta_i(I) = \{A_i\}$ for $\in \{0, 1\}$.

Definition 4.3.1 (The Inference Symbols of the Finitary Proof System Z^*)**.** We assume the existence of a primitive recursive set Ax of sequents satisfying the following conditions:

- for all $\Delta \in \mathsf{Ax}$, Δ is a set of literals,

- if there is a substitution instance Δ_0 of Δ, then $\Delta_0 \in \mathsf{Ax}$,
- if $\Delta \in \mathsf{Ax}$ and Δ is closed, then $\Delta \cap \mathsf{TRUE}_0 \neq \emptyset$.

The inference symbols of Z^* are as follows.

$$(\mathsf{Ax}_\Delta)\ \frac{}{\Delta} \quad \text{with } \Delta \in \mathsf{Ax}$$

$$(\textstyle\bigwedge_{A_0 \wedge A_1})\ \frac{A_0 \quad A_1}{A_0 \wedge A_1} \qquad (\textstyle\bigvee^k_{A_0 \vee A_1})\ \frac{A_k}{A_0 \vee A_1} \quad \text{with } k \in \{0,1\}$$

$$(\textstyle\bigwedge^y_{\forall x A})\ \frac{A(x/y)}{\forall x A} \qquad (\textstyle\bigvee^t_{\exists x A})\ \frac{A(x/t)}{\exists x A}$$

$$(\mathsf{Ind}^{y,t}_F)\ \frac{\neg F, F(y/S(y))}{\neg F(y/0), F(y/t)} \qquad (\mathsf{R}_C)\ \frac{C \quad \neg C}{\emptyset} \qquad (\mathsf{E})\ \frac{\emptyset}{\emptyset}$$

A typical example of Ax_Δ is obtained by taking $\Delta = \{0 = 0\}$ or $\{x = x\}$. At a first glance, the E-rule is redundant, but this rule is needed for "expressing" the cut-elimination for the corresponding infinitary system Z^∞ inside Z^* (cf. Theorem 4.3.2). Similarly, the R-rule corresponds to one-step reduction for Z^∞ (cf. Theorem 4.3.1).

Definition 4.3.2 (Z^*-quasi-derivations)**.** If $\mathcal{I}$ is an inference symbol of Z^* with $|\mathcal{I}| = \{0, \dots, n-1\}$ and $h_0, \dots, h_{n-1}$ are Z^*-quasi-derivations, then $h := \mathcal{I} h_0 \cdots h_{n-1}$ is a Z^*-quasi-derivation with

$$\Gamma(h) := \bigcup_{0 \le i \le n-1} (\Gamma(h_i) \setminus \Delta_i(\mathcal{I})) \cup \Delta(\mathcal{I}).$$

Definition 4.3.3 (Z^*-derivations)**.** If $\mathcal{I}$ is an inference symbol of Z^* with $|\mathcal{I}| = \{0, \dots, n-1\}$ and $h := \mathcal{I} h_0 \cdots h_{n-1}$ such that $h_0, \dots, h_{n-1}$ are Z^*-derivations and the following conditions are satidfied:

1. if $\mathcal{I} = \bigwedge^y_{\forall x A}$ then $y \notin \mathrm{FV}(\Gamma(h))$,
2. if $\mathcal{I} = \mathsf{Ind}^{y,t}_F$ then $y \notin \mathrm{FV}(\Gamma(h))$,
3. if $\mathcal{I} = \bigvee^t_{\exists x A}$ then $\mathrm{FV}(t) \subseteq \mathrm{FV}(\Gamma(h))$,
4. if $\mathcal{I} = \mathsf{R}_C$ then $\mathrm{FV}(C) \subseteq \mathrm{FV}(\Gamma(h))$,

then h is a Z^*-derivation.

We denote Z^*-derivations by h possibly with suffixes. A Z^*-derivation h is *closed* if and only if $\Gamma(h)$ is closed. The conditions (1) and (2) of the definition above impose the standard proviso of eigenvariables. By the conditions (3) and (4), it holds that if h is closed and has the form $\mathcal{I}h_0 \cdots h_{n-1}$ such that $\mathcal{I}$ is neither $\bigwedge^y_{\forall x A(x)}$ nor $\mathsf{Ind}^{y,t}_F$, then $h_0, \ldots, h_{n-1}$ are closed as well. If h is closed and has the form $\bigwedge^y_{\forall x A(x)} h_0$ or $\mathsf{Ind}^{y,t}_F h_0$, then $FV(\Gamma(h_0)) \subseteq \{y\}$.

We assign ordinals up to ε_0 to Z^*-derivations. This assignment is motivated by the corresponding ordinal in the infinitary system Z^∞. For a more detailed informal idea behind this, see Remark 4.3.1. If α, β are ordinals, then $\alpha \sharp \beta$ means the *natural sum* of them.

Definition 4.3.4 (Ordinal Assignment of Z^*-derivations)**.**

$$\mathrm{o}(h) := \begin{cases} \mathrm{o}(h_0) \sharp \mathrm{o}(h_1) & \text{if } I = \mathsf{R}_C, \\ \mathrm{o}(h_0) \times \omega & \text{if } I = \mathsf{Ind}^{y,t}_F, \\ \omega^{\mathrm{o}(h_0)} & \text{if } I = \mathsf{E}, \\ (\sup_{i \in |I|} \mathrm{o}(h_i)) + 1 & \text{otherwise.} \end{cases}$$

$$\deg(h) := \begin{cases} \max\{\mathrm{rk}(C), \deg(h_0), \deg(h_1)\} & \text{if } I = \mathsf{R}_C, \\ \max\{\mathrm{rk}(F), \deg(h_0)\} & \text{if } I = \mathsf{Ind}^{y,t}_F, \\ \deg(h_0) - 1 & \text{if } I = \mathsf{E}, \\ \sup_{i \in |I|} \deg(h_i) & \text{otherwise.} \end{cases}$$

Define the result $h(x/n)$ of substituting a numeral n for a free variable x in a Z^*-derivation h as follows.

- If $h = \mathsf{Ax}_\Delta$, then $h(x/n) := \mathsf{Ax}_{\Delta(x/n)}$.
- If $h = \bigwedge_C h_0 h_1$, then $h(x/n) := \bigwedge_{C(x/n)} h_0(x/n) h_1(x/n)$.
- If $h = \bigwedge^y_C h_0$ and
  - $x = y$, then $h(x/n) := h$,
  - $x \neq y$, then $h(x/n) := \bigwedge^y_{C(x/n)} h_0(x/n)$.
- If $h = \bigvee^t_C h_0$, then $h(x/n) := \bigvee^{t(x/n)}_{C(x/n)} h_0(x/n)$.
- If $h = \mathsf{Ind}^{y,t}_F h_0$ and
  - $x = y$, then $h(x/n) := h$,
  - $x \neq y$, then $h(x/n) := \mathsf{Ind}^{y,t(x/n)}_{F(x/n)} h_0(x/n)$.

- If $h = \mathsf{R}_C h_0 h_1$, then $h(x/n) := \mathsf{R}_{C(x/n)} h_0(x/n) h_1(x/n)$.
- If $h = \mathsf{E}h_0$, then $h(x/n) := \mathsf{E}h_0(x/n)$.

We define the system Z as the subsystem of Z^* obtained by omitting the rule E.

Next, we introduce the infinitary proof system Z^∞. Since this system contains Schütte's ω-rule (denoted by $\bigwedge_{\forall x A}$), it suffices to consider only closed terms and formulas. After proving the cut-elimination theorem for it (Theorems 4.3.1 and 4.3.2), we recall the canonical embedding from Z^* to Z^∞ by Schütte.

Definition 4.3.5 (The Inference Symbols of the Infinitary Proof System Z^∞)**.** The inference symbols of Z^∞ are as follows:

$$(\mathsf{Ax}_A)\ \frac{}{A}\ \text{ with } A \in \mathsf{TRUE}_0 \qquad (\textstyle\bigwedge_{\forall x A})\ \frac{\ldots A(x/n) \ldots (n \in \mathbb{N})}{\forall x A}$$

$$(\mathsf{Cut}_C)\ \frac{C \quad \neg C}{\emptyset} \qquad (\mathsf{Rep})\ \frac{\emptyset}{\emptyset}$$

and $\bigwedge_{A_0 \wedge A_1}$, $\bigvee^k_{A_0 \vee A_1}$, $\bigvee^t_{\exists x A}$ as in Z^*.

The Rep-rule is needed for defining finite notations for infinitary derivations in a primitive recursive way (cf. Remark 4.3.2).

Below, we define the notion of Z^∞-derivations in a precise way. Note that the ordinal $\mathrm{o}(\mathtt{d})$ and the degree $\deg(\mathtt{d})$ of a given Z^∞-derivation $\mathtt{d}$ mean the size of $\mathtt{d}$ as a tree and the cut-rank of $\mathtt{d}$, respectively.

The figure

$$\frac{\ldots \Gamma_i \ldots (i \in I)}{\Gamma}\ \mathcal{I}$$

is a *correct* $\mathfrak{S}$*-inference* if and only if $\mathcal{I} \in \mathfrak{S}$, $|\mathcal{I}| = I$, $\Delta(\mathcal{I}) \subseteq \Gamma$ and $\Gamma_i \subseteq \Gamma \cup \Delta_i(\mathcal{I})$ for every $i \in |\mathcal{I}|$.

Definition 4.3.6 (Z^∞-derivations)**.** If $\mathcal{I}$ is an inference symbol of Z^∞, $(\mathtt{d}_i)_{i \in I}$ is a family of Z^∞-derivations, Γ is a sequent and α is an ordinal such that

$$\frac{\ldots \Gamma(\mathtt{d}_i) \ldots (i \in I)}{\Gamma}\ \mathcal{I}$$

is a correct Z^∞-inference and $\mathrm{o}(\mathtt{d}_i) \prec \alpha$ for every $i \in I$, then the tree $\mathtt{d}$

$$\frac{\ldots \mathtt{d}_i \ldots (i \in I)}{\mathcal{I} : \Gamma : \alpha}$$

is a Z^∞-derivation and

$$\Gamma(\mathtt{d}) := \Gamma,\ \mathrm{last}(\mathtt{d}) := \mathcal{I},\ \mathrm{o}(\mathtt{d}) := \alpha,\ \mathtt{d}(i) := \mathtt{d}_i,$$

$$\deg(\mathtt{d}) := \begin{cases} \max\{\mathrm{rk}(C)+1, \deg(\mathtt{d}_0), \deg(\mathtt{d}_1)\} & \text{if } \mathcal{I} = \mathsf{Cut}_C, \\ \sup_{i\in I} \deg(\mathtt{d}_i) & \text{else.} \end{cases}$$

We write $\mathtt{d} \vdash^\alpha_m \Gamma$ if $\Gamma(\mathtt{d}) \subseteq \Gamma,\ \mathrm{o}(\mathtt{d}) = \alpha$ and $\deg(\mathtt{d}) \le m$. In what follows, we may assume $\Gamma(\mathtt{d}) = \Gamma$ unless otherwise stated.

The point of the following theorem is that we can derive Γ from Γ, C and $\Gamma, \neg C$ without increasing the degree of the derivation, under a certain condition.

Theorem 4.3.1. *For every formula C, there is an operator $\mathcal{R}_C$ such that if $\mathtt{d}_0 \vdash^\alpha_m \Gamma, C$, $\mathtt{d}_1 \vdash^\beta_m \Gamma, \neg C$ and $\mathrm{rk}(C) \le m$, then $\mathcal{R}_C(\mathtt{d}_0, \mathtt{d}_1) \vdash^{\alpha\sharp\beta}_m \Gamma$.*

Proof. By double induction on $\mathtt{d}_0$ and $\mathtt{d}_1$. Let $\mathcal{I}_0, \mathcal{I}_1$ be the last inference rules of $\mathtt{d}_0, \mathtt{d}_1$ respectively. Then $\mathtt{d}_k = \mathcal{I}_k(\mathtt{d}_{ki})_{i\in|\mathcal{I}_k|}$.

If $C \notin \Delta(\mathcal{I}_0)$, we set

$$\mathcal{R}_C(\mathtt{d}_0, \mathtt{d}_1) := \mathcal{I}_0(\mathcal{R}_C(\mathtt{d}_{0i}, \mathtt{d}_1))_{i\in|\mathcal{I}_0|} \vdash^{\alpha\sharp\beta}_m \Gamma.$$

The case $\neg C \notin \Delta(\mathcal{I}_1)$ is treated in the same way.

Hence, we consider the case $C \in \Delta(\mathcal{I}_0)$ and $\neg C \in \Delta(\mathcal{I}_1)$. Although all cases according to the shape of C must be considered, we treat the only case when $C = \forall x A(x)$. In this case, $\mathtt{d}_0 = \bigwedge_{\forall x A(x)}(\mathtt{d}_{0n})_{n\in\omega}$ and $\mathtt{d}_1 = \bigvee^k_{\exists\neg A(x)}(\mathtt{d}_{10})$ for some k. By IH, we have $\mathcal{R}_C(\mathtt{d}_{0k}, \mathtt{d}_1) \vdash^{\alpha_k\sharp\beta}_m \Gamma, A(k)$ with $\alpha_k\sharp\beta < \alpha\sharp\beta$. Moreover, again by IH, we have $\mathcal{R}_C(\mathtt{d}_0, \mathtt{d}_{10}) \vdash^{\alpha\sharp\beta_0}_m \Gamma, \neg A(k)$ with $\alpha\sharp\beta_0 < \alpha\sharp\beta$. By inserting a cut over $A(k)$, we get the desired derivation:

$$\mathcal{R}_C(\mathtt{d}_0, \mathtt{d}_1) := \mathsf{Cut}_{A(k)}(\mathcal{R}_C(\mathtt{d}_{0k}, \mathtt{d}_1), \mathcal{R}_C(\mathtt{d}_0, \mathtt{d}_{10})) \vdash^{\alpha\sharp\beta}_m \Gamma.$$

We remark that $\mathrm{rk}(C) = \mathrm{rk}(\forall x A(x)) > \mathrm{rk}(A(k))$ so that $\deg(\mathcal{R}_C(\mathtt{d}_0, \mathtt{d}_1)) \le m$ holds. $\square$

Theorem 4.3.2. *There is an operator $\mathcal{E}$ such that if $\mathtt{d} \vdash^\alpha_{m+1} \Gamma$, then $\mathcal{E}(\mathtt{d}) \vdash^{\omega^\alpha}_m \Gamma$.*

Proof. By induction on $\mathtt{d}$. Let $\mathcal{I}$ be its last inference symbol.

First we consider the crucial case $\mathcal{I} = \mathsf{Cut}_C$ so that $\mathtt{d} = \mathsf{Cut}_C(\mathtt{d}_0, \mathtt{d}_1)$. By IH, we have $\mathcal{E}(\mathtt{d}_0) \vdash^{\omega^{\alpha_0}}_m \Gamma, C$ and $\mathcal{E}(\mathtt{d}_1) \vdash^{\omega^{\alpha_1}}_m \Gamma, \neg C$ with $\alpha_0, \alpha_1 < \alpha$. By applying Theorem 4.3.1, we get $\mathcal{R}_C(\mathcal{E}(\mathtt{d}_0), \mathcal{E}(\mathtt{d}_1)) \vdash^{\omega^{\alpha_0}\sharp\omega^{\alpha_1}}_m \Gamma$. Since $\omega^{\alpha_0}\sharp\omega^{\alpha_1} < \omega^\alpha$, we may define

$$\mathcal{E}(\mathtt{d}) := \mathsf{Rep}(\mathcal{R}_C(\mathcal{E}(\mathtt{d}_0), \mathcal{E}(\mathtt{d}_1))) \vdash^{\omega^\alpha}_m \Gamma.$$

In other cases, we define (using IH)

$$\mathcal{E}(\mathtt{d}) := \mathcal{I}(\mathcal{E}(\mathtt{d}_i))_{i\in|\mathcal{I}|} \vdash^{\omega^\alpha}_m \Gamma.$$

For example, if $\mathcal{I} = \bigwedge_{\forall x A(x)}$, then $\mathcal{E}(\mathtt{d}_i) \vdash^{\omega^{\alpha_0}}_m \Gamma, A(i)$ with $\alpha_0 < \alpha$ for each $i < \omega$. From this, applying $\bigwedge_{\forall x A(x)}$, we get $\mathcal{I}(\mathcal{E}(\mathtt{d}_i))_{i\in|\mathcal{I}|} \vdash^{\omega^\alpha}_m \Gamma$. □

We set $\mathtt{d}_A$ as a fixed cut-free Z^∞-derivation with $\Gamma(\mathtt{d}_A) = \{A, \neg A\}$, $\deg(\mathtt{d}_A) = 0$ and $\mathrm{o}(\mathtt{d}_A) = 2 \cdot \mathrm{rk}(A)$. Such a $\mathtt{d}_A$ may be defined by induction on A.

Next, we define the canonical embedding from Z^*-derivations into Z^∞-derivations.

Definition 4.3.7 (Translation of Z^*-derivations into Z^∞-derivations)**.** We define the Z^∞-derivation h^∞ for every closed Z^*-derivation h. Let Γ be $\Gamma(h)$ and α be $\mathrm{o}(h)$.

1. $(\mathsf{Ax}_\Delta)^\infty := \overline{\mathsf{Ax}_A : \Gamma : \alpha}$, where A is the arbitrarily fixed element of $\Delta \cap \mathsf{TRUE}_0$.
2. $(\bigwedge^y_{\forall x A} h)^\infty := \dfrac{\ldots h(y/i)^\infty \ldots (i \in \mathbb{N})}{\bigwedge_{\forall x A} : \Gamma : \alpha}$.
3. $(\mathsf{R}_C h_0 h_1)^\infty := \mathcal{R}_C(h_0^\infty, h_1^\infty)$.
4. $(\mathsf{E}h)^\infty := \mathcal{E}(h^\infty)$.
5. $(\mathsf{Ind}^{y,n}_F h)^\infty := \dfrac{\mathsf{e}_n}{\mathsf{Rep} : \Gamma : \alpha}$ where
$$\mathsf{e}_0 := \mathtt{d}_{F(x/0)},\ \mathsf{e}_1 := h(y/0)^\infty,\ \mathsf{e}_{i+1} := \mathcal{R}_{F(x/i)}(\mathsf{e}_i, h(y/i)^\infty).$$
6. Otherwise: $\mathcal{I} h_0 \cdots h_{n-1}^\infty := \dfrac{h_0^\infty \ldots h_{n-1}^\infty}{\mathcal{I} : \Gamma : \alpha}$.

We insert Rep in Clause 5 for making our definition of finite notations for infinitary derivations primitive recursive. For details, see Remark 4.3.2.(4) after Definition 4.3.8.

Remark 4.3.1. We see that the ordinal assignment for Z^*-derivations (cf. Definition 4.3.4) comes from the corresponding theorems for Z^∞-derivations. For example, the clause that $\mathrm{o}(\mathsf{R}_C(h_0, h_1)) = \mathrm{o}(h_0)\sharp\mathrm{o}(h_1)$ in Definition 4.3.4 is motivated by Theorem 4.3.1. Following this, we defined $(\mathsf{R}_C h_0 h_1)^\infty := \mathcal{R}_C(h_0^\infty, h_1^\infty)$ as in Clause 3 of Definition 4.3.7.

In what follows, we use the following notations. For every $k \in \mathbb{N}$, we set

$$C[k] := \begin{cases} A(x/k), & \text{if } C = QxA \text{ with } Q \in \{\forall, \exists\}, \\ A_k, & \text{if } C = A_0 \circ A_1 \text{ with } \circ \in \{\wedge, \vee\} \text{ and } k \in \{0,1\}, \\ \text{undefined}, & \text{if } C = A_0 \circ A_1 \text{ with } \circ \in \{\wedge, \vee\} \text{ and } k \notin \{0,1\}. \end{cases}$$

In addition, we define h_A by induction on the length of A as a Z-derivation such that

$$\Gamma(h_A) = \{A, \neg A\},\ \deg(h_A) = 0,\ \mathrm{o}(h_A) = 1 + 2 \cdot \mathrm{rk}(A).$$

Next, we define finite notations for infinitary derivations, i.e., the two functions $\mathrm{tp}(h)$ and $h[i]$ with $i \in |\mathrm{tp}(h)|$ for a Z^*-derivation h. To give a simple and short definition, we first define the expression h^+ and then obtain $\mathrm{tp}(h)$ and $h[i]$ from h^+. A definition of $\mathrm{tp}(h)$ and $h[i]$ in this manner can be found in [Buc10].

Definition 4.3.8. For an arbitrary closed Z^*-derivation h, $\mathrm{tp}(h)$ and $h[i]$ with $i \in |\mathrm{tp}(h)|$ are defined as follows.

1. For every closed Z^*-derivation h of the form Ax_Δ, define

$$h^+ := \mathrm{tp}(h) := \mathsf{Ax}_A,$$

   where A is the arbitrarily fixed element of $\Delta \cap \mathsf{TRUE}_0$.

2. Let h be a closed Z^*-derivation with $h \neq \mathsf{Ax}_\Delta$. By induction on the built-up of h, we define h^+ as the expression $\mathcal{I}(h_i)_{i \in |\mathcal{I}|}$ for some inference symbol $\mathcal{I}$ of Z^∞ and some sequence $(h_i)_{i \in |\mathcal{I}|}$ of closed Z^*-derivations. Then, set

$$\mathrm{tp}(h) := \mathcal{I},\ h[i] := h_i \text{ for every } i \in |\mathcal{I}|.$$

   (a) $(\bigwedge_C h_0 h_1)^+ := \bigwedge_C (h_i)_{i \in \{0,1\}}$.

   (b) $(\bigwedge_C^y h_0)^+ := \bigwedge_C (h_0(y/i))_{i \in \mathbb{N}}$.

   (c) $(\bigvee_C^k h_0)^+ := \bigvee_C^k h_0$.

   (d) $(\mathsf{Ind}_F^{y,n} h_0)^+ := \mathsf{Rep} e_n$, where $e_0 := h_{F(x/0)}$, $e_1 := h_0(y/0)$ and $e_{i+1} := \mathsf{R}_{F(x/i)} e_i h_0(y/i)$.

   (e) Let h be $\mathsf{E}h'$ and h'^+ be $\mathcal{I}((h_i)_{i \in |\mathcal{I}|})$, then

$$(\mathsf{E}h')^+ := \begin{cases} \mathsf{Rep}(\mathsf{R}_C \mathsf{E}h_0 \mathsf{E}h_1), & \text{if } \mathcal{I} = \mathsf{Cut}_C, \\ \mathcal{I}((\mathsf{E}h_i)_{i \in |\mathcal{I}|}) & \text{else.} \end{cases}$$

(f) Let h be $\mathsf{R}_C h_0 h_1$ and h_l^+ be $\mathcal{I}_l((h_{li})_{i\in|\mathcal{I}_l|})$ for $l \in \{0,1\}$, then

$$(\mathsf{R}_C h_0 h_1)^+ := \begin{cases} \mathcal{I}_0((\mathsf{R}_C h_{0i} h_1)_{i\in|\mathcal{I}_0|}), & \text{if } C \notin \Delta(\mathcal{I}_0), \\ \mathcal{I}_1((\mathsf{R}_C h_0 h_{1i})_{i\in|\mathcal{I}_1|}), & \text{if } \neg C \notin \Delta(\mathcal{I}_1), \\ \mathsf{Cut}_{C[k]}(\mathsf{R}_C h_{0k} h_1 \mathsf{R}_C h_0 h_{10}), & \text{if } C \in \Delta(\mathcal{I}_0), \neg C \in \Delta(\mathcal{I}_1) \text{ and } \mathcal{I}_1 = \bigvee^k_{\neg C}, \\ \mathsf{Cut}_{C[k]}(\mathsf{R}_C h_{00} h_1 \mathsf{R}_C h_0 h_{1k}), & \text{if } C \in \Delta(\mathcal{I}_0), \neg C \in \Delta(\mathcal{I}_1) \text{ and } \mathcal{I}_0 = \bigvee^k_{C}. \end{cases}$$

Remark 4.3.2. For readers' convenience, let us make several remarks about Definition 4.3.8.

1. If $h = \mathsf{Ax}_\Delta$, then $h[i]$ is not defined since $\mathrm{tp}(h) = \mathsf{Ax}_A$ and $|\mathsf{Ax}_A| = \emptyset$ for some A.

2. Clauses (a),(b),(c),(d) are motivated by the definition of $(\cdot)^\infty$ (Cf. Definition 4.3.7).

3. In Clause (f), we may suppose that if both $C \notin \Delta(\mathcal{I}_0)$ and $\neg C \notin \Delta(\mathcal{I}_1)$ hold, then $(\mathsf{R}_C h_0 h_1)^+$ is defined according to the first case:

$$(\mathsf{R}_C h_0 h_1)^+ := \mathcal{I}_0((\mathsf{R}_C h_{0i} h_1)_{i\in|\mathcal{I}_0|}).$$

   Moreover, note that if $C \in \Delta(\mathcal{I}_0)$, $\neg C \in \Delta(\mathcal{I}_1)$ and $\mathcal{I}_1 = \bigvee^k_{\neg C}$ hold, then $\mathcal{I}_0$ must be $\bigwedge_C$ because C and $\neg C$ are principal formulas of $\mathcal{I}_0$ and $\mathcal{I}_1$, respectively.

4. Now, we can explain the reason why we inserted Rep in Definition 4.3.7 when $h = \mathsf{Ind}^{y,n}_F h_0$. If we do not insert Rep, then we might define $\mathrm{tp}(h) := \mathrm{tp}(\mathsf{Cut}_{F(x/i)}(\mathsf{e}_n, h(y/i)^\infty)))$. However, this is not even recursive because $\mathrm{rk}(\mathsf{Cut}_{F(x/i)}(\mathsf{e}_n, h(y/i)^\infty))$ might be bigger than that of h.

5. In Clause (e), we insert Rep for making our definition primitive recursive. The reason is very similar to the last remark.

The next theorem says that the definition of $\mathrm{tp}(h)$ and $h[i]$ satisfies the desired properties: We can recover some useful information of a finitary derivation from the corresponding infinitary one primitive recursively.

Theorem 4.3.3 (Cf. Buchholz 1997, Theorem 3)**.** *For each closed Z^*-derivation h, the following holds:*

1. $\dfrac{\dots\Gamma(h[i])\dots(i \in |\mathrm{tp}(h)|)}{\Gamma(h)}\ \mathrm{tp}(h)$ *is a correct Z^∞-inference.*

2. *If* $\mathrm{tp}(h) = \mathsf{Cut}_C$*, then* $\mathrm{rk}(C) < \deg(h)$*.*

3. $\deg(h[i]) \leq \deg(h)$ *for all* $i \in |\mathrm{tp}(h)|$.

4. $\mathrm{o}(h[i]) \prec \mathrm{o}(h)$ *for all* $i \in |\mathrm{tp}(h)|$.

Proof. By induction on the length of h. To explain how the definitions of $\mathrm{tp}(h), h[i]$ work, we focus on the following cases. Other cases are treated in a similar way.

First, let $h = \bigwedge^{y}_{\forall x A(x)} h_0$. In this case, $\mathrm{tp}(h) = \bigwedge_{\forall x A(x)}$ and $h[i] = h_0(y/i)$. The clause (1) holds because $\Gamma(h[i]) = \Gamma, \forall x A(x), A(i)$ for every $i \in \mathbb{N}$ and $\Gamma(h) = \Gamma, \forall x A(x)$. Other clauses are proved easily.

Next, suppose that $h = \mathsf{R}_C h_0 h_1$ with $C \in \Delta(\mathrm{tp}(h_0))$ and $\neg C \in \Delta(\mathrm{tp}(h_1))$ and $C = \forall x A(x)$. From this it follows that $\mathrm{tp}(h_0) = \bigwedge_{\forall x A(x)}$, $\mathrm{tp}(h_1) = \bigvee^{k}_{\exists x \neg A(x)}$ and $\mathrm{tp}(h) = \mathsf{Cut}_{C[k]}$. This is one of the crucial cases of the cut-elimination in Z^∞. We show that

$$\frac{\Gamma(\mathsf{R}_C h_0[k] h_1) \quad \Gamma(\mathsf{R}_C h_0 h_1[0])}{\Gamma(\mathsf{R}_C h_0 h_1)} \; \mathsf{Cut}_{C[k]}$$

is a correct Z^∞-inference. By IH,

$$\frac{\ldots \Gamma(h_0[i]) \ldots (i \in \mathbb{N})}{\Gamma(h_0)} \; \bigwedge_{\forall x A(x)} \qquad \frac{\Gamma(h_1[0])}{\Gamma(h_1)} \; \bigvee^{k}_{\exists x \neg A(x)}$$

are correct Z^∞-inferences. Then, $\forall x A(x) \in \Gamma(h_0)$, $\exists x \neg A(x) \in \Gamma(h_1)$, $\Gamma(h_0[i]) \subseteq \Gamma(h_0) \cup \{A(x/i)\}$ and $\Gamma(h_1[0]) \subseteq \Gamma(h_1) \cup \{\neg A(x/k)\}$. Therefore, it follows that

$$\Gamma(\mathsf{R}_C h_0[k] h_1) \subseteq \Gamma(\mathsf{R}_C h_0 h_1) \cup \{C[k]\}, \; \Gamma(\mathsf{R}_C h_0 h_1[0]) \subseteq \Gamma(\mathsf{R}_C h_0 h_1) \cup \{(\neg C)[k]\}.$$

The clause (2) holds since $\mathrm{rk}(C[k]) < \mathrm{rk}(C) \leq \deg(h)$. It is easy to see that the clauses (3) hold.

Finally, let us see why the clause (4) holds. By Definition 4.3.4,

$$\mathrm{o}(h) = \mathrm{o}(h_0) \sharp \mathrm{o}(h_1).$$

On the other hand,

$$\mathrm{o}(h[0]) = \mathrm{o}(\mathsf{R}_C(h_0[k], h_1)) = \mathrm{o}(h_0[k]) \sharp \mathrm{o}(h_1).$$

Then, $\mathrm{o}(h[0]) \prec \mathrm{o}(h)$ holds since we have $\mathrm{o}(h_0[k]) \prec \mathrm{o}(h_0)$ by IH. In the same way, we see $\mathrm{o}(h[1]) \prec \mathrm{o}(h)$ holds.

As the last example, let $h = \mathsf{E} h_0$ with $\mathrm{tp}(h_0) = \mathsf{Cut}_C$. In this case, $\mathrm{tp}(h) = \mathsf{Rep}$ and $h[0] = \mathsf{R}_C(\mathsf{E} h_0[0], \mathsf{E} h_0[1])$. Now, $\Gamma(h[0]) = \Gamma(h)$. Note that

$\Gamma(h_0[0]) = \Gamma(h), C$ and $\Gamma(h_0[1]) = \Gamma(h), \neg C$. This proves the clause (1). The clause (2) and (3) are trivial. As to the clause (4), we compute

$$\mathrm{o}(h) = \omega^{\mathrm{o}(h_0)},$$

and

$$\mathrm{o}(h[0]) = \omega^{\mathrm{o}(h_0[0])} \sharp \omega^{\mathrm{o}(h_0[1])}.$$

Then, $\mathrm{o}(h[0]) \prec \mathrm{o}(h)$ holds since $\mathrm{o}(h_0) \prec \mathrm{o}(h_0[i])$ holds for $i \in \{0, 1\}$ by IH. □

4.4 Contentual and Formal Aspects of Gentzen's 1936 Proof

The aim of this section is to show that the 1936 proof is not only contentual but also formal and that contentual aspects of the 1936 proofs are formed by its formal aspects. First, we formulate and prove the main lemma of the 1936 proof, using *normalization trees*. Next, we argue that the 1936 proof is both contentual and formal, by showing that the main lemma of the 1936 proof implies both the main lemma of the 1935 proof and the main lemma of the 1938 proof. Finally, we explain that the 1936 proof's contentual aspects are formed by its formal aspects.

For every Z^*-derivation h, we use the function $\phi(h)$ in the same sense in [Buc97], i.e., $\phi(h)$ denote the Z-derivation obtained by deleting all E's in h. In addition, define

$$\mathsf{Z}_0^* := \{h \mid h \text{ is a } \mathsf{Z}^*\text{-derivation with } \deg(h) = 0\}.$$

In this section, we often consider only the derivation in Z_0^*, but this is not substantial restriction: We obtain from an arbitrary Z^*-derivation h a derivation $h' \in \mathsf{Z}_0^*$ with the same endsequent as h's by taking h' as

$$h' := \mathsf{E}^m h := \underbrace{\mathsf{E} \dots \mathsf{E}}_{m \text{ times}} h$$

with $\deg(h) = m$.

The following is our formulation of reduction steps of the 1936 proof.

Definition 4.4.1 (Reduction Steps of the 1936 Proof)**.** For every closed Z_0^*-derivation h and every infinite sequence $\langle h_n \rangle_{n \in \mathbb{N}}$ of closed Z_0^*-derivations, the predicate $RED(h, \langle h_n \rangle_{n \in \mathbb{N}})$ holds if and only if either

- $\mathrm{tp}(h) \neq \mathsf{Ax}_A$ for any A, and for all $i \in \mathbb{N}$,

  if $i \in |\mathrm{tp}(h)|$ then $h_i = h[i]$, otherwise $\mathrm{tp}(h_i) = \mathsf{Ax}_{A'}$ for some A',

  or

- $\mathrm{tp}(h) = \mathsf{Ax}_A$ for some A and $h_i = h$ for all i.

As suggested implicitly in [Ara02] and observed in [Buc15, Aki10], an application of the function $h[i]$ corresponds to an application of a reduction step of the 1936 proof. More precisely, for a closed Z_0^*-derivation h, the step $\phi(h) \mapsto \phi(h[i])$ is a 1936 reduction step to $\phi(h)$.[20] This motivates the above definition: If $RED(h, \langle h_n \rangle_{n\in\mathbb{N}})$ holds, then each of $\phi(h_i)$'s with $i \in |\mathrm{tp}(h)|$ is obtained by an application of a 1936 reduction step to $\phi(h)$. The cases of $i \notin |\mathrm{tp}(h)|$ are inessential.

For example, $RED(h, \langle h_n \rangle_{n\in\mathbb{N}})$ holds if h is a closed Z_0^*-derivation $\bigwedge^y_{\forall x A(x)} h'$ described diagrammatically as

$$\dfrac{\begin{matrix}\vdots\, h' \\ \Gamma, A(y)\end{matrix}}{\Gamma, \forall x A(x)}\ \bigwedge\nolimits^y_{\forall x A(x)}$$

and h_i is $h'(y/i)$ described as

$$\begin{matrix}\vdots\, h'(y/i) \\ \Gamma, A(i)\end{matrix}$$

for all $i \in \mathbb{N}$. Here, each $\phi(h_i)$ is obtained from $\phi(h)$ with an application of a reduction step defined in [Gen36, §14.23]. Another example is the following one: $RED(h, \langle h_n \rangle_{n\in\mathbb{N}})$ holds if h is a closed Z_0^*-derivation $\mathsf{E}(\mathsf{R}_{\forall x A(x)}(\bigwedge^y_{\forall x A(x)} h', \bigvee^k_{\exists x \neg A(x)} h''))$ described as

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\, h' \\ \Gamma, \forall x A(x), A(y)\end{matrix}}{\Gamma, \forall x A(x)}\ \bigwedge\nolimits^y_{\forall x A(x)} \qquad \dfrac{\begin{matrix}\vdots\, h'' \\ \Gamma, \exists x \neg A(x), \neg A(k)\end{matrix}}{\Gamma, \exists x \neg A(x)}\ \bigvee\nolimits^k_{\exists x \neg A(x)}}{\Gamma}\ \mathsf{R}_{\forall x A(x)}}{\Gamma}\ \mathsf{E}$$

and h_0 is

[20] To formulate reduction steps of the 1936 proof by means of finite notations for infinitary derivations, we need to insert sufficiently many E-rules into a given Z-derivation h. Moreover, we also need the function ϕ to delete the inserted E-rules, since, of course, the proof system Gentzen used to provide the 1936 proof does not include the E-rule.

$$\begin{aligned}\mathsf{R}_{A(k)}\{&\mathsf{E}(\mathsf{R}_{\forall x A(x)}((\textstyle\bigwedge^{y}_{\forall x A(x)} h')[k], \textstyle\bigvee^{k}_{\exists x \neg A(x)} h'')),\\ &\quad \mathsf{E}(\mathsf{R}_{\forall x A(x)}(\textstyle\bigwedge^{y}_{\forall x A(x)} h', (\textstyle\bigvee^{k}_{\exists x \neg A(x)} h'')[0]))\}\end{aligned}$$

described as

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\; h'(y/k)\\ \Gamma, \forall x A(x), A(k)\end{array} \quad \begin{array}{c}\vdots\\ \Gamma, \exists x \neg A(x)\end{array}}{\Gamma, A(k)}\,\mathsf{R}_{\forall x A(x)}}{\Gamma, A(k)}\,\mathsf{E} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma, \forall x A(x)\end{array} \quad \begin{array}{c}\vdots\; h''\\ \Gamma, \exists x \neg A(x), \neg A(k)\end{array}}{\Gamma, \neg A(k)}\,\mathsf{R}_{\forall x A(x)}}{\Gamma, \neg A(k)}\,\mathsf{E}}{\Gamma}\,\mathsf{R}_{A(k)}$$

and $h_i = \mathsf{Ax}_{\{0=0\}}$ for all $i \in \mathbb{N} \setminus \{0\}$. In this case, $\phi(h_0)$ is obtained from $\phi(h)$ with an application of the reduction step defined in [Gen36, §14.25].[21]

Let $\mathbb{N}^{<\omega}$ be the set of all finite sequences of natural numbers, and $\vec{u}, \vec{v}, \vec{w}$ be variables for elements of $\mathbb{N}^{<\omega}$. As is known, a primitive recursive coding of elements of $\mathbb{N}^{<\omega}$ as natural numbers can be given, so we treat functions of the domain $(\mathbb{N}^{<\omega})^n$ with $n > 0$ (resp. of the range $(\mathbb{N}^{<\omega})^n$) as functions of the domain $\mathbb{N}^n$ (resp. of the range $\mathbb{N}^n$). Let $\vec{u} * \vec{v}$ be a primitive recursive concatenation function from $(\mathbb{N}^{<\omega})^2$ to $\mathbb{N}^{<\omega}$. Infinite sequences of natural numbers are denoted by α. For an infinite sequence α of natural numbers, the initial segment $\bar{\alpha}(n)$ of length n is defined as

$$\bar{\alpha}(n) := \langle \alpha(0), \ldots, \alpha(n-1) \rangle.$$

We denote by T a function from $\mathbb{N}^{<\omega}$ to the set of all closed Z_0^*-derivations, and abbreviate the value $T(\vec{u})$ of T for $\vec{u}$ as $T_{\vec{u}}$.

Definition 4.4.2 (Local Correctness and Well-Foundedness on T)**.** For every T,

- T is *locally correct* if and only if for every $\vec{u} \in \mathbb{N}^{<\omega}$, $RED(T_{\vec{u}}, \langle T_{\vec{u}*\langle n \rangle} \rangle_{n \in \mathbb{N}})$ holds,

- T is *well-founded* if and only if

  for every infinite sequence α of natural numbers, there is a natural number n such that for some A, $\mathrm{tp}(T_{\bar{\alpha}(n)}) = \mathsf{Ax}_A$ holds.

[21]Strictly speaking, the above step is a special case of the reduction step that Gentzen formulated in [Gen36, §14.25], since he used not the cut-rule such as R_C but the "chain-rule (Kettenschluss)," one of whose instances is the cut-rule. Recently, Buchholz analyzed the chain-rule by means of finite notations for infinitary derivations ([Buc15]).

Roughly speaking, when T is locally correct, T is a tree growing up along reduction steps of the 1936 proof. If T is well-founded, then every branch of T always has a node $T_{\vec{u}}$ such that $\mathrm{tp}(T_{\vec{u}}) = \mathsf{Ax}_A$ holds for some A. Informally, this means that every branch of a well-founded T has a leaf node. Note that if a locally correct T has a node $T_{\vec{u}}$ such that $\mathrm{tp}(T_{\vec{u}}) = \mathsf{Ax}_A$ for some A, then $\mathrm{tp}(T_{\vec{u}*\vec{v}}) = \mathsf{Ax}_A$ for every $\vec{v}$. In other words, if a branch of a locally correct T has a leaf node, then each of the nodes above this leaf node is also a leaf node.

Now we define the main notion of this section.

Definition 4.4.3 (Normalization Trees)**.** For every T and every closed Z_0^*-derivation h, T is a *normalization tree* of h if and only if T is locally correct and $T_{\langle\rangle} = h$.

A typical example T of a normalization tree can be described as follows, if we omit the irrelevant nodes of T, i.e., the nodes $T_{\vec{u}*\langle i\rangle}$ such that $i \notin |\mathrm{tp}(T_{\vec{u}})|$.

$$\begin{tikzcd}[column sep=small]
\vdots & & \vdots & & \vdots & & \vdots & \\
h[0][0] & & h[0][1] & & h[1][0] & & h[1][1] & \\
& h[0] \arrow[ul] \arrow[ur] & & & & h[1] \arrow[ul] \arrow[ur] & & \cdots \\
& & & h \arrow[ull] \arrow[urr] & & & &
\end{tikzcd}$$

Lemma 4.4.1. *For every T, every $\vec{u} \in \mathbb{N}^{<\omega}$ and every formula C,* $\mathrm{tp}(T_{\vec{u}}) \neq \mathsf{Cut}_C$ *holds.*

Proof. Suppose that $\mathrm{tp}(T_{\vec{u}}) = \mathsf{Cut}_C$. By Theorem 4.3.3, $\mathrm{rk}(C) < \deg(T_{\vec{u}})$ holds, but this is impossible because $\deg(T_{\vec{u}}) = 0$ by definition. $\square$

The main lemma of the 1936 proof says that every reduction sequence formed by reduction steps of the 1936 proof terminates with a normal derivation. In our setting, the lemma says that for every closed Z_0^*-derivation h, there is a well-founded normalization tree of h. Note that the truth of the endsequent of a closed Z_0^*-derivation h with $\mathrm{tp}(h) = \mathsf{Ax}_A$ is verified in a primitive recursive way because $\Gamma(h) \cap \mathsf{TRUE}_0 \neq \emptyset$ by Theorem 4.3.3.(1), which is proved by structural induction on h. For the purpose of Section 4.5, we also see that every well-founded normalization tree defined in the proof below is a $< \varepsilon_0$-recursive function. For the definition of $< \varepsilon_0$-recursive functionals of level ≤ 2, see [Sch77, §4.1].

Proposition 4.4.1 (Main Lemma of the 1936 Proof)**.** *For every closed Z_0^*-derivation h, there is a well-founded normalization tree T of h such that T is a $< \varepsilon_0$-recursive function.*

Proof. If $\mathrm{tp}(h) = \mathsf{Ax}_A$ for some A, then define $T_{\vec{u}} := h$ for every $\vec{u}$. It is obvious that T is both a well-founded normalization tree of h and a primitive recursive function.

Consider the case that $\mathrm{tp}(h) \neq \mathsf{Ax}_A$ for any A. We use transfinite induction up to ε_0. Assume that a well-founded normalization tree of h' is defined for every closed Z_0^*-derivation h' with $\mathrm{o}(h') \prec \mathrm{o}(h) \prec \varepsilon_0$. By Theorem 4.3.3, there is a well-founded normalization tree $T^{h[i]}$ of $h[i]$ for every $i \in |\mathrm{tp}(h)|$. Set the tree T^{Ax} as

$$T^{\mathsf{Ax}}(\vec{u}) := \mathsf{Ax}_{\{0=0\}} \text{ for every } \vec{u}.$$

Define T as follows:

$$T(\vec{u}) := \begin{cases} h, & \text{if } \vec{u} = \langle\rangle, \\ T^{h[i]}(\vec{v}), & \text{if } \vec{u} = \langle i \rangle * \vec{v} \text{ and } i \in |\mathrm{tp}(h)|, \\ T^{\mathsf{Ax}}(\vec{v}), & \text{if } \vec{u} = \langle i \rangle * \vec{v} \text{ and } i \notin |\mathrm{tp}(h)|. \end{cases}$$

Then, T is a well-founded normalization tree of h. Moreover, it can be shown that T is a $< \varepsilon_0$-recursive function, because $\mathrm{tp}(h)$ and $h[i]$ are primitive recursive. □

Next, we show that Proposition 4.4.1 implies both the main lemma of the 1935 proof and the main lemma of the 1938 proof. Let us start with our definition for reduction steps of the 1935 proof.

Definition 4.4.4 (Reduction Steps of the 1935 Proof)**.** For every closed sequent Γ and every infinite sequence $\langle \Gamma_n \rangle_{n \in \mathbb{N}}$ of closed sequents, the predicate $RED^{sq}(\Gamma, \langle \Gamma_n \rangle_{n \in \mathbb{N}})$ holds if and only if there is a formula $C \in \Gamma$ such that

1. $C \in \mathsf{TRUE}_0$ and $\Gamma_n = \Gamma$ for every $n \in \mathbb{N}$, or

2. C is of the form $\forall x A(x)$ and $\Gamma_n \subseteq \Gamma \cup \{A(n)\}$ for every $n \in \mathbb{N}$, or

3. C is of the form $A_0 \wedge A_1$ and $\Gamma_i \subseteq \Gamma \cup \{A_i\}$ for every $i \in \{0, 1\}$, or

4. C is of the form $\exists x A(x)$ and $\Gamma_0 \subseteq \Gamma \cup \{A(k)\}$ for some $k \in \mathbb{N}$, or

5. C is of the form $A_0 \vee A_1$ and $\Gamma_0 \subseteq \Gamma \cup \{A_i\}$ for some $i \in \{0, 1\}$.

As Definition 4.4.1, if $RED^{sq}(\Gamma, \langle \Gamma_n \rangle_{n \in \mathbb{N}})$ holds, then each of Γ_i's is obtained from Γ with an application of a reduction step of the 1935 proof. A typical example is as follows: $RED^{sq}(\Gamma, \langle \Gamma_n \rangle_{n \in \mathbb{N}})$ holds whenever $\Gamma = \{\forall x(x = x)\}$ and $\Gamma_n = \{\forall x(x = x), n = n\}$ hold. Here each of Γ_i's is obtained from Γ with an application of the reduction step being defined in [Gen74, §13.21]. This example can be described as follows.

$$\begin{tikzcd}
\forall x(x=x), 0=0 & \forall x(x=x), 1=1 & \forall x(x=x), 2=2 & \forall x(x=x), 3=3 & \cdots \\
 & & \forall x(x=x) \arrow[ull] \arrow[ul] \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

We denote by $\mathcal{T}$ a function from $\mathbb{N}^{<\omega}$ to the set of all closed sequents, and often abbreviate $\mathcal{T}(\vec{u})$ as $\mathcal{T}_{\vec{u}}$. As the case of the 1936 proof, we define the local correctness and well-foundedness on $\mathcal{T}$.

Definition 4.4.5 (Local Correctness and Well-Foundedness on $\mathcal{T}$)**.** For every $\mathcal{T}$,

- $\mathcal{T}$ is *locally correct* if and only if for every $\vec{u} \in \mathbb{N}^{<\omega}$, $RED^{sq}(\mathcal{T}_{\vec{u}}, \langle \mathcal{T}_{\vec{u} * \langle n \rangle} \rangle_{n \in \mathbb{N}})$ holds.

- $\mathcal{T}$ is *well-founded* if and only if for every infinite sequence α of natural numbers, there is a natural number n such that $\mathcal{T}_{\bar{\alpha}(n)} \cap \mathsf{TRUE}_0 \neq \emptyset$.

The main notion of the 1935 proof is the notion of *reduction procedures*, as we have seen in Section 4.2. The following is its definition in our setting.

Definition 4.4.6 (Reduction Procedures)**.** We say $\mathcal{T}$ is a *reduction procedure* for Γ if and only if

- $\mathcal{T}_{\langle\rangle} = \Gamma$,

- $\mathcal{T}$ is locally correct and well-founded.

A reduction procedure is a tree growing up along reduction steps of the 1935 proof, every branch of which always has a node $\mathcal{T}_{\vec{u}}$ that contains at least one element of TRUE_0.

We derive the main lemma of the 1935 proof from Proposition 4.4.1 by means of *monotone bar induction*, which we used in Section 2.5.

Monotone Bar Induction. Let B and P be predicates on $\mathbb{N}^{<\omega}$. If the following four conditions hold:

1. for every infinite sequence α of natural numbers, there is a natual number n such that $B(\bar{\alpha}(n))$ holds,

2. for every $\vec{u}$ and $\vec{v}$, if $B(\vec{u})$ holds then $B(\vec{u} * \vec{v})$ holds,
3. for every $\vec{u}$, if $B(\vec{u})$ holds then $P(\vec{u})$ holds,
4. for every $\vec{u}$, if $P(\vec{u} * \langle n \rangle)$ holds for all n then $P(\vec{u})$ holds,

then $P(\langle\rangle)$ holds.

The idea of our proof for the main lemma of the 1935 proof is as follows.[22] Given a closed Z_0^*-derivation h, a reduction procedure for $\Gamma(h)$ is obtained from a normalization tree T of h by extracting the endsequent of $T_{\vec{u}}$ from $T_{\vec{u}}$ for every $\vec{u}$.

Lemma 4.4.2. *For every closed Z_0^*-derivation h, if there is a well-founded normalization tree of h, then there is also a reduction procedure for $\Gamma(h)$.*

Proof. By monotone bar induction (MBI). Assume that there is a well-founded normalization tree T of h, and define the predicates B and P as follows:

$B(\vec{u})$ if and only if $\mathrm{tp}(T_{\vec{u}}) = \mathsf{Ax}_A$ for some A,

$P(\vec{u})$ if and only if there is a reduction procedure for $\Gamma(T_{\vec{u}})$.

Then, Premise 1 of MBI holds by the well-foundedness of T, and Premise 2 of MBI holds by the local correctness of T.

Next, we show that Premise 3 of MBI holds. If $B(\vec{u})$ holds, i.e., $\mathrm{tp}(T_{\vec{u}}) = \mathsf{Ax}_A$ for some A, then

$$\frac{}{\Gamma(T_{\vec{u}})}\ \mathsf{Ax}_A$$

is a correct Z^∞-inference by Theorem 4.3.3, so $\Gamma(T_{\vec{u}}) \cap \mathsf{TRUE}_0 \neq \emptyset$. Thus, the tree $\mathcal{T}$ being defined as $\mathcal{T}(\vec{v}) := \Gamma(T_{\vec{u}})$ for every $\vec{v}$ is a reduction procedure for $\Gamma(T_{\vec{u}})$.

Finally, we show that Premise 4 of MBI holds. Assume that $P(\vec{u} * \langle n \rangle)$ holds for every n, i.e., there is a reduction procedure $\mathcal{T}^n$ for each $\Gamma(T_{\vec{u} * \langle n \rangle})$.

Consider the case that $\mathrm{tp}(T_{\vec{u}}) = \mathsf{Rep}$. Then, by Theorem 4.3.3,

$$\frac{\Gamma(T_{\vec{u} * \langle 0 \rangle})}{\Gamma(T_{\vec{u}})}\ \mathsf{Rep}$$

is a correct Z^∞-inference, so $\mathcal{T}^0$ is also a reduction procedure for $\Gamma(T_{\vec{u}})$.

[22] One can read off this idea in Gentzen's explanation for reduction steps of the 1936 steps. See [Gen36, §14.2].

Consider in turn the case that $\mathrm{tp}(T_{\vec{u}}) = \bigwedge_{A_0 \wedge A_1}$. Again, by Theorem 4.3.3,

$$\frac{\Gamma(T_{\vec{u}*\langle 0\rangle}) \quad \Gamma(T_{\vec{u}*\langle 1\rangle})}{\Gamma(T_{\vec{u}})} \bigwedge_{A_0 \wedge A_1}$$

is a correct Z^∞-inference. Set the tree $\mathcal{T}^{\mathsf{Ax}}$ as

$$\mathcal{T}^{\mathsf{Ax}}(\vec{v}) := \{0 = 0\} \text{ for every } \vec{v}.$$

Then, define $\mathcal{T}$ as follows:

$$\mathcal{T}(\vec{v}) := \begin{cases} \Gamma(T_{\vec{u}}), & \text{if } \vec{v} = \langle\rangle, \\ \mathcal{T}^i(\vec{w}), & \text{if } \vec{v} = \langle i \rangle * \vec{w} \text{ and } i \in \{0,1\}, \\ \mathcal{T}^{\mathsf{Ax}}(\vec{w}), & \text{if } \vec{v} = \langle i \rangle * \vec{w} \text{ and } i \notin \{0,1\}. \end{cases}$$

It is obvious that $\mathcal{T}$ is a reduction procedure for $\Gamma(T_{\vec{u}})$. The remaining cases are similar to this case. Note that $\mathrm{tp}(T_{\vec{u}}) \neq \mathsf{Cut}_C$ for any C by Lemma 4.4.1.

Therefore, $P(\langle\rangle)$ holds by MBI, so there is a reduction procedure for $\Gamma(T_{\langle\rangle}) = \Gamma(h)$. $\square$

Proposition 4.4.2 (Main Lemma of the 1935 Proof)**.** *For every closed* Z_0^*-*derivation* h*, there is a reduction procedure for* $\Gamma(h)$.

Proof. By Proposition 4.4.1 and Lemma 4.4.2. $\square$

The above proof of Proposition 4.4.2 shows that the 1936 proof is a contentual correctness proof. Let h be an arbitrary Z-derivation. For the sake of simplicity, we assume that h is closed. A Z_0^*-derivation h' with the same endsequent as h's is obtained from h by inserting sufficiently many E-rules into the bottom of h. By Proposition 4.4.1, namely, the main lemma of the 1936 proof, there is a well-founded normalization tree T of h'. Then, by Lemma 4.4.2, we can extract a reduction procedure $\mathcal{T}$ for $\Gamma(h') = \Gamma(h)$ from T with a proof for the well-foundedness of $\mathcal{T}$.

We define the statability of a reduction procedure as follows: For every reduction procedure $\mathcal{T}'$ for Γ, $\mathcal{T}'$ is *statable* for Γ if and only if both $\mathcal{T}'$ and a proof for its well-foundedness are obtained. Therefore, by the argument above, a reduction procedure is statable for $\Gamma(h)$. In other words, $\Gamma(h)$ is correct in the sense of (GI). Note that we avoided the use of the principle of the excluded middle for non-decidable predicates in our arguments.[23] In this way, the main lemma of the 1936 proof verifies that every Z-derivation has the correct endsequent, so this proof is contentual.

[23]In the future works, we will investigate the exact strength of Gentzen's finitist standpoint and examine whether our argument can be formalized in this standpoint.

For our claim that the main lemma of the 1936 proof implies the main lemma of the 1938 proof, the following observation by Buchholz ([Buc97]) is crucial: Let h be a closed Z_0^*-derivation for a numeric equation $n = m$, then the step $\phi(h) \mapsto \phi(h[0])$ is a main reduction step of the 1938 proof.[24] For example, the following step, which is called a "operational reduction (Verknüpfungs-Reduktion)," is one of the main reduction steps of the 1938 proof.[25]

$$
\begin{prooftree}
\AxiomC{$\vdots\ h'$}
\noLine
\UnaryInfC{$\Delta_0, \forall x A(x), A(y)$}
\RightLabel{$\bigwedge^y_{\forall x A(x)}$}
\UnaryInfC{$\Delta_0, \forall x A(x)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma', \forall x A(x)$}
\AxiomC{$\vdots\ h''$}
\noLine
\UnaryInfC{$\Delta_1, \exists x \neg A(x), \neg A(k)$}
\RightLabel{$\bigvee^k_{\exists x \neg A(x)}$}
\UnaryInfC{$\Delta_1, \exists x \neg A(x)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma', \exists x \neg A(x)$}
\RightLabel{$\mathsf{R}_{\forall x A(x)}$}
\BinaryInfC{Γ'}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{Γ}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$n = m$}
\end{prooftree}
\qquad \triangleright
$$

$$
\begin{prooftree}
\AxiomC{$\vdots\ h'(y/k)$}
\noLine
\UnaryInfC{$\Delta_0, A(k), \forall x A(x)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma', A(k), \forall x A(x)$}
\AxiomC{$\vdots\ h''$}
\noLine
\UnaryInfC{$\Delta_1, \exists x \neg A(x), \neg A(k)$}
\RightLabel{$\bigvee^k_{\exists x \neg A(x)}$}
\UnaryInfC{$\Delta_1, \exists x \neg A(x)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma', \exists x \neg A(x)$}
\RightLabel{$\mathsf{R}_{\forall x A(x)}$}
\BinaryInfC{$\Gamma', A(k)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma, A(k)$}
\AxiomC{$\vdots\ h'$}
\noLine
\UnaryInfC{$\Delta_0, \forall x A(x), A(y)$}
\RightLabel{$\bigwedge^y_{\forall x A(x)}$}
\UnaryInfC{$\Delta_0, \forall x A(x)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma', \forall x A(x)$}
\AxiomC{$\vdots\ h''$}
\noLine
\UnaryInfC{$\Delta_1, \neg A(k), \exists x \neg A(x)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma', \neg A(k), \exists x \neg A(x)$}
\RightLabel{$\mathsf{R}_{\forall x A(x)}$}
\BinaryInfC{$\Gamma', \neg A(k)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma, \neg A(k)$}
\RightLabel{$\mathsf{R}_{A(k)}$}
\BinaryInfC{Γ}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$n = m$}
\end{prooftree}
$$

Note that all 1938 reduction steps have no branching, in contrast to reduction steps of the 1935 proof and the 1936 proof. In addition, Gentzen introduced the notion of an *end-piece* (Endstück) of a derivation to formulate reduction steps of the 1938 proof.[26] For our purpose, it suffices to explain the notion of an end-piece informally: The end-piece of a Z^*-derivation h consists of the sequents that we encounter when we ascend from the endsequent of h and stop as soon as we arrive at the lower sequent of an inference symbol $\mathcal{I}$ of the form $\bigwedge_A$, $\bigwedge^y_A$ or $\bigvee^k_A$. For example, the end-pieces of the following

[24]Here, we slightly changed Buchholz's actual observation. The actual observation is that for a supposed Z_0^*-derivation h of the *empty sequent* $\emptyset$, the step $\phi(h) \mapsto \phi(h[0])$ is a main reduction step of the 1938 proof.

[25]For the operational reductions, see [Gen38b, §3.5].

[26]Cf. [Gen38b, §3.2]. In [Gen69], Szabo translated "Endstück" as "ending".

Z^*-derivations

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, A(y), \forall x A(x)}}{\Gamma, \forall x A(x)} \bigwedge^{y}_{\forall x A(x)} \quad \dfrac{\dfrac{\vdots}{\Gamma, \neg A(k), \exists x \neg A(x)}}{\Gamma, \exists x \neg A(x)} \bigvee^{k}_{\exists x \neg A(x)}}{\dfrac{\Gamma}{\Gamma}\ \mathsf{E}} \mathsf{R}_{\forall x A(x)} \qquad \dfrac{\overline{\Delta, 0=0}\ \mathsf{Ax}_{\{\Delta, 0=0\}} \quad \overline{\Delta, 0\neq 0}\ \mathsf{Ax}_{\{\Delta, 0\neq 0\}}}{\Delta} \mathsf{R}_{0=0}$$

are

$$\dfrac{\Gamma, \forall x A(x) \quad \Gamma, \exists x \neg A(x)}{\dfrac{\Gamma}{\Gamma}\ \mathsf{E}} \mathsf{R}_{\forall x A(x)} \qquad \dfrac{\overline{\Delta, 0=0}\ \mathsf{Ax}_{\{\Delta, 0=0\}} \quad \overline{\Delta, 0\neq 0}\ \mathsf{Ax}_{\{\Delta, 0\neq 0\}}}{\Delta} \mathsf{R}_{0=0}$$

respectively.

As seen above, a step $\phi(h) \mapsto \phi(h[i])$ for an *arbitrary* closed Z_0^*-derivation h is a reduction step of the 1936 proof. Thus, one can consider reduction steps of the 1938 proof to be *special cases* of reduction steps of the 1936 proof. That is to say, each of reduction step of the 1938 proof is nothing but a 1936 reduction step to some closed Z-derivation of a *numeric equation*. On the basis of this fact, one can see that the main lemma of the 1938 proof is also a special case of the main lemma of the 1936 proof. This means that the 1936 proof assigns a normal derivation to every derivable numeric equation in first-order arithmetic as the 1938 proof does. In our setting, the main lemma of the 1938 proof is formulated as the following proposition, which is a special case of Proposition 4.4.1, namely, the main lemma of the 1936 proof.

Proposition 4.4.3 (Main Lemma of the 1938 Proof)**.** *For every closed Z_0^*-derivation h with the endsequent of the form $n = m$, there is a well-founded normalization tree T of h.*

In the remaining part of this section, we aim to justify the claim that the main lemma of the 1938 proof can be formulated as above. To achieve this aim, we explain the relationship between normalization trees and reduction steps of the 1938 proof in detail: We explain that each well-founded normalization tree for some closed Z_0^*-derivation with a numeric conclusion corresponds to consecutive applications of 1938 reduction steps.

Let us begin with the definition for the notion of *nominal forms*, which Buchholz introduced to verify his observation we have seen before.[27]

[27] Cf. [Buc97]. Buchholz defined also the function $\mathsf{hgt}^*(\mathfrak{a})$ of a nominal form $\mathfrak{a}$, to describe the ordinal assignment in the 1938 proof by means of finite notations for infinitary derivations. Here, we do not need to describe it, so we omit the definition of this function.

Definition 4.4.7 (Nominal Forms). *Nominal forms* are defined as follows:

1. $*$ is a nominal form and $\mathsf{Cut}(*) := \emptyset$,

2. if $\mathfrak{a}$ is a nominal form and h is a Z^*-derivation, then $\mathsf{E}^m\mathsf{R}_C\mathfrak{a}h$ and $\mathsf{E}^m\mathsf{R}_C h\mathfrak{a}$ are nominal forms, and

$$\mathsf{Cut}(\mathsf{E}^m\mathsf{R}_C\mathfrak{a}h) := \mathsf{Cut}(\mathfrak{a}) \cup \{C\}, \quad \mathsf{Cut}(\mathsf{E}^m\mathsf{R}_C h\mathfrak{a}) := \mathsf{Cut}(\mathfrak{a}) \cup \{\neg C\}.$$

We denote a nominal form by $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$. When q is a nominal form or a Z^*-derivation, $\mathfrak{a}\{q\}$ is the result of substituting q for $*$ in $\mathfrak{a}$. For example, the nominal form

$$\mathfrak{a} = \mathsf{E}^m(\mathsf{R}_{C_0}(\mathsf{E}^n(\mathsf{R}_{C_1}(*, h_1)), h_0))$$

is of the following figure:

$$
\begin{array}{c}
\dfrac{\begin{array}{c}
\dfrac{\dfrac{\dfrac{* \qquad \begin{array}{c}\vdots\; h_1 \\ \neg C_1, \Gamma, C_0\end{array}}{\Gamma, C_0}\,\mathsf{R}_{C_1}}{\Gamma, C_0}\,\mathsf{E}}{\Gamma, C_0}\,\mathsf{E} \\
\vdots \\
\Gamma, C_0
\end{array}
\qquad\qquad
\begin{array}{c}\vdots\; h_0 \\ \neg C_0, \Gamma\end{array}}{\dfrac{\dfrac{\Gamma}{\Gamma}\,\mathsf{E}}{\Gamma}\,\mathsf{E}}\,\mathsf{R}_{C_0} \\
\vdots \\
\Gamma
\end{array}
$$

It is easily observed that for every nominal form $\mathfrak{a}$ and every Z^*-derivation h, the endsequent of h is included in the end-piece of $\mathfrak{a}\{h\}$. This observation is crucial for our argument below.

For readers' convenience, we carry out the details of the proofs of Lemma 4.4.3, Proposition 4.4.4 and Proposition 4.4.5. We suggest a reader being familiar with [Buc97] or not being interested in the details to skip them.

Lemma 4.4.3 (cf. Buchholz 1997, Lemma 3). *Let h be a closed Z^*-derivation.*

1. *If* $\mathrm{tp}(h) = \mathsf{Rep}$, *then there are* $\mathfrak{a}, h_0, h_1$ *such that* $h = \mathfrak{a}\{h_0\}$, $h[0] = \mathfrak{a}\{h_0[0]\}$ *and*

   either $h_0 = \mathsf{E}^m\mathsf{Ind}_F^{y,t}h_1$ *or*

   $h_0 = \mathsf{E}^{m+1}h_1$ *and* $\mathrm{tp}(h_1) = \mathsf{Cut}_B$ *for some* B.

2. *If* $\mathrm{tp}(h) = \mathsf{Cut}_B$, *then there are* $\mathfrak{a}, C, h_0, h_1$ *such that* $h = \mathfrak{a}\{\mathsf{R}_C h_0 h_1\}$, $h[i] = \mathfrak{a}\{(\mathsf{R}_C h_0 h_1)[i]\}$ *and either*

   (a) $\mathrm{tp}(h_0) = \bigwedge_C$ & $\mathrm{tp}(h_1) = \bigvee^k_{\neg C}$ & $B = C[k]$ *or*

   (b) $\mathrm{tp}(h_0) = \bigvee^k_C$ & $\mathrm{tp}(h_1) = \bigwedge_{\neg C}$ & $B = C[k]$.

3. *If* $\mathrm{tp}(h) = \bigwedge_{\forall x A(x)}$, *then there are* $\mathfrak{a}, h_0$ *such that* $\forall x A(x) \notin \mathsf{Cut}(\mathfrak{a})$, $h = \mathfrak{a}\{\bigwedge^y_{\forall x A(x)} h_0\}$ *and* $h[i] = \mathfrak{a}\{h_0(y/i)\}$.

4. *If* $\mathrm{tp}(h) = \bigwedge_{A_0 \wedge A_1}$, *then there are* $\mathfrak{a}, h_0, h_1$ *such that* $A_0 \wedge A_1 \notin \mathsf{Cut}(\mathfrak{a})$, $h = \mathfrak{a}\{\bigwedge_C h_0 h_1\}$ *and* $h[i] = \mathfrak{a}\{h_i\}$.

5. *If* $\mathrm{tp}(h) = \bigvee^k_C$, *then there are* $\mathfrak{a}, h_0$ *such that* $C \notin \mathsf{Cut}(\mathfrak{a})$, $h = \mathfrak{a}\{\bigvee^k_C h_0\}$ *and*

   $h[0] = \mathfrak{a}\{h_0\}$.

6. *If* $\mathrm{tp}(h) = \mathsf{Ax}_A$, *then there is a nominal form* $\mathfrak{a}$ *such that* $A \notin \mathsf{Cut}(\mathfrak{a})$ *and* $h = \mathfrak{a}\{\mathsf{Ax}_\Delta\}$ *with* $A \in \Delta$.

Proof. By induction on the length of h.

(1) Since $\mathrm{tp}(h) = \mathsf{Rep}$, h is of either the form $\mathsf{E}^m \mathsf{Ind}^{y,t}_F h'$ or $\mathsf{E}^n h''$ with $\mathrm{last}(h'') \neq \mathsf{E}, \mathsf{Ind}$. If $h = \mathsf{E}^m \mathsf{Ind}^{y,t}_F h'$, then take $\mathfrak{a}$ as $*$, h as h_0 and h' as h_1.

Consider the case that $h = \mathsf{E}^n h''$ with $\mathrm{last}(h'') \neq \mathsf{E}, \mathsf{Ind}$. If $\mathrm{tp}(h'') = \mathsf{Cut}_B$ for some B, then $n > 0$ holds, so take $*$ as $\mathfrak{a}$, h as h_0 and h'' as h_1. If $\mathrm{tp}(h'') \neq \mathsf{Cut}_B$ for any B, then $h'' = \mathsf{R}_C h'_0 h'_1$ and $\mathrm{tp}(h'') = \mathrm{tp}(h'_i) = \mathsf{Rep}$ for some $i \in \{0, 1\}$ by the definition of tp. We consider only the case that $i = 0$, since the other case is similar. By IH, there are $\mathfrak{b}, h''_0, h''_1$ such that $h'_0 = \mathfrak{b}\{h''_0\}$, $h'_0[0] = \mathfrak{b}\{h''_0[0]\}$ and either $h''_0 = \mathsf{E}^m \mathsf{Ind}^{y,t}_F h''_1$ or ($h''_0 = \mathsf{E}^{m+1} h''_1$ & $\mathrm{tp}(h''_1) = \mathsf{Cut}_B$ for some B). Take $\mathsf{E}^n \mathsf{R}_C \mathfrak{b} h'_1$ as $\mathfrak{a}$, h''_0 as h_0 and h''_1 as h_1. Then,

$$h = \mathsf{E}^n \mathsf{R}_C(h'_0, h'_1) = \mathsf{E}^n \mathsf{R}_C(\mathfrak{b}\{h''_0\}, h'_1) = \mathfrak{a}\{h''_0\}$$

and

$$h[0] = (\mathsf{E}^n \mathsf{R}_C(h'_0, h'_1))[0] = \mathsf{E}^n \mathsf{R}_C(h'_0[0], h'_1) = \mathsf{E}^n \mathsf{R}_C(\mathfrak{b}\{h''_0[0]\}, h'_1) = \mathfrak{a}\{h''_0[0]\}.$$

(2) Assume that $\mathrm{tp}(h) = \mathsf{Cut}_B$. Then, $h = \mathsf{R}_C h'_0 h'_1$ and either

$\mathrm{tp}(h'_0) = \bigwedge_C$, $\mathrm{tp}(h'_1) = \bigvee^k_{\neg C}$, $B = C[k]$, or

$$\mathrm{tp}(h'_0) = \bigvee^k_C,\ \mathrm{tp}(h'_1) = \bigwedge_{\neg C},\ B = C[k], \text{ or}$$

$$\mathrm{tp}(h'_0) = \mathsf{Cut}_B,\ h[i] = \mathsf{R}_C(h'_0[i], h'_1), \text{ or}$$

$$\mathrm{tp}(h'_1) = \mathsf{Cut}_B,\ h[i] = \mathsf{R}_C(h'_0, h'_1[i]).$$

In the first two cases, take $*$ as $\mathfrak{a}$, h'_0 as h_0 and h'_1 as h_1. Below we consider the third case and omit the fourth case, since the latter is similar to the former.

Assume that $\mathrm{tp}(h'_0) = \mathsf{Cut}_B,\ h[i] = \mathsf{R}_C(h'_0[i], h'_1)$. Then, by IH, there are $\mathfrak{b}, D, h''_0, h''_1$ such that $h'_0 = \mathfrak{b}\{\mathsf{R}_D h''_0 h''_1\}$, $h'_0[i] = \mathfrak{b}\{(\mathsf{R}_D h''_0 h''_1)[i]\}$ and either

(a) $\mathrm{tp}(h''_0) = \bigwedge_D$ & $\mathrm{tp}(h''_1) = \bigvee^k_{\neg D}$ & $B = D[k]$ or

(b) $\mathrm{tp}(h''_0) = \bigvee^k_D$ & $\mathrm{tp}(h''_1) = \bigwedge_{\neg D}$ & $B = D[k]$.

Take $\mathsf{R}_C \mathfrak{b} h_1$ as $\mathfrak{a}$, h''_0 as h_0 and h''_1 as h_1. Then,

$$h = \mathsf{R}_C h'_0 h'_1 = \mathsf{R}_C(\mathfrak{b}\{\mathsf{R}_D h''_0 h''_1\}, h'_1) = \mathfrak{a}\{\mathsf{R}_D h''_0 h''_1\}$$

and

$$h[i] = \mathsf{R}_C(h'_0[i], h'_1) = \mathsf{R}_C(\mathfrak{b}\{(\mathsf{R}_D h''_0 h''_1)[i]\}, h'_1) = \mathfrak{a}\{(\mathsf{R}_D h''_0 h''_1)[i]\}.$$

(3) By the definition of $\mathrm{tp}(h)$, either $h = \bigwedge^y_{\forall x A(x)} h'$ or $h = \mathsf{E}^n \mathsf{R}_D h'_0 h'_1$ with $\mathrm{tp}(h'_i) = \bigwedge_{\forall x A(x)}$ for some $i \in \{0, 1\}$. In the former case, take $*$ as $\mathfrak{a}$ and h'_0 as h_0.

In the latter case, we may assume without the loss of generality that $i = 0$, then $D \neq \forall x A(x)$ by the definition of $\mathrm{tp}(h'_i)$. By IH, there are $\mathfrak{b}, h''_0$ such that $\forall x A(x) \notin \mathsf{Cut}(\mathfrak{b})$, $h'_0 = \mathfrak{b}\{\bigwedge^y_{\forall x A(x)} h''_0\}$ and $h'_0[i] = \mathfrak{b}\{h''_0(y/i)\}$. Take $\mathsf{E}^n \mathsf{R}_D \mathfrak{b} h'_1$ as $\mathfrak{a}$ and h''_0 as h_0, then $\forall x A(x) \notin \mathsf{Cut}(\mathfrak{a})$ and $h = \mathsf{E}^n \mathsf{R}_D(h'_0, h'_1) = \mathsf{E}^n \mathsf{R}_D(\mathfrak{b}\{\bigwedge^y_{\forall x A(x)} h''_0\}, h'_1) = \mathfrak{a}\{\bigwedge^y_{\forall x A(x)} h''_0\}$. Moreover,

$$h[i] = \mathsf{E}^n \mathsf{R}_D(h'_0[i], h'_1) = \mathsf{E}^n \mathsf{R}_D(\mathfrak{b}\{h''_0(y/i)\}, h'_1) = \mathfrak{a}\{h''_0(y/i)\}.$$

In the cases of (4), (5) and (6), we can prove the assertion in the similar way to the case of (3). □

We set

$$\mathbf{0}^n := \begin{cases} \langle \underbrace{0, 0, \ldots, 0}_{n \text{ times}} \rangle, & \text{if } n > 0, \\ \langle\rangle, & \text{else.} \end{cases}$$

Note that $T_{\mathbf{0}^k}$ denotes the node of T labeled by $\mathbf{0}^k$.

Proposition 4.4.4 (cf. Buchholz 1997, Theorem 5). *Let T be a normalization tree of some closed $h \in \mathsf{Z}_0^*$ with the endsequent of the form $n = m$. For every $k \in \mathbb{N}$, it holds that* $\mathrm{tp}(T_{\mathbf{0}^k}) = \mathsf{Ax}_{n=m}$ *or* Rep, *and*

1. if $\mathrm{tp}(T_{\mathbf{0}^k}) = \mathsf{Ax}_{n=m}$, *then there is a nominal form* $\mathfrak{a}$ *such that* $n = m \notin \mathsf{Cut}(\mathfrak{a})$ *and* $T_{\mathbf{0}^k} = \mathfrak{a}\{\mathsf{Ax}_\Delta\}$ *with* $n = m \in \Delta$, *and*

2. if $\mathrm{tp}(T_{\mathbf{0}^k}) = \mathsf{Rep}$, *then one of the following statements holds for some nominal forms* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}_0, \mathfrak{c}_1$. *(The symmetric cases are suppressed.)*

- *(a)* $T_{\mathbf{0}^k} = \mathfrak{a}\{\mathsf{E}^m \mathsf{Ind}_F^{y,t} h_0\}$ *and* $T_{\mathbf{0}^{k+1}} = \mathfrak{a}\{\mathsf{E}^m((\mathsf{Ind}_F^{y,t} h_0)[0])\}$.
- *(b)* $T_{\mathbf{0}^k} = \mathfrak{a}\{\mathsf{E}^{m+1}\mathfrak{b}\{\mathsf{R}_C(\mathfrak{c}_0\{h_0'\}, \mathfrak{c}_1\{h_1'\})\}\}$ *and*
  $T_{\mathbf{0}^{k+1}} = \mathfrak{a}\{\mathsf{E}^m \mathsf{R}_{C[n]}(\mathsf{E}\mathfrak{b}\{\mathsf{R}_C(\mathfrak{c}_0\{h_0(y/n)\}, \mathfrak{c}_1\{h_1'\})\}, \mathsf{E}\mathfrak{b}\{\mathsf{R}_C(\mathfrak{c}_0\{h_0'\}, \mathfrak{c}_1\{h_1\})\})\}$,
  where $C = \forall x A(x)$, $C \notin \mathsf{Cut}(\mathfrak{c}_0)$, $\neg C \notin \mathsf{Cut}(\mathfrak{c}_1)$, $h_0' = \bigwedge_C^y h_0$ *and* $h_1' = \bigvee_{\neg C}^n h_1$.
- *(c)* $T_{\mathbf{0}^k} = \mathfrak{a}\{\mathsf{E}^{m+1}\mathfrak{b}\{\mathsf{R}_C(\mathfrak{c}_0\{h_0\}, \mathfrak{c}_1\{h_1\})\}\}$ *and*
  $T_{\mathbf{0}^{k+1}} = \mathfrak{a}\{\mathsf{E}^m \mathsf{R}_{C[i]}(\mathsf{E}\mathfrak{b}\{\mathsf{R}_C(\mathfrak{c}_0\{h_{0i}\}, \mathfrak{c}_1\{h_1\})\}, \mathsf{E}\mathfrak{b}\{\mathsf{R}_C(\mathfrak{c}_0\{h_0\}, \mathfrak{c}_1\{h_{10}\})\})\}$,
  where $C = A_0 \wedge A_1$, $C \notin \mathsf{Cut}(\mathfrak{c}_0)$, $\neg C \notin \mathsf{Cut}(\mathfrak{c}_1)$, $h_0 = \bigwedge_C h_{00} h_{01}$ *and* $h_1 = \bigvee_{\neg C}^i h_{10}$.

Proof. By Theorem 4.3.3.(1) and Lemma 4.4.1, it follows that for all $\vec{u} \in \mathbb{N}^{<\omega}$, if $\Gamma(T_{\vec{u}}) = \{n = m\}$, then $\mathrm{tp}(T_{\vec{u}}) = \mathsf{Rep}$ or $\mathsf{Ax}_{n=m}$. We show that $\Gamma(T_{\mathbf{0}^k}) = \{n = m\}$ for every $k \in \mathbb{N}$, by induction on k. When $\mathbf{0}^k = \langle\rangle$, we have $\Gamma(T_{\langle\rangle}) = \Gamma(h) = \{n = m\}$. Assume that $\Gamma(T_{\mathbf{0}^k}) = \{n = m\}$. Then, $\mathrm{tp}(T_{\mathbf{0}^k}) = \mathsf{Rep}$ or $\mathsf{Ax}_{n=m}$. In each case, we can show that $\Gamma(T_{\mathbf{0}^{k+1}}) = \{n = m\}$ by the definition of normalization trees and Theorem 4.3.3.(1).

Below we set $\vec{u} := \mathbf{0}^k$ for an arbitrary $k \in \mathbb{N}$.

(1) Assume that $\mathrm{tp}(T_{\vec{u}}) = \mathsf{Ax}_{n=m}$. By Lemma 4.4.3.(6), there is a nominal form $\mathfrak{a}$ such that $n = m \notin \mathsf{Cut}(\mathfrak{a})$ and $T_{\vec{u}} = \mathfrak{a}\{\mathsf{Ax}_\Delta\}$ with $n = m \in \Delta$.

(2) Assume that $\mathrm{tp}(T_{\vec{u}}) = \mathsf{Rep}$. By Lemma 4.4.3.(1), there are $\mathfrak{c}, h_0'', h_1''$ such that $T_{\vec{u}} = \mathfrak{c}\{h_0''\}$, $T_{\vec{u}}[0] = T_{\vec{u}*\langle 0\rangle} = \mathfrak{c}\{h_0''[0]\}$ and either $h_0'' = \mathsf{E}^m \mathsf{Ind}_F^{y,t} h_1''$ or ($h_0'' = \mathsf{E}^{m+1} h_1''$ & $\mathrm{tp}(h_1'') = \mathsf{Cut}_B$ for some B). If $h_0'' = \mathsf{E}^m \mathsf{Ind}_F^{y,t} h_1''$, then the statement (a) holds by taking $\mathfrak{c}$ as $\mathfrak{a}$ and h_0'' as h_0. Assume that $h_0'' = \mathsf{E}^{m+1} h_1''$ and $\mathrm{tp}(h_1'') = \mathsf{Cut}_B$ for some B. Then, by Lemma 4.4.3.(2), there are $\mathfrak{b}_0, C, h_{10}'', h_{11}''$ such that $h_1'' = \mathfrak{b}_0\{\mathsf{R}_C h_{10}'' h_{11}''\}$, $h_1''[i] = \mathfrak{b}_0\{(\mathsf{R}_C h_{10}'' h_{11}'')[i]\}$ and either

(†) $\mathrm{tp}(h_{10}'') = \bigwedge_C$ & $\mathrm{tp}(h_{11}'') = \bigvee_{\neg C}^n$ & $B = C[n]$ or

(††) $\mathrm{tp}(h''_{10}) = \bigvee^n_C$ & $\mathrm{tp}(h''_{11}) = \bigwedge_{\neg C}$ & $B = C[n]$.

We consider only the case that (†) holds and $C = \forall x A(x)$. Each of the other cases can be dealt with in a similar way. By Lemma 4.4.3.(3) and (5), there are $\mathfrak{a}_0$, $\mathfrak{a}_1$, h^*_{10} and h^*_{11} such that

$$C \notin \mathsf{Cut}(\mathfrak{a}_0),\ h''_{10} = \mathfrak{a}_0\{\textstyle\bigwedge^y_C h^*_{10}\},\ h''_{10}[n] = \mathfrak{a}_0\{h^*_{10}(y/n)\} \text{ and}$$

$$\neg C \notin \mathsf{Cut}(\mathfrak{a}_1),\ h''_{11} = \mathfrak{a}_1\{\textstyle\bigvee^n_{\neg C} h^*_{11}\},\ h''_{11}[0] = \mathfrak{a}_1\{h^*_{11}\}.$$

Thus,

$$T_{\vec{u}} = \mathfrak{c}\{h''_0\} = \mathfrak{c}\{\mathsf{E}^{m+1}h''_1\} = \mathfrak{c}\{\mathsf{E}^{m+1}\mathfrak{b}_0\{\mathsf{R}_C h''_{10} h''_{11}\}\} =$$
$$\mathfrak{c}\{\mathsf{E}^{m+1}\mathfrak{b}_0\{\mathsf{R}_C(\mathfrak{a}_0\{\textstyle\bigwedge^y_C h^*_{10}\}, \mathfrak{a}_1\{\bigvee^n_{\neg C} h^*_{11}\})\}\}$$

and

$$T_{\vec{u}*\langle 0\rangle} = \mathfrak{c}\{(\mathsf{E}^{m+1}h''_1)[0]\} = \mathfrak{c}\{\mathsf{E}^m((\mathsf{E}h''_1)[0])\} = \mathfrak{c}\{\mathsf{E}^m(\mathsf{R}_{C[n]}(\mathsf{E}h''_1[0], \mathsf{E}h''_1[1]))\}.$$

Then,

$$\mathfrak{c}\{\mathsf{E}^m(\mathsf{R}_{C[n]}(\mathsf{E}h''_1[0], \mathsf{E}h''_1[1]))\} =$$
$$\mathfrak{c}\{\mathsf{E}^m(\mathsf{R}_{C[n]}(\mathsf{E}\mathfrak{b}_0\{(\mathsf{R}_C h''_{10} h''_{11})[0]\}, \mathsf{E}\mathfrak{b}_0\{(\mathsf{R}_C h''_{10} h''_{11})[1]\}))\} =$$
$$\mathfrak{c}\{\mathsf{E}^m(\mathsf{R}_{C[n]}(\mathsf{E}\mathfrak{b}_0\{\mathsf{R}_C(h''_{10}[n], h''_{11})\}, \mathsf{E}\mathfrak{b}_0\{\mathsf{R}_C(h''_{10}, h''_{11}[0])\}))\} =$$
$$\mathfrak{c}\{\mathsf{E}^m(\mathsf{R}_{C[n]}(\mathsf{E}\mathfrak{b}_0\{\mathsf{R}_C(\mathfrak{a}_0\{h^*_{10}(y/n)\}, \mathfrak{a}_1\{\textstyle\bigvee^n_{\neg C} h^*_{11}\})\}, \mathsf{E}\mathfrak{b}_0\{\mathsf{R}_C(\mathfrak{a}_0\{\bigwedge^y_C h^*_{10}\}, \mathfrak{a}_1\{h^*_{11}\})\}))\}.$$

Set

$$\mathfrak{a} := \mathfrak{c},\ \mathfrak{b} := \mathfrak{b}_0,\ \mathfrak{c}_i := \mathfrak{a}_i \text{ for } i \in \{0,1\},\ h_0 := h^*_{10} \text{ and } h_1 := h^*_{11}.$$

Then, the statement (b) holds. □

Before we proceed, let us make the following remark. As observed in [AT13], reduction steps of the 1936 proof can be reformulated by slightly generalizing Proposition 4.4.4. Let FO be the set of all Z^∞-inference symbols of the form $\bigwedge_C$ or $\bigvee^k_C$, and FFO be the set of all Z^*-inference symbols of the form $\bigwedge^y_C$, $\bigwedge_C$ or $\bigvee^k_C$.

Proposition 4.4.5 (cf. Akiyoshi and Takahashi 2013, Theorem 8)**.** *Let T be a normalization tree of some closed $h \in \mathsf{Z}^*_0$. For every $\vec{u}$, one of the following statements holds:*

1. *if $\mathrm{tp}(T_{\vec{u}}) = \mathsf{Ax}_A$ for some A, then there is a nominal form $\mathfrak{a}$ such that $A \notin \mathsf{Cut}(\mathfrak{a})$ and $T_{\vec{u}} = \mathfrak{a}\{\mathsf{Ax}_\Delta\}$ with $A \in \Delta$, and*

2. *if $\mathrm{tp}(T_{\vec{u}}) = \mathsf{Rep}$, then one of the following statements holds for some nominal forms $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}_0, \mathfrak{c}_1$. (The symmetric cases are suppressed.)*

(a) $T_{\vec{u}} = \mathfrak{a}\{\mathsf{E}^m \mathsf{Ind}_F^{y,t} h_0\}$ *and* $T_{\vec{u}*\langle 0\rangle} = \mathfrak{a}\{\mathsf{E}^m((\mathsf{Ind}_F^{y,t} h_0)[0])\}$.

(b) $T_{\vec{u}} = \mathfrak{a}\{\mathsf{E}^{m+1}\mathfrak{b}\{\mathsf{R}_C(\mathfrak{c}_0\{h'_0\}, \mathfrak{c}_1\{h'_1\})\}\}$ *and*
$T_{\vec{u}*\langle 0\rangle} = \mathfrak{a}\{\mathsf{E}^m \mathsf{R}_{C[k]}(\mathsf{E}\mathfrak{b}\{\mathsf{R}_C(\mathfrak{c}_0\{h_0(y/k)\}, \mathfrak{c}_1\{h'_1\})\}, \mathsf{E}\mathfrak{b}\{\mathsf{R}_C(\mathfrak{c}_0\{h'_0\}, \mathfrak{c}_1\{h_1\})\})\}$, *where* $C = \forall x A(x)$, $C \notin \mathsf{Cut}(\mathfrak{c}_0)$, $\neg C \notin \mathsf{Cut}(\mathfrak{c}_1)$, $h'_0 = \bigwedge_C^y h_0$ *and* $h'_1 = \bigvee_{\neg C}^k h_1$.

(c) $T_{\vec{u}} = \mathfrak{a}\{\mathsf{E}^{m+1}\mathfrak{b}\{\mathsf{R}_C(\mathfrak{c}_0\{h_0\}, \mathfrak{c}_1\{h_1\})\}\}$ *and*
$T_{\vec{u}*\langle 0\rangle} = \mathfrak{a}\{\mathsf{E}^m \mathsf{R}_{C[i]}(\mathsf{E}\mathfrak{b}\{\mathsf{R}_C(\mathfrak{c}_0\{h_{0i}\}, \mathfrak{c}_1\{h_1\})\}, \mathsf{E}\mathfrak{b}\{\mathsf{R}_C(\mathfrak{c}_0\{h_0\}, \mathfrak{c}_1\{h_{10}\})\})\}$, *where* $C = A_0 \wedge A_1$, $C \notin \mathsf{Cut}(\mathfrak{c}_0)$, $\neg C \notin \mathsf{Cut}(\mathfrak{c}_1)$, $h_0 = \bigwedge_C h_{00}h_{01}$ *and* $h_1 = \bigvee_{\neg C}^i h_{10}$.

3. if $\mathrm{tp}(T_{\vec{u}}) \in FO$, *then there is a nominal form* $\mathfrak{a}$ *such that* $T_{\vec{u}} = \mathfrak{a}\{\mathcal{I}(h_j)_{j\in|\mathcal{I}|}\}$, $\mathcal{I} \in FFO$, $\Delta(\mathcal{I}) \cap \mathsf{Cut}(\mathfrak{a}) = \emptyset$ *and for every* $i \in |\mathrm{tp}(T_{\vec{u}})|$,

$$T_{\vec{u}*\langle i\rangle} = \begin{cases} \mathfrak{a}\{h_0(y/i)\}, & \text{if } \mathcal{I} = \bigwedge_C^y, \\ \mathfrak{a}\{h_i\}, & \text{else.} \end{cases}$$

Proof. (1) Similar to the proof of Proposition 4.4.4.(1).

(2) This case is the same as Proposition 4.4.4.(2).

(3) We consider only the case that $\mathrm{tp}(T_{\vec{u}}) = \bigwedge_{\forall x A(x)}$. By Lemma 4.4.3.(3), there are $\mathfrak{b}, h^*$ such that $T_{\vec{u}} = \mathfrak{b}\{\bigwedge_{\forall x A(x)}^y h^*\}$, $\Delta(\bigwedge_{\forall x A(x)}^y) \cap \mathsf{Cut}(\mathfrak{b}) = \emptyset$, and $T_{\vec{u}*\langle k\rangle} = (T_{\vec{u}})[k] = \mathfrak{b}\{h^*(y/k)\}$ for all $k \in \mathbb{N}$. We are done. □

Let T be a well-founded normalization tree of some closed $h \in \mathsf{Z}_0^*$ with the endsequent of the form $n = m$. Now we argue that T corresponds to consecutive applications of 1938 reduction steps. First, since $\mathrm{tp}(T_{\mathbf{0}^k}) = \mathsf{Ax}_{n=m}$ or Rep for every $k \in \mathbb{N}$, T essentially has no branching: If we omit its irrelevant nodes, then T is described as follows.

$$T_{\langle\rangle} \longrightarrow T_{\langle 0\rangle} \longrightarrow T_{\langle 0,0\rangle} \longrightarrow \cdots$$

This accords with the fact that reduction steps of the 1938 proof has no branching.

Second, each transition from $\phi(T_{\mathbf{0}^k})$ to $\phi(T_{\mathbf{0}^{k+1}})$ corresponds to a reduction step of the 1938 proof, which is formulated with end-pieces. Consider the case (2a) of Proposition 4.4.4. There $\phi(T_{\mathbf{0}^k})$ and $\phi(T_{\mathbf{0}^{k+1}})$ are as follows.

$$\begin{array}{c} \vdots\, h_0 \\ \dfrac{\neg F(y), F(S(y))}{\neg F(0), F(k)}\ \mathsf{Ind}_F^{y,k} \\ \vdots\, \mathfrak{a} \\ n = m \end{array} = \phi(T_{\mathbf{0}^k}) \qquad \triangleright$$

$$
\begin{array}{c}
\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma, \neg F(0), F(0)\end{array} \quad \begin{array}{c}\vdots\, h_0(y/0)\\ \Gamma, \neg F(0), F(1)\end{array}}{\Gamma, \neg F(0), F(1)}\,\mathsf{R}_{F(0)} \qquad \begin{array}{c}\vdots\, h_0(y/1)\\ \Gamma, \neg F(1), F(2)\end{array}}{\Gamma, \neg F(0), F(2)}\,\mathsf{R}_{F(1)}\\
\vdots\\
\Gamma, \neg F(0), F(k)\\
\vdots\, \mathfrak{a}\\
n = m
\end{array}
\quad = \phi(T_{\mathbf{0}^{k+1}})
$$

This transition is done exactly in the end-pieces of $\phi(T_{\mathbf{0}^k})$ and $\phi(T_{\mathbf{0}^{k+1}})$. Thus, it is the same as the "CJ-reduction" of the 1938 proof formulated in [Gen38b, §3.3]. Furthermore, in the case (2b) of Proposition 4.4.4, $\phi(T_{\mathbf{0}^k})$ and $\phi(T_{\mathbf{0}^{k+1}})$ are of the following forms, respectively. Let C be $\forall x A(x)$.

$$
\begin{array}{c}
\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\vdots\, h_0\\ \Delta_0, A(y), \forall x A(x)\end{array}}{\Delta_0, \forall x A(x)}\,\bigwedge^y_C\\ \vdots\, \mathfrak{c}_0\\ \Gamma', \forall x A(x)\end{array} \qquad \begin{array}{c}\dfrac{\begin{array}{c}\vdots\, h_1\\ \Delta_1, \neg A(k), \exists x \neg A(x)\end{array}}{\Delta_1, \exists x \neg A(x)}\,\bigvee^k_{\neg C}\\ \vdots\, \mathfrak{c}_1\\ \Gamma', \exists x \neg A(x)\end{array}}{\Gamma'}\,\mathsf{R}_C\\
\vdots\, \mathfrak{b}\\
\Gamma\\
\vdots\, \mathfrak{a}\\
n = m
\end{array}
\quad = \phi(T_{\mathbf{0}^k}) \qquad \triangleright
$$

$$
\begin{array}{c}
\dfrac{\dfrac{\begin{array}{c}\vdots\, h_0(y/k)\\ \Delta_0, A(k), \forall x A(x)\\ \vdots\, \mathfrak{c}_0\\ \Gamma', A(k), \forall x A(x)\end{array} \quad \begin{array}{c}\dfrac{\begin{array}{c}\vdots\, h_1\\ \Delta_1, \neg A(k), \exists x \neg A(x)\end{array}}{\Delta_1, \exists x \neg A(x)}\,\bigvee^k_{\neg C}\\ \vdots\, \mathfrak{c}_1\\ \Gamma', \exists x \neg A(x)\end{array}}{\begin{array}{c}\Gamma', A(k)\\ \vdots\, \mathfrak{b}\\ \Gamma, A(k)\end{array}}\,\mathsf{R}_C \qquad \dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\vdots\, h_0\\ \Delta_0, A(y), \forall x A(x)\end{array}}{\Delta_0, \forall x A(x)}\,\bigwedge^y_C\\ \vdots\, \mathfrak{c}_0\\ \Gamma', \forall x A(x)\end{array} \quad \begin{array}{c}\vdots\, h_1\\ \Delta_1, \neg A(k), \exists x \neg A(x)\\ \vdots\, \mathfrak{c}_1\\ \Gamma', \neg A(k), \exists x \neg A(x)\end{array}}{\begin{array}{c}\Gamma', \neg A(k)\\ \vdots\, \mathfrak{b}\\ \Gamma, \neg A(k)\end{array}}\,\mathsf{R}_C}{\Gamma}\,\mathsf{R}_{C[k]}\\
\vdots\, \mathfrak{a}\\
n = m
\end{array}
\quad = \phi(T_{\mathbf{0}^{k+1}})
$$

with $\forall x A(x) \notin \mathsf{Cut}(\mathfrak{c}_0)$, $\exists x \neg A(x) \notin \mathsf{Cut}(\mathfrak{c}_1)$. This is one of the operational reductions formulated in [Gen38b, §3.5]. The case (2c) corresponds to another operational reduction.

Third, T gives a normal derivation of $n = m$. By the definition of well-foundedness, T has a node $T_{\mathbf{0}^l}$ such that $\mathrm{tp}(T_{\mathbf{0}^l}) = \mathsf{Ax}_A$ for some A. By Proposition 4.4.4.(1), A is of the form $n = m$ and $T_{\mathbf{0}^l}$ is of the form $\mathfrak{a}\{\mathsf{Ax}_\Delta\}$ with $n = m \in \Delta$. Since $n = m \notin \mathsf{Cut}(\mathfrak{a})$, $\Gamma(T_{\mathbf{0}^l})$ includes $n = m$. In fact, $\Gamma(T_{\mathbf{0}^l}) = \{n = m\}$ holds as we have seen in the proof of Proposition 4.4.4.

Thus, $T_{\mathbf{0}^l}$ is a Z_0^*-derivation of $n = m$. Moreover, we can consider $T_{\mathbf{0}^l}$ a normal derivation, because the endsequent of $T_{\mathbf{0}^l}$ is shown to be true in a primitive recursive way.

From these arguments, we conclude that T corresponds to consecutive applications of reduction steps of the 1938 proof and that the formulation of its main lemma as Proposition 4.4.3 is justified. Proposition 4.4.3 assigns a normal derivation to every derivable numeric equation of first-order arithmetic, and the main lemma of the 1936 does so as well, because the 1938 proof is a special case of the 1936 proof.

Then, we can explain how contentual and formal aspects of the 1936 proof relate to each other. The foregoing argument indicated that a normalization tree of closed Z_0^*-derivation h is a generalization of syntactic transformation in the 1938 proof: Normalization trees transform not only a derivation of a numeric equation but also a derivation with an arbitrary conclusion, whereas the 1938 proof's procedure transforms the former kind of a derivation only. In other words, a normalization tree is a generalization of the cut elimination procedure in the 1938 proof. This constitutes formal aspects of the 1936 proof.

Remember that for every closed Z_0^*-derivation h, we can extract both a reduction procedure $\mathcal{T}$ for the sequent $\Gamma(h)$ and a proof for the well-foundedness of $\mathcal{T}$ from a normalization tree of h. This means that the correctness of $\Gamma(h)$ in the sense of (GI) is shown by the normalization tree. Therefore, we can say that the correctness of $\Gamma(h)$ is in fact shown in terms of a procedure obtained by generalizing the cut elimination procedure in the 1938 proof. That is to say, the 1936 proof's contentual aspects are formed by its formal aspects.

Let us summarize the argument in this section. First, we have argued that the 1936 proof is a contentual correctness proof, by verifying that the main lemma of the 1936 proof (Proposition 4.4.1) with an additional lemma (Lemma 4.4.2) implies the main lemma of the 1935 proof (Proposition 4.4.2). This consequence from the main lemma of the 1936 proof means that the proof shows all axioms and theorems of first-order arithmetic to be correct in the sense of (GI), as the 1935 proof does. Next, we have argued that the 1936 proof is also a formal correctness proof. By utilizing Buchholz's observation in [Buc97], we showed that reduction steps of the 1936 proof for derivations of numeric equations have the same structures as the ones of reductions steps of the 1938 proof. This enabled us to see that the main lemma of the 1938 proof (Proposition 4.4.3) is a special case of the main lemma of the 1936 proof. Accordingly, the 1936 proof assigns a normal derivation to every derivable numeric equation in first-order arithmetic as the 1938 proof does. Finally, we have explained the relation between the 1936 proof's contentual aspects and its formal aspects: the former are given by the latter.

4.5 No-counterexample Interpretation

In this section, we indicate a consequence of our arguments in the previous section: We prove the theorem called the *no-counterexample interpretation*, following Schwichtenberg's proof [Sch77, §4] in our setting.[28]

Theorem 4.5.1 (No-Counterexample Interpretation)**.** *If a closed formula*

$$\exists x_1 \forall y_1 \dots \exists x_n \forall y_n p(x_1, y_1, \dots, x_n, y_n)$$

is derivable in Z*, then there are* $< \varepsilon_0$*-recursive functionals* $\mathbf{G}_1, \dots, \mathbf{G}_n$ *such that for every* k*-ary function* $\mathbf{F}_k$ *on natural numbers* $(1 \leq k \leq n)$*,*

$$\mathbf{P}(\mathbf{G}_1\vec{\mathbf{F}}, \mathbf{F}_1(\mathbf{G}_1\vec{\mathbf{F}}), \mathbf{G}_2\vec{\mathbf{F}}, \mathbf{F}_2(\mathbf{G}_1\vec{\mathbf{F}}, \mathbf{G}_2\vec{\mathbf{F}}), \dots, \mathbf{G}_n\vec{\mathbf{F}}, \mathbf{F}_n(\mathbf{G}_1\vec{\mathbf{F}}, \dots, \mathbf{G}_n\vec{\mathbf{F}}))$$

holds, where $\mathbf{G}_k\vec{\mathbf{F}} := \mathbf{G}_k(\mathbf{F}_1, \dots, \mathbf{F}_n)$ *for all* k $(1 \leq k \leq n)$*.*

For the sake of simplicity, we concentrate on the case that $n = 2$ in the theorem above. Let an arbitrary unary function $\mathbf{F}_1$ and an arbitrary binary function $\mathbf{F}_2$ on natural numbers be given.

First, we extend the language L. Let $L(f_1, f_2)$ be the language obtained by extending L with a unary function-variable f_1 and a binary function-variable f_2. We denote the set of all free number-variables in an expression θ in $L(f_1, f_2)$ by $FV(\theta)$. An expression θ is *closed* if $FV(\theta) = \emptyset$. Let $\mathsf{TRUE}_0^{\mathbf{F}_1,\mathbf{F}_2}$ be the set of all true literals of $L(f_1, f_2)$ under the assignment $f_i \mapsto \mathbf{F}_i$ for $i \in \{1, 2\}$. For every closed term t of $L(f_1, f_2)$, we denote the numeral n such that $t = n \in \mathsf{TRUE}_0^{\mathbf{F}_1,\mathbf{F}_2}$ by $[\![t]\!]$.

On the basis of the language $L(f_1, f_2)$, we define the finitary proof system $\mathsf{Z}^*(\mathbf{F}_1, \mathbf{F}_2)$. We assume the existence of a set $\mathsf{Ax}^{\mathbf{F}_1,\mathbf{F}_2}$ of sequents satisfying the following conditions:

- for all $\Delta \in \mathsf{Ax}^{\mathbf{F}_1,\mathbf{F}_2}$, Δ is a set of literals,
- if there is a substitution instance Δ_0 of Δ, then $\Delta_0 \in \mathsf{Ax}^{\mathbf{F}_1,\mathbf{F}_2}$,
- if $\Delta \in \mathsf{Ax}^{\mathbf{F}_1,\mathbf{F}_2}$ and $FV(\Delta) = \emptyset$, then $\Delta \cap \mathsf{TRUE}_0^{\mathbf{F}_1,\mathbf{F}_2} \neq \emptyset$.

[28] The difference is our use of finite notations for infinitary derivations instead of heavy coding.

Define the inference symbol Ax_Δ of $\mathsf{Z}^*(\mathbf{F}_1, \mathbf{F}_2)$ as

$$(\mathsf{Ax}_\Delta)\ \overline{\Delta}$$

with $\Delta \in \mathsf{Ax}^{\mathbf{F}_1,\mathbf{F}_2}$. Furthermore, we add the following inference symbol:

$$(\mathsf{S}^t_s)\ \frac{A(x/t)}{A(x/s)}$$

with $[\![t]\!] = [\![s]\!]$. The remaining inference symbols $\bigwedge_{A_0 \wedge A_1}$, $\bigwedge^y_{\forall x A}$, $\bigvee^i_{A_0 \vee A_1}$, $\bigvee^t_{\exists x A}$, $\mathsf{Ind}^{y,t}_F$, R_C and E of $\mathsf{Z}^*(\mathbf{F}_1, \mathbf{F}_2)$ are defined in the same manner as Z^*.

Hereafter, a $\mathsf{Z}^*(\mathbf{F}_1, \mathbf{F}_2)$-derivation is denoted by h possibly with suffixes. Again, we suppose that every $\mathsf{Z}^*(\mathbf{F}_1, \mathbf{F}_2)$-derivation satisfies Free Variables-Conditions. We define the ordinal assignment $\mathrm{o}(h)$ and the degree $\deg(h)$ of a $\mathsf{Z}^*(\mathbf{F}_1, \mathbf{F}_2)$-derivation h with $h = \mathsf{S}^t_s h_0$ as follows: $\mathrm{o}(h) := \mathrm{o}(h_0)$ and $\deg(h) := \deg(h_0)$. The other cases are treated in the same manner as Z^*.

Note that if a closed formula of the form

$$\exists x_1 \forall y_1 \exists x_2 \forall y_2 p(x_1, y_1, x_2, y_2)$$

is derivable in Z, then we can verify only by first-order logic that the formula

$$\exists x_1 \exists x_2 p(x_1, f_1(x_1), x_2, f_2(x_1, x_2))$$

is derivable in $\mathsf{Z}^*(\mathbf{F}_1, \mathbf{F}_2)$. In particular, we have no need of the $\mathsf{Z}^*(\mathbf{F}_1, \mathbf{F}_2)$-axioms about $\mathbf{F}_1$ and $\mathbf{F}_2$ in deriving the formula above.

The infinitary proof system $\mathsf{Z}^\infty(\mathbf{F}_1, \mathbf{F}_2)$ with the language $L(f_1, f_2)$ is defined in the same way as Z^∞ except the inference symbols Ax_A. The inference symbol Ax_A of $\mathsf{Z}^\infty(\mathbf{F}_1, \mathbf{F}_2)$ is defined as

$$(\mathsf{Ax}_A)\ \overline{A}$$

with $A \in \mathsf{TRUE}_0^{\mathbf{F}_1,\mathbf{F}_2}$. We denote a $\mathsf{Z}^\infty(\mathbf{F}_1, \mathbf{F}_2)$-derivation by $\mathtt{d}$ possibly with suffixes.

By the following lemma, we define an operator for substitution in $\mathsf{Z}^\infty(\mathbf{F}_1, \mathbf{F}_2)$-derivations that corresponds to the inference symbol S^t_s of $\mathsf{Z}^*(\mathbf{F}_1, \mathbf{F}_2)$.

Lemma 4.5.1. *For all terms t and s with $[\![t]\!] = [\![s]\!]$, there is an operator $\mathcal{S}^t_s$ such that if $\mathtt{d} \vdash^\alpha_m \Gamma, A(x/t)$, then $\mathcal{S}^t_s(\mathtt{d}) \vdash^\alpha_m \Gamma, A(x/s)$.*

Proof. By induction on α. □

The following two theorems can be shown for $\mathsf{Z}^\infty(\mathbf{F}_1, \mathbf{F}_2)$ in a similar way to the case of Z^∞. To prove the first theorem, we use the previous lemma.

Theorem 4.5.2. *For every formula C, there is an operator $\mathcal{R}_C$ such that if* $\mathsf{d}_0 \vdash^{\alpha}_{m} \Gamma, C$, $\mathsf{d}_1 \vdash^{\beta}_{m} \Gamma, \neg C$ *and* $\mathrm{rk}(C) \leq m$, *then* $\mathcal{R}_C(\mathsf{d}_0, \mathsf{d}_1) \vdash^{\alpha\#\beta}_{m} \Gamma$.

Theorem 4.5.3. *There is an operator $\mathcal{E}$ such that if* $\mathsf{d} \vdash^{\alpha}_{m+1} \Gamma$ *then* $\mathcal{E}(\mathsf{d}) \vdash^{\omega^\alpha}_{m} \Gamma$.

We define the translation of $\mathsf{Z}^*(\mathbf{F}_1, \mathbf{F}_2)$ into $\mathsf{Z}^\infty(\mathbf{F}_1, \mathbf{F}_2)$. Let h be a closed $\mathsf{Z}^*(\mathbf{F}_1, \mathbf{F}_2)$-derivation, Γ be $\Gamma(h)$ and α be $\mathrm{o}(h)$.

1. If $h = \mathsf{Ax}_\Delta$, then $(\mathsf{Ax}_\Delta)^\infty := \overline{\mathsf{Ax}_A : \Gamma : \alpha}$, where A is the arbitrarily fixed element of $\Delta \cap \mathsf{TRUE}_0^{\mathbf{F}_1,\mathbf{F}_2}$.

2. If $h = \mathsf{S}^t_s h_0$, then $(\mathsf{S}^t_s h_0)^\infty := \mathcal{S}^t_s(h_0^\infty)$ with $\Gamma(\mathcal{S}^t_s(h_0^\infty)) = \Gamma$.

3. If $h = \mathsf{Ind}^{y,t}_F h_0$ and $[\![t]\!] = n$, then $(\mathsf{Ind}^{y,t}_F h_0)^\infty := \dfrac{\mathcal{S}^n_t(\mathsf{e}_n)}{\mathsf{Rep} : \Gamma : \alpha}$, where

$$\mathsf{e}_0 := \mathsf{d}_{F(x/0)},\ \mathsf{e}_1 := h_0(y/0)^\infty,\ \mathsf{e}_{i+1} := \mathcal{R}_{F(x/i)}(\mathsf{e}_i, h_0(y/i)^\infty)$$

   and $\Gamma(\mathsf{e}_n) = \{\Delta, \neg F(0), F(n)\}$, $\Gamma(\mathcal{S}^n_t(\mathsf{e}_n)) = \{\Delta, \neg F(0), F(t)\} = \Gamma$.

4. Otherwise, define h^∞ in the same manner as the translation of Z^* into Z^∞.

In addition, we define the two functionals $\mathrm{tp}^{\mathbf{F}_1,\mathbf{F}_2}(h)$ and $h(\mathbf{F}_1, \mathbf{F}_2)[i]$ of level ≤ 2 for closed $\mathsf{Z}^*(\mathbf{F}_1, \mathbf{F}_2)$-derivations. Note that these are primitive recursive functional with $\mathbf{F}_1$ and $\mathbf{F}_2$ as function-arguments. For closed terms t and s, the expression $\theta(t \mapsto s)$ is the result of replacing some occurrences of t in θ with s.

Let h be a closed $\mathsf{Z}^*(\mathbf{F}_1, \mathbf{F}_2)$-derivation.

1. If $h = \mathsf{Ax}_\Delta$, then $h^+ := \mathsf{Ax}_A$ and $\mathrm{tp}^{\mathbf{F}_1,\mathbf{F}_2}(h) := \mathsf{Ax}_A$, where A is the arbitrarily fixed element of $\Delta \cap \mathsf{TRUE}_0^{\mathbf{F}_1,\mathbf{F}_2}$.

2. If $h = \mathsf{S}^t_s h'$ and $h'^+ = \mathcal{I}((h_i)_{i \in |\mathcal{I}|})$, then $(\mathsf{S}^t_s h')^+ := \mathcal{I}(t \mapsto s)((\mathsf{S}^t_s h_i)_{i \in |\mathcal{I}(t/s)|})$.

3. If $h = \mathsf{Ind}^{y,t}_F h_0$ and $[\![t]\!] = n$, then $(\mathsf{Ind}^{y,t}_F h_0)^+ := \mathsf{Rep}(\mathsf{S}^n_t e_n)$,

   where $e_0 := h_{F(x/0)}$, $e_1 := h_0(y/0)$ and $e_{i+1} := \mathsf{R}_{F(x/i)} e_i h_0(y/i)$.

4. If $h = \mathsf{R}_C h_0 h_1$ and $h_l^+ = \mathcal{I}_l((h_{li})_{i \in |\mathcal{I}_l|})$ for $l \in \{0, 1\}$, then

$$(\mathsf{R}_C h_0 h_1)^+ := \begin{cases} \mathcal{I}_0((\mathsf{R}_C h_{0i} h_1)_{i\in|\mathcal{I}_0|}), & \text{if } C \notin \Delta(\mathcal{I}_0), \\ \mathcal{I}_1((\mathsf{R}_C h_0 h_{1i})_{i\in|\mathcal{I}_1|}), & \text{if } \neg C \notin \Delta(\mathcal{I}_1), \\ \mathsf{Cut}_{C[i]}(\mathsf{R}_C h_{0i} h_1 \mathsf{R}_C h_0 h_{10}), & \text{if } C \in \Delta(\mathcal{I}_0), \neg C \in \Delta(\mathcal{I}_1) \\ & \quad \text{and } \mathcal{I}_1 = \bigvee^i_{C_0\vee C_1}, \\ \mathsf{Cut}_{C[i]}(\mathsf{R}_C h_{00} h_1 \mathsf{R}_C h_0 h_{1i}), & \text{if } C \in \Delta(\mathcal{I}_0), \neg C \in \Delta(\mathcal{I}_1) \\ & \quad \text{and } \mathcal{I}_0 = \bigvee^i_{C_0\vee C_1}, \\ \mathsf{Cut}_{C[k]}(\mathsf{R}_C h_{0k} h_1 \mathsf{R}_C h_0 \mathsf{S}^t_k h_{10}), & \text{if } C \in \Delta(\mathcal{I}_0), \neg C \in \Delta(\mathcal{I}_1), \\ & \quad \mathcal{I}_1 = \bigvee^t_{\exists x C_0(x)} \text{ and } [\![t]\!] = k, \\ \mathsf{Cut}_{C[k]}(\mathsf{R}_C \mathsf{S}^t_k h_{00} h_1 \mathsf{R}_C h_0 h_{1k}), & \text{if } C \in \Delta(\mathcal{I}_0), \neg C \in \Delta(\mathcal{I}_1), \\ & \quad \mathcal{I}_0 = \bigvee^t_{\exists x C_0(x)} \text{ and } [\![t]\!] = k. \end{cases}$$

5. Otherwise, define h^+ in the same manner as $\mathrm{tp}(h)$ and $h[i]$.

By induction on the build-up of a given closed $\mathsf{Z}^*(\mathbf{F}_1, \mathbf{F}_2)$-derivation h, we have the following theorem.

Theorem 4.5.4 (Cf. Theorem 4.3.3)**.** *Let h be a closed $\mathsf{Z}^*(\mathbf{F}_1, \mathbf{F}_2)$-derivation.*

1. $\dfrac{\ldots \Gamma(h(\mathbf{F}_1, \mathbf{F}_2)[i]) \ldots (i \in |\mathrm{tp}^{\mathbf{F}_1,\mathbf{F}_2}(h)|)}{\Gamma(h)}\ \mathrm{tp}^{\mathbf{F}_1,\mathbf{F}_2}(h)$ *is a correct $\mathsf{Z}^\infty(\mathbf{F}_1, \mathbf{F}_2)$-inference.*
2. *If* $\mathrm{tp}^{\mathbf{F}_1,\mathbf{F}_2}(h) = \mathsf{Cut}_C$*, then* $\mathrm{rk}(C) < \deg(h)$.
3. $\deg(h(\mathbf{F}_1, \mathbf{F}_2)[i]) \leq \deg(h)$ *for all* $i \in |\mathrm{tp}^{\mathbf{F}_1,\mathbf{F}_2}(h)|$.
4. $\mathrm{o}(h(\mathbf{F}_1, \mathbf{F}_2)[i]) \prec \mathrm{o}(h)$ *for all* $i \in |\mathrm{tp}^{\mathbf{F}_1,\mathbf{F}_2}(h)|$.

Define the set $\mathsf{Z}^*_0(\mathbf{F}_1, \mathbf{F}_2)$ as

$$\mathsf{Z}^*_0(\mathbf{F}_1, \mathbf{F}_2) := \{h \mid h \text{ is a } \mathsf{Z}^*(\mathbf{F}_1, \mathbf{F}_2)\text{-derivation with } \deg(h) = 0\}.$$

Hereafter, we denote by T a functional of level ≤ 2 such that it takes $\mathbf{F}_1, \mathbf{F}_2, \vec{u}$ as arguments and outputs a closed $\mathsf{Z}^*_0(\mathbf{F}_1, \mathbf{F}_2)$-derivation. The local correctness, the well-foundedness and normalization trees are defined in a similar manner to Section 4.4. Then, we have the following lemma and proposition.

Lemma 4.5.2 (Cf. Lemma 4.4.1)**.** *For every T, every $\vec{u} \in \mathbb{N}^{<\omega}$ and every formula C, $\mathrm{tp}^{\mathbf{F}_1,\mathbf{F}_2}(T_{\vec{u}}) \neq \mathsf{Cut}_C$ holds.*

Proposition 4.5.1 (Cf. Proposition 4.4.1)**.** *For every closed $\mathsf{Z}^*_0(\mathbf{F}_1, \mathbf{F}_2)$-derivation h, there is a well-founded normalization tree T of h such that T is a $< \varepsilon_0$-recursive functional with $\mathbf{F}_1$ and $\mathbf{F}_2$ as function-arguments.*

In what follows, our proofs for Lemma 4.5.3 and 4.5.4 are inspired by Buchholz' proofs for [Buc91, Lemma 4.4 and Theorem 4.5]. We use the following notions:

- $\mathsf{INS}_p := \{\exists x_1 \exists x_2 p(x_1, f_1(x_1), x_2, f_2(x_1, x_2))\} \cup$
  $\{\exists x_2 p(t_1, f_1(t_1), x_2, f_2(t_1, x_2)) \mid t_1 \text{ is a closed term}\} \cup$
  $\{p(t_1, f_1(t_1), t_2, f_2(t_1, t_2)) \mid t_1, t_2 \text{ are closed terms}\}$.
- For every inference symbol $\mathcal{I}$ of $\mathsf{Z}^\infty(\mathbf{F}_1, \mathbf{F}_2)$,

$$\mathsf{WIT}_p(\mathcal{I}) := \begin{cases} \langle t_1, t_2 \rangle, & \text{if } \mathcal{I} \text{ is of the form } \mathsf{Ax}_{p(t_1, f_1(t_1), t_2, f_2(t_1, t_2))}, \\ \langle \rangle, & \text{else.} \end{cases}$$

- $w_p := \{\langle t_1, t_2 \rangle \mid p(t_1, f_1(t_1), t_2, f_2(t_1, t_2)) \in \mathsf{TRUE}_0^{\mathbf{F}_1, \mathbf{F}_2}\}$.

Lemma 4.5.3. *Let h be a closed $\mathsf{Z}_0^*(\mathbf{F}_1, \mathbf{F}_2)$-derivation with*

$$\Gamma(h) = \{\exists x_1 \exists x_2 p(x_1, f_1(x_1), x_2, f_2(x_1, x_2))\}.$$

If T is a normalization tree of h and $\mathrm{o}(T_{\mathbf{0}^{k+1}}) \not\prec \mathrm{o}(T_{\mathbf{0}^k})$, then

$$\mathrm{tp}^{\mathbf{F}_1, \mathbf{F}_2}(T_{\mathbf{0}^k}) = \mathsf{Ax}_{p(t_1, f_1(t_1), t_2, f_2(t_1, t_2))}$$

for some closed terms t_1, t_2.

Proof. By Lemma 4.5.2 and Theorem 4.5.4.(1), it holds that if $\Gamma(T_{\mathbf{0}^k}) \subseteq \mathsf{INS}_p$, then

(†) $\mathrm{tp}^{\mathbf{F}_1, \mathbf{F}_2}(T_{\mathbf{0}^k}) \in \{\mathsf{Rep}\} \cup \{\mathsf{Ax}_A \mid A \text{ is of the form } p(t_1, f_1(t_1), t_2, f_2(t_1, t_2))\} \cup$
$\{\bigvee^{t_1}_{\exists x_1 \exists x_2 p(x_1, f_1(x_1), x_2, f_2(x_1, x_2))} \mid t_1 \text{ is a closed term}\} \cup$
$\{\bigvee^{t_2}_{\exists x_2 p(t, f_1(t_1), x_2, f_2(t_1, x_2))} \mid t_1, t_2 \text{ are closed terms}\}$

for all $k \in \mathbb{N}$. We show that $\Gamma(T_{\mathbf{0}^k}) \subseteq \mathsf{INS}_p$ by induction on k. The base case is obvious. Assume as IH that $\Gamma(T_{\mathbf{0}^k}) \subseteq \mathsf{INS}_p$, then (†) holds for $T_{\mathbf{0}^k}$. By Theorem 4.5.4.(1) again, it holds that $\Gamma(T_{\mathbf{0}^{k+1}}) \subseteq \mathsf{INS}_p$.

Assume that $\mathrm{o}(T_{\mathbf{0}^{k+1}}) \not\prec \mathrm{o}(T_{\mathbf{0}^k})$. Then, $\mathrm{tp}^{\mathbf{F}_1, \mathbf{F}_2}(T_{\mathbf{0}^k}) = \mathsf{Ax}_A$ for some $A \in \mathsf{TRUE}_0^{\mathbf{F}_1, \mathbf{F}_2}$ by Theorem 4.5.4.(4). Since $\Gamma(T_{\mathbf{0}^k}) \subseteq \mathsf{INS}_p$, A is of the form $p(t_1, f_1(t_1), t_2, f_2(t_1, t_2))$. □

Lemma 4.5.4. *Let h be a closed $\mathsf{Z}_0^*(\mathbf{F}_1, \mathbf{F}_2)$-derivation with*

$$\Gamma(h) = \{\exists x_1 \exists x_2 p(x_1, f_1(x_1), x_2, f_2(x_1, x_2))\}.$$

Then, there is a well-founded normalization tree T of h such that the following three statements hold:

1. *For every* $k \in \mathbb{N}$, $\mathrm{o}(T_{\mathbf{0}^k}) \preceq \mathrm{o}(h)$ *holds,*

2. $\mathsf{WIT}_p(\mathrm{tp}^{\mathbf{F}_1,\mathbf{F}_2}(T_{\mathbf{0}^m})) \in w_p$*, where* $m = \min\{k \mid \mathrm{o}(T_{\mathbf{0}^{k+1}}) \not\prec \mathrm{o}(T_{\mathbf{0}^k})\}$*,*

3. *let* $\mathcal{G}_{\mathbf{F}_1,\mathbf{F}_2}$ *be the functional of level* ≤ 2 *with* $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *as function-arguments, which is defined as*

$$\mathcal{G}_{\mathbf{F}_1,\mathbf{F}_2}(T_{\mathbf{0}^n}) := \begin{cases} \mathcal{G}_{\mathbf{F}_1,\mathbf{F}_2}(T_{\mathbf{0}^{n+1}}), & \textit{if } \mathrm{o}(T_{\mathbf{0}^{n+1}}) \prec \mathrm{o}(T_{\mathbf{0}^n}) \preceq \mathrm{o}(h) \prec \varepsilon_0 \\ \mathsf{WIT}_p(\mathrm{tp}^{\mathbf{F}_1,\mathbf{F}_2}(T_{\mathbf{0}^n})), & \textit{else,} \end{cases}$$

then $\mathcal{G}_{\mathbf{F}_1,\mathbf{F}_2}$ *is a* $< \varepsilon_0$*-recursive functional and* $\mathcal{G}_{\mathbf{F}_1,\mathbf{F}_2}(T_{\mathbf{0}^0}) \in w_p$ *holds.*

Proof. By Proposition 4.5.1, there is a normalization tree T of h such that T is a $< \varepsilon_0$-recursive functional with $\mathbf{F}_1$ and $\mathbf{F}_2$ as function-arguments.

(1) By Theorem 4.5.4.(4), $\mathrm{o}(T_{\mathbf{0}^k}) \preceq \mathrm{o}(h) \prec \varepsilon_0$ holds for every $k \in \mathbb{N}$.

(2) By the well-foundedness of T, there is a $k \in \mathbb{N}$ such that $\mathrm{o}(T_{\mathbf{0}^{k+1}}) \not\prec \mathrm{o}(T_{\mathbf{0}^k})$ holds. The assertion holds by Lemma 4.5.3.

(3) By (1), (2) and the definition of $\mathcal{G}_{\mathbf{F}_1,\mathbf{F}_2}(T_{\mathbf{0}^n})$, we can see that $\mathcal{G}_{\mathbf{F}_1,\mathbf{F}_2}(T_{\mathbf{0}^0}) \in w_p$ and that $\mathcal{G}_{\mathbf{F}_1,\mathbf{F}_2}$ is a $< \varepsilon_0$-recursive functional. Note that $\mathrm{tp}^{\mathbf{F}_1,\mathbf{F}_2}(h)$ and $h(\mathbf{F}_1, \mathbf{F}_2)[i]$ are primitive recursive functional and T is a $< \varepsilon_0$-recursive functional. □

Now we prove Theorem 4.5.1.

Proof. Let $\mathcal{G}_{\mathbf{F}_1,\mathbf{F}_2}$ be the functional defined in Lemma 4.5.4.(3) and $\mathbf{p}_0$, $\mathbf{p}_1$ be primitive recursive projection functions for pairs. Define

$$\mathbf{G}_0(\mathbf{F}_1, \mathbf{F}_2) := \mathbf{p}_0(\mathcal{G}_{\mathbf{F}_1,\mathbf{F}_2}(T_{\mathbf{0}^0})),\ \mathbf{G}_1(\mathbf{F}_1, \mathbf{F}_2) := \mathbf{p}_1(\mathcal{G}_{\mathbf{F}_1,\mathbf{F}_2}(T_{\mathbf{0}^0})).$$

Then,

$$\mathbf{P}(\mathbf{G}_0(\mathbf{F}_1, \mathbf{F}_2), \mathbf{F}_1(\mathbf{G}_0(\mathbf{F}_1, \mathbf{F}_2)), \mathbf{G}_1(\mathbf{F}_1, \mathbf{F}_2), \mathbf{F}_2(\mathbf{G}_0(\mathbf{F}_1, \mathbf{F}_2), \mathbf{G}_1(\mathbf{F}_1, \mathbf{F}_2)))$$

holds by Lemma 4.5.4.(3). □

4.6 Conclusion of Chapter 4

First in this chapter, we have answered the following question induced by Sieg's paper [Sie12]: Is Gentzen's 1936 proof both a contentual correctness proof and a formal correctness proof? We have argued that the 1936 proof is both contentual and formal because the main lemma of the 1936 proof implies both the main lemma of the 1935 proof and the main lemma of the 1938 proof. In other words, the main lemma of the 1936 proof not only verifies the correctness of axioms and theorems of first-order arithmetic, but also assigns a normal derivation to each derivable numeric equation of first-order arithmetic. (Here, according to Gentzen, A is correct if and only if a reduction procedure is statable for it.)

Next, we have answered the question of how contentual and formal aspects of the 1936 proof relate to each other: Contentual aspects of the 1936 proof are formed by its formal aspects. In other words, the correctness of a derivable formula A of first-order arithmetic is in fact shown by means of syntactic transformation of a given derivation of the formula.

Finally, we have noted a consequence of our argument for the answers above. Our argument has given another proof of Kreisel's no-counterexample interpretation.

As mentioned in the introduction of this chapter, we have focused not only on the mathematical side of Gentzen's consistency proofs, but also on their conceptual side. Our whole argument indicates that contentual and formal aspects of Gentzen's consistency proofs are fruitful products of an interaction between philosophical ideas and mathematical proof-theoretical methods.

Concluding Remarks

We have discussed Gentzen's interpretation of first-order arithmetical formulas, which he formulated in his consistency proofs. In our concluding remarks, we note that the following question underlies the entire discussion in this thesis: What significance do Gentzen's consistency proofs hold? Let us recall the arguments of each chapter in the perspective of this question.

In Chapter 2, we examined Gentzen's response to the Brouwer-style objection against the significance of consistency proofs. The objection claimed that consistency proofs hold no significance because such proofs do not enable us to interpret classical mathematics as correct: Consistency proofs do not provide each theorem of classical mathematics with a sense such that the theorem is correct according to it. Gentzen's 1935 and 1936 proofs not only proved the consistency of first-order arithmetic, but also responded to this objection. Gentzen's interpretation of arithmetical formulas provided each theorem of first-order classical arithmetic with a sense from Gentzen's finitist standpoint such that the theorem is correct according to this sense.

We noted the following significance of the 1935 proof: This proof provided each theorem of first-order classical arithmetic with a sense that is admissible to intuitionists. We formulated Gentzen's interpretation of arithmetical formulas with the notion of spreads, which are infinite trees in intuitionistic mathematics, and proved the main lemma of the 1935 proof by means of monotone bar induction. Remember that monotone bar induction is an induction principle of intuitionistic mathematics. As we have emphasized in Section 2.1, Hilbert's 1920s proof theory did not include the significance above. Thus, Gentzen's proof theory is distinguished from Hilbert's proof theory in this period here.

In Chapter 3, we discussed a Gentzen-style interpretation of implication. Several authors in the early 20th century remarked on how to interpret implication in mathematics. For example, Hilbert, Bernays and Brouwer were concerned with the problem of interpreting implication. Gentzen also dealt with this problem and pointed out certain circularity concerning implication: He proposed an interpretation of implication and then showed that an argument for the soundness of modus ponens on this interpretation includes trivial circular reasoning. Gentzen said that it was one of the main objectives of his 1935 and 1936 proofs to avoid this circularity, but he did not explicitly present his interpretation of implication. Moreover, he did not argue that his interpretation avoids circularity.

According to our arguments in Chapter 3, the 1935 proof has the following significance: This proof gives a Gentzen-style solution to the problem of interpreting implication. We formulated the Gentzen-style interpretation of implication in [Taka15], using Tait's definition of reduction procedures, and argued that this interpretation avoids the circularity Gentzen urged against. To avoid this circularity, it sufficed to show that there is a reduction procedure for the conclusion of an instance I of the cut rule, appealing to the soundness of the instances of lower "complexity" than I. We demonstrated that the step-by-step argument with the two induction principles shows this. Although Gentzen's arguments themselves included a gap, his methods, together with contemporary devices, provided an interpretation of implication that avoids circularity.

In Chapter 4, we have discussed the contentual and formal aspects of Gentzen's 1936 consistency proof. Sieg, in his recent paper [Sie12], clarified the distinction between contentual correctness proofs and formal correctness proofs that was made in Gentzen's unpublished manuscripts. Moreover, Sieg observed that Gentzen's 1936 proof is intermediate between contentual correctness proofs and formal correctness proofs. We posed the following two questions and answered them: Is the 1936 proof both contentual and formal in Gentzen's sense? If so, how do the contentual and formal aspects of the 1936 proof relate to each other? We answered the first question affirmatively, by showing that the main lemma of the 1936 proof implies both of the main lemmas of the 1935 proof and the 1938 proof, which are contentual and formal, respectively. For the second question, we claimed that contentual aspects of the 1936 proof are formed by its formal aspects: In the 1936 proof, the correctness of theorems of first-order arithmetic is shown by normalization trees, which constitute the 1936 proof's formal aspects.

Our arguments noted the following significance of the 1936 proof: This proof shows that one can assign a finitist sense to each theorem of first-order arithmetic not only by the 1935 proof's method, but also by a generalization of the cut elimination procedure in the 1938 proof. We saw that one can assign such a sense to each theorem of first-order arithmetic by means of normalization trees and that normalization trees were obtained by generalizing the cut elimination procedure in the 1938 proof.

Then, we can formulate both Gentzen's response to the Brouwer-style objection and the Gentzen-style interpretation of implication, which we discussed in Chapters 2 and 3, by using normalization trees. This provides us with a connection between these two conceptual roles of Gentzen's interpretation of arithmetical formulas and the cut elimination method employed for the 1938 proof. In other words, we obtain a connection between the two aims above of Gentzen's research in 1935 and a result that he reached in 1938.

In this thesis, we investigated Gentzen's interpretation of arithmetical formulas in the perspective of the foundations of mathematics. Our arguments indicated Gentzen's intense interest in foundational issues. He wrote much about the significance of consistency proofs[29] and realized his foundational ideas through ingenious proof-theoretic methods. As seen in Chapter 4, Gentzen left unpublished manuscripts, in which we can find his attempt to deal with foundational issues. It remains an important work for the future to evaluate Gentzen's unpublished manuscripts, and we hope that this thesis provides a clue to the new conception of Gentzen's consistency proofs that his manuscripts might give.

[29] For example, see [Gen36, §§9-10].

Bibliography

[Aki10] R. Akiyoshi. "Gentzen's First Consistency Proof Revisited." In *CARLS Series of Advanced Study of Logic and Sensibility Vol.4*, 315-24. Tokyo: Keio University, 2010.

[AT13] R. Akiyoshi and Y. Takahashi. "Reading Gentzen's Three Consistency Proofs Uniformly." (in Japanese) *Kagaku kisoron kenkyū* (*Journal of the Japan Association for Philosophy of Science*) 41 (2013): 1-22.

[AR01] J. Avigad and E. H. Reck. "“Clarifying the nature of the infinite”: the development of metamathematics and proof theory." Carnegie Mellon Technical Report CMU-PHIL-120, 2001.

[Ara02] T. Arai. "Review: Three papers on proof theory by W. Buchholz and S. Tupailo." *The Bulletin of Symbolic Logic* 8 (2002): 437-439.

[Ber35] P. Bernays. "Sur le platonisme dans les mathématiques." L'Enseignement Mathématique 34 (1935): 52-69.

[Ber67] P. Bernays. "Hilbert, David." In *The Encyclopedia of Philosophy* Vol. 3, edited by P. Edwards, 496-504. New York: Macmillan, 1967.

[Ber70] P. Bernays. "On the original Gentzen consistency proof for number theory." In *Intuitionism and proof theory*, edited by A. Kino, J. Myhill, and R. Vesley, 409-17. Amsterdam: North-Holland, 1970.

[Ber83] P. Bernays. "On platonism in mathematics." In *Philosophy of mathematics: Selected readings, second edition*, edited by P. Benacerraf and H. Putnam, 258-71. Cambridge: Cambridge University Press, 1983.

[Bro22] L. E. J. Brouwer. "Intuitionistische Mengenlehre." *KNAW Proceedings* 23 (1922): 949-54.

[Bro23] L. E. J. Brouwer. "Über die Bedeutung des Satzes vom ausgeschlossenen Dritten in der Mathematik, insbesondere in der Funktionentheorie." *Journal für die reine und angewandte Mathematik* 154 (1923): 1-7.

[Bro25] L. E. J. Brouwer. "Zur Begründung der intuitionistischen Mathematik I." *Mathematische Annalen* 93 (1925): 244-57.

[Bro28] L. E. J. Brouwer. "Intuitionistische Betrachtungen über den Formalismus." *KNAW Proceedings* 31 (1928): 374-9.

[Bro33] L. E. J. Brouwer. "Willen, Weten, Spreken." *Euclides* 9 (1933): 177-93. English translation in [vSt90, pp.418-431].

[Buc91] W. Buchholz. "Notation systems for infinitary derivations." *Archive for Mathematical Logic* 30 (1991): 277-96.

[Buc97] W. Buchholz. "Explaining Gentzen's consistency proof within infinitary proof theory." In *Computational Logic and Proof Theory: 5th Kurt Gödel Colloquium, KGC'97*, edited by G. Gottlob, A Leitsch and D. Mundici. Vol.1289 of Lecture Notes in Computer Science, 4-17. Berlin: Springer, 1997.

[Buc01] W. Buchholz. "Explaining the Gentzen-Takeuti reduction steps: a second-order system." *Archive for Mathematical Logic* 40 (2001): 255-72.

[Buc10] W. Buchholz. "Another Reduction of Classical ID_ν to Constructive ID^i_ν." In *Ways of Proof Theory*, edited by R. Schindler, 183-98. Frankfult: ontos verlag, 2010.

[Buc15] W. Buchholz. "On Gentzen's first consistency proof for arithmetic." In *Gentzen's Centenary: The Quest for Consistency*, edited by R. Kahle and M. Rathjen, 63-87. Switzerland: Springer International Publishing, 2015.

[Coq95] T. Coquand. "A Semantics of Evidence for Classical Arithmetic." *The Journal of Symbolic Logic* 60 (1995): 325-37.

[Det90] M. Detlefsen. "On an Alleged Refutation of Hilbert's Program Using Gödel's First Incompleteness Theorem." *Journal of Philosophical Logic* 19 (1990): 343-77.

[Det15] M. Detlefsen. "Gentzen's Anti-Formalist Views." In *Gentzen's Centenary: The Quest for Consistency*, edited by R. Kahle and M. Rathjen, 25-44. Switzerland: Springer International Publishing, 2015.

[Dum00] M. Dummett. *Elements of Intuitionism*, second edition. Oxford: Clarendon Press, 2000.

[Ewa96] W. B. Ewald. *From Kant to Hilbert*, Vol. 2. Oxford: Clarendon Press, 1996.

[Gen36] G. Gentzen. "Die Widerspruchsfreiheit der reinen Zahlentheorie." *Mathematische Annalen* 112 (1936): 493-565. English translation in [Gen69, ch.4].

[Gen38a] G. Gentzen. "Die gegenwärtige Lage in der mathematischen Grundlagenforshung." *Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften* 4 (1938): 5-18. English translation in [Gen69, ch.7].

[Gen38b] G. Gentzen. "Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie." *Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften* 4 (1938): 19-44. English translation in [Gen69, ch.8].

[Gen69] G. Gentzen. *The Collected Papers of Gerhard Gentzen*, edited by M. E. Szabo. Amsterdam: North-Holland, 1969.

[Gen74] G. Gentzen. "Der erste Widerspruchsfreiheitsbeweis für die klassische Zahlentheorie." *Archiv für mathematische Logik und Grundlagenforschung* 16 (1974): 97-118.

[Göd38] K. Gödel. "Lecture at Zilsel's." In [Göd95, pp.86-113].

[Göd95] K. Gödel. *Collected Works. III: Unpublished essays and lectures*, edited by S. Feferman, J. Dawson, S. Kleene, G. Moore, R. Solovay, and J. van Heijenoort. Oxford: Oxford University Press, 1995.

[Hey34] A. Heyting. "Mathematische Grundlagenforschung, Intuitionismus, Beweistheorie." *Ergebnisse d. Math. und ihrer Grenzgebiete* 3 (1934).

[Hal90] M. Hallett. "Physicalism, reductionism and Hilbert." In *Physicalism in Mathematics*, edited by A. Irvine, 182-256. Dordrecht: D. Reidel Publishing Co., 1990.

[vHe67] J. van Heijenoort. *From Frege to Gödel.* Harvard University Press, 1967.

[Hil00] D. Hilbert. "Mathematische Probleme." *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Math.-Phys. Klasse* (1900): 25397.

[Hil05] D. Hilbert. "Über die Grundlagen der Logik und der Arithmetik." *Verhandlungen des dritten internationalen Mathematiker-Kongresses in Heidelberg vom 8. bis 13. August 1904*, 174-85. Leipzig: Teubner, 1905.

[Hil22] D. Hilbert. "Neubegründung der Mathematik: Erste Mitteilung." *Abhandlungen aus dem Seminar der Hamburgischen Universität* 1 (1922): 157-77.

[Hil26] D. Hilbert. "Über das Unendliche." *Mathematische Annalen* 95 (1926): 161-90.

[Hil28] D. Hilbert. "Die Grundlagen der Mathematik." *Abhandlungen aus dem Seminar der Hamburgischen Universität* 6 (1928): 65-85.

[Hil31] D. Hilbert. "Beweis des tertium non datur." *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen Mathematische-physikalische Klasse* (1931): 120-5.

[HB1934] David Hilbert and Paul Bernays. *Grundlagen der Mathematik*, Vol. 1. Berlin: Springer, 1934.

[HB2011] David Hilbert and Paul Bernays. *Foundations of Mathematics I, Part A: Prefaces and §§1–2*, edited by Claus-Peter Wirth, J. Siekmann, M. Gabbay and D. Gabbay. London: College Publications, 2011.

[Kan06] H. Kaneko. "Undetachability of Propositional Content and Its Process of Construction." *Annals of the Japan Association for Philosophy of Science* 14 (2006): 101-12.

[Kre51] G. Kreisel. "On the Interpretation of Non-Finitist Proofs – Part I." *The Journal of Symbolic Logic* 16 (1951): 241-67.

[Kre71] G. Kreisel. "Review: The Collected Papers of Gerhard Gentzen. by M. E. Szabo." *The Journal of Philosophy* 68 (1971): 238-65.

[Man98] P. Mancosu, ed. *From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920s.* Oxford: Oxford University Press, 1998.

[Men07] E. Menzler-Trott. *Logic's lost genius : the life of Gerhard Gentzen*, translated by C. Smoryński and E. Griffor. American Mathematical Society, 2007.

[Min78] G. E. Mints. "Finite investigations of transfinite derivations." *Journal of Soviet Mathematics* 10 (1978): 548-96.

[Neg80] M. Negri. “Constructive sequent reduction in Gentzen’s first consistency proof for arithmetic.” In *Italian Studies in the Philosophy of Science*, edited by Maria Luisa Dalla Chiara, 153-68. Dordrecht: Reidel, 1980.

[Oka88] M. Okada. “On a Theory of Weak Implications.” *The Journal of Symbolic Logic* 53 (1988): 200-211.

[Oka08] M. Okada. “Some remarks on difference between Gentzen’s finitist and Heyting’s intuitionist approaches toward intuitionistic logic and arithmetic.” *Annals of the Japan Association for Philosophy of Science* 16 (2008): 1-18.

[vPl09a] J. von Plato. “Gentzen’s logic.” In *Handbook of the History of Logic. Volume 5. Logic from Russell to Church*, edited by D. Gabbay and J. Woods, 667-721. Amsterdam: Elsevier, 2009.

[vPl09b] J. von Plato. “Gentzen’s original proof of the consistency of arithmetic revisited.” In *Acts of Knowledge – History, Philosophy and Logic*, edited by G. Primiero and S. Rahman, 151-71. London: College Publications, 2009.

[Poi06] H. Poincaré. “Les mathématiques et la logique.” *Revue de métaphysigue et de morale* 14 (1906): 17-34.

[Sch77] H. Schwichtenberg. “Proof-Theory: Some Applications of Cut-Elimination.” In *Handbook of mathematical logic*, edited by J. Barwise, 867-95. Amsterdam: North-Holland, 1977.

[Sie12] W. Sieg. “In the Shadow of Incompleteness: Hilbert and Gentzen.” In *Epistemology versus Ontology: essays on the philosophy and foundations of mathematics in honour of Per Martin-Löf*, edited by P. Dybjer, S. Lindström, E. Palmgren and G. Sundholm, 87-127. London: Springer, 2012.

[SP95] W. Sieg and C. Parsons. “Introductory note to **1938a*.” In [Göd95, pp.62-85].

[SR05] W. Sieg and M. Ravaglia. “David Hilbert and Paul Bernays, Grundlagen der Mathematik.” In *Landmark Writings in Western Mathematics, 1640-1940*, edited by I. Grattan-Guinness, 981-99. Amsterdam: Elsevier, 2005.

[vSt90] W. P. van Stigt. *Brouwer's Intuitionism*. Amsterdam: North-Holland, 1990.

[Sun83] G. Sundholm. *Proof theory: A survey of the omega rule*. Oxford D. Phil. dissertation, 1983.

[Tai01] W. W. Tait. "Gödel's Unpublished Papers on Foundations of Mathematics." *Philosophia Mathematica* 9 (2001): 87-126.

[Tai02] W. W. Tait. "Remarks on finitism." In *Reflections on the Foundations of Mathematics: Essays in honor of Solomon Feferman*, edited by W. Sieg, R. Sommer and C. Talcott., 407-16. Vol. 15 of Lecture Notes in Logic, 407-16. Urbana: Association for Symbolic Logic, 2002.

[Tai05] W. W. Tait. "Gödel's reformulation of Gentzen's first consistency proof for arithmetic: the no-counterexample interpretation." *The Bulletin of Symbolic Logic* 11 (2005): 225-38.

[Tai15] W. W. Tait. "Gentzen's original consistency proof and the Bar Theorem." In *Gentzen's Centenary: The Quest for Consistency*, edited by R. Kahle and M. Rathjen, 213-28. Switzerland: Springer International Publishing, 2015.

[Taka15] Y. Takahashi. "Gentzen's 1935 Consistency Proof and an Interpretation of Implication." (in Japanese) *Tetsugaku* (*Philosophy*), edited by Mita Philosophy Society, 135 (2015): 45-58.

[Taka16] Y. Takahashi. "A Philosophical Significance of Gentzen's 1935 Consistency Proof for First-Order Arithmetic: on a Circularity of Implication." (in Japanese) *Kagaku Tetsugaku* (*PHILOSOPHY OF SCIENCE*), 49 (2016): 49-66.

[Take87] G. Takeuti. *Proof Theory*, second edition. Amsterdam: North-Holland, 1987.

[TD88] A. S. Troelstra and D. van Dalen. *Constructivism in Mathematics*, Vol. 1. Amsterdam: Elsevier, 1988.

[Wey25] H. Weyl. "Die heutige Erkenntnislage in der Mathematik." *Symposion* 1 (1925): 1-32, reprinted in [Wey68, pp.511-542]. English translation in [Man98, pp.123-142].

[Wey68] H. Weyl. *Gesammelte Abhandlungen*, Vpl. 2, edited by K. Chandrasekharan. Berlin: Springer Verlag, 1968.

[Yas80] M. Yasugi. “Gentzen reduction revisited.” *Publications of RIMS* 16 (1980): 1-33.

[Zac03] R. Zach. “The Practice of Finitism: Epsilon Calculus and Consistency Proofs in Hilbert's Program.” *Synthese* 137 (2003): 211-59.