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New Proofs of Some Basic Theorems on Stationary Point Processes

Nariyuki Minami*

Summary — We give new proofs of three basic theorems on stationary point processes on the real line — theorems of Khintchine, Korolyuk, and Dobrushin. Moreover we give a direct construction of the Palm measure for a class of point processes which includes stationary ones as special cases.

1. Introduction

The purpose of this note is to give new proofs, based on a same simple idea, to some basic theorems on stationary point processes on the real line $\mathbb{R}$, as stated in standard treatises on point processes such as Daley and Vere-Jones (see §3.3 of [3]).

To begin with, let us introduce necessary definitions and notation. By $M_p$, we denote the set of all integer-valued Radon measures on $\mathbb{R}$. Namely $M_p$ is the totality of all measures $N(dx)$ on $\mathbb{R}$ such that for any bounded Borel set $B$, $N(B)$ is a non-negative integer. Let us call any such measure a counting measure. For a counting measure $N \in M_p$, let us define

$$X(t) := N((0, t]) \quad (t \geq 0), \quad = -N((t, 0)) \quad (t < 0).$$

Then the function $X(t)$ is right-continuous, integer-valued, locally bounded and non-decreasing. Hence $X(t)$ is piecewise constant on $\mathbb{R}$ and the set $\Delta$, finite or countably infinite, of its points of discontinuity has no accumulation points other than $\pm \infty$. Thus the points in $\Delta$ can be ordered as

$$\cdots < x_{-1} < x_0 \leq 0 < x_1 < x_2 < \cdots,$$

so that if we let $m_n := X(x_n) - X(x_n - 0)$, then $N(dx)$ can be represented as

$$N(dx) = \sum_n m_n \delta_{x_n}(dx),$$

where $\delta_a$ denotes the unit mass placed at $a$. Each $m_n$ is a positive integer and is called

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* 南就将, 慶應義塾大学医学部数学教室 (〒 223-8521 横浜市港北区日吉 4-1-1) : School of medicine, Keio University, Hiyoshi, Kohoku-ku, Yokohama 223-8521, Japan [Received Mar. 3, 2012]
the *multiplicity* of the point $x_n$. In general, either $N ([0, \infty))$ or $N ((-\infty, 0))$ can be finite, in which case either $\{x_n\}_{n=0}^\infty$ or $\{x_n\}_{n=0}^\infty$ is a finite sequence. If in the former [resp. latter] case $\{x_n\}_{n=0}^\infty$ [resp. $\{x_n\}_{n=0}^\infty$] terminates with $x_\nu$, then we will set $x_n = \infty$ [resp. $x_n = -\infty$] for $n > \nu$ [resp. $n < \nu$]. When $m_n = 1$ for all $n$ such that $x_n \neq \pm \infty$, the counting measure $N$ is said to be *simple*. For each $N \in M_p$ with representation (2), let us associate a simple counting measure $N^*$ defined by

$$N^*(dx) = \sum_n \delta_{x_n}(dx).$$

(3)

In order to make $M_p$ a measurable space, we define $\mathcal{M}_p$ to be the $\sigma$-algebra of subsets of $M_p$ generated by all mappings of the form

$$M_p \ni N \mapsto N (B) \in [0, \infty]$$

(4)

for all Borel sets $B \subset \mathbb{R}$. Then we see that $x_n$, $m_n$ and $N^*$ are all measurable functions of $N$, as the following lemma shows.

**Lemma 1**  
(i) The set

$$C := \{N \in M_p : N ((-\infty, 0]) = N ((0, \infty)) = \infty\} = \{N \in M_p : x_n \text{ is finite for all } n\}$$

belongs to $\mathcal{M}_p$.

(ii) For each integer $n$, $x_n$ and $m_n$ are $\mathcal{M}_p$-measurable functions of $N$.

(iii) The mapping $M_p \ni N \mapsto N^* \in M_p$ is $\mathcal{M}_p / \mathcal{M}_p$-measurable.

**Proof.** (i) The assertion is obvious from the definition of $\mathcal{M}_p$, since we can write

$$C = \bigcap_{k=1}^\infty \bigcup_{n=1}^\infty \{N \in M_p : N ((-n, 0]) > k, N ((0, n]) > k\}.$$ 

(ii) The measurability of $x_1$ follows from the relation

$$\{N \in M_p : x_1 > t\} = \{N \in M_p : N ((0, t]) = 0\},$$

which holds for all $t \geq 0$. Now for each $k \geq 1$, define

$$x_1^{(k)} := \sum_{j=1}^k \frac{j}{2k} 1_{[\frac{j}{2k-1}, \frac{j}{2k})} (x_1) + \infty \cdot 1_{\{x_1 = \infty\}}.$$ 

Then we see that $x_1^{(k)}$ is measurable in $N$ and that $x_1^{(k)} \downarrow x_1$ as $k \to \infty$. By the right-continuity of $X(t) = N ((0, t])$ at $t > 0$, we have, as $k \to \infty$,

$$1_{\{x_1 < \infty\}} \cdot X(x_1^{(k)}) = \sum_{j=1}^\infty 1_{[\frac{j}{2k-1}, \frac{j}{2k})} (x_1) X\left(\frac{j}{2k}\right) \to X(x_1) = m_1,$$

which shows the measurability of $m_1$ in $N$.

Next let $Y(t) := X(t) - X(t \wedge x_1)$. This is measurable in $N$ for all $t \geq 0$, since
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\[X(t \land x_i) = X(t)1_{[x_i, \infty)} + X(x_i)1_{(x_i, x_i]}\].

If we apply the above argument to \(X(t)\) instead of \(X(t)\), we can verify the measurability of \(x_2\) and \(m_2\) in \(N\), and the argument can be iterated to give the measurability of all \(x_n\) and \(m_n\).

(iii) For each \(j = 0, 1, 2, \ldots\) and \(t > 0\), the sets

\[\{N \in M_p : N^\bullet ((0, t]) = j\} = \{N \in M_p : x_j \leq t < x_{j+1}\}\]

and

\[\{N \in M_p : N^\bullet ((-t, 0]) = j\} = \{N \in M_p : x_{-j} \leq t < x_{-j+1}\}\]

belong to \(M_p\). Now for each \(n \geq 1\), let \(G_n\) be the class of all Borel subsets \(B\) of \([-n, n]\) such that the mapping

\[M_p \ni N \mapsto N^\bullet (B) \in [0, \infty)\]  \hspace{1cm} (5)

is measurable. Then \(G_n\) is seen to be a \(\lambda\)-system which contains the class of intervals

\[I := [(0, t] : 0 < t \leq t] \cup [(-t, 0] : 0 < t \leq n]\]

which forms a \(\pi\)-system. Hence by Dynkin’s \(\pi\)-\(\lambda\) theorem (see e.g. Durrett [2]), \(G_n\) contains all Borel subsets of \([-n, n]\). Since \(n \geq 1\) is arbitrary, and since we can write \(N^\bullet (B) = \lim_{n \to \infty} N^\bullet (B \cap [-n, n])\), the mapping (5) is measurable for all Borel subsets of \(\mathbb{R}\).

**Remark 1.** By an argument similar to (iii), it is easy to show that \(M_p\) is generated by mappings \(M_p \ni N \mapsto X(t)\) for all \(t\), where \(X(t)\) is defined in (1).

**Definition 1** A point process \(N_\omega\) is a random variable defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and taking values in the measurable space \((M_p, \mathcal{M}_p)\).

**Definition 2** A point process \(N_\omega\) is said to be crudely stationary if for any bounded interval \(I\) and for any \(x \in \mathbb{R}\), \(N_\omega(I)\) and \(N_\omega(I + x)\) are identically distributed. Its mean density is the expectation value \(m := \mathbb{E}[N_\omega((0, 1])] \leq \infty\).

**Definition 3** A point process \(N_\omega\) is said to be stationary if for any \(C \in \mathcal{M}_p\) and \(x \in \mathbb{R}\), one has the identity

\[P(N_\omega(\cdot) \in C) = P(N_\omega(x + \cdot) \in C)\].

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Obviously, $N_\omega$ is crudely stationary if it is stationary.

Remark 2. By another application of $\pi\lambda$ theorem, one can show without difficulty that $N_\omega$ is stationary if and only if for any finite family of Borel subsets $B_1, \ldots, B_n$ of $\mathbb{R}$, and of non-negative integers $k_1, \ldots, k_n$, the identity

$$P(N_\omega(B_i) = k_i, i = 1, \ldots, n) = P(N_\omega(x + B_i) = k_i, i = 1, \ldots, n)$$

holds for any $x \in \mathbb{R}$.

2. Basic theorems and their proofs

Our argument is based on the following lemma, which is an immediate consequence of Definition 2.

Lemma 2 Let the point process $N_\omega$ be crudely stationary. Then for any bounded interval $I$ and for any non-negative integer $k$,

$$P(N_\omega(I) = k) = \int_0^1 P(N_\omega(x + I) = k) dx = E \left[ \int_0^1 1_{\{N_\omega(I) = k\}} dx \right].$$

Proposition 1 (Khintchine's theorem) For any crudely stationary point process $N_\omega$, the limit

$$\lambda := \lim_{h \to 0} \frac{1}{h} P(N_\omega((0, h]) > 0)$$

exists and satisfies $\lambda \leq m$. $\lambda$ is called the intensity of the point process $N_\omega$.

Proof. Let $N_\omega$ be represented as (2) and define the point process $N_\omega^*$ by (3). If we set $\nu(\omega) := N_\omega^*(0, 1]$, it satisfies $x_{\nu(\omega)}(\omega) \leq 1 < x_{\nu(\omega)+1}(\omega)$. Obviously we have

$$\{ x \in (0, 1] : N_\omega([x, x + h]) > 0 \} = (0, 1] \cap \bigcup_{j=1}^{\nu(\omega)+1} [x_j(\omega) - h, x_j(\omega))$$

$$= (0, 1] \cap \bigcup_{j=1}^{\nu(\omega)+1} J^\omega_j(h) = \sum_{j=1}^{\nu(\omega)+1} (0, 1] \cap (J^\omega_j(h) \setminus J^\omega_{j-1}(h)),$$

where we have set $J^\omega_j(h) := [x_j(\omega) - h, x_j(\omega))$ and $J_0 = 0$. Hence

$$\frac{1}{h} \int_0^1 1_{\{N_\omega([x, x + h]) > 0\}} dx = \frac{1}{h} \sum_{j=1}^{\nu(\omega)+1} (0, 1] \cap (J^\omega_j(h) \setminus J^\omega_{j-1}(h))$$

$$= \sum_{j=1}^{\nu(\omega)+1} \frac{1}{h} \{ (1 \wedge x_j(\omega)) - (0 \vee x_{j-1}(\omega) \vee (x_j(\omega) - h) \},$$
where for a Borel subset $B$ of $\mathbb{R}$, $|B|$ denotes its Lebesgue measure and for a real number $a$, $a_+ := \max\{a, 0\}$ denotes its positive part. Now it is easy to see that for $1 \leq j \leq \nu(\omega)$,
\[ \frac{1}{h} \left( 1 \wedge x_j(\omega) \right) - \left( 0 \vee x_{j-1}(\omega) \vee (x_j(\omega) - h) \right), \]
as $h \searrow 0$, and that for $j = \nu(\omega) + 1$,
\[ \frac{1}{h} \left( \left( 1 \wedge x_{\nu(\omega)+1}(\omega) \right) - \left( 0 \vee x_{\nu(\omega)}(\omega) \vee (x_{\nu(\omega)+1}(\omega) - h) \right) \right), \]
is bounded by 1 and tends to 0 as $h \searrow 0$. Thus we can apply the monotone convergence theorem, the dominated convergence theorem and Lemma 2, to obtain
\[ \lim_{h \searrow 0} \frac{1}{h} \mathbb{P}(N_\omega((0, h]) > 0) = \mathbb{E} \left[ \frac{1}{h} \sum_{j=1}^{\nu(\omega)+1} \mathbb{I}_{(0, 1]}(J_j(\omega) \setminus J_j(\nu(\omega)+1)(h)) \right] \]
\[ \to \mathbb{E} \left[ \sum_{j=1}^{\nu(\omega)} \mathbb{I}_{(0, 1]} \right] = \mathbb{E} \left[ N_\omega^*(0, 1) \right], \]
as $h \searrow 0$. Thus the desired limit $\lambda$ exists and is equal to $\mathbb{E}[N_\omega^*(0, 1)]$. Clearly it satisfies the inequality $\lambda \leq \mathbb{E}[N_\omega((0, 1])] = m$.

**Corollary 1** If $N_\omega$ is simple, then $\lambda = m$. When $m < \infty$, the converse is also true.

**Proof.** $N_\omega$ is simple if and only if $N_\omega^* = N_\omega$ almost surely, which obviously implies $\lambda = m$. On the other hand, if $\lambda = m < \infty$, then
\[ \mathbb{E}[N_\omega((0, 1]) - N_\omega^*(0, 1)]) = m - \lambda = 0. \]
But $N_\omega((0, 1]) - N_\omega^*(0, 1)] \geq 0$ in general, so that $N_\omega((0, 1]) = N_\omega^*(0, 1)$ almost surely. The same argument is valid if the interval $(0, 1]$ is replaced by $(n, n + 1]$, so that $N_\omega((n, n + 1]) = N_\omega^*(n, n + 1)$ almost surely for all integers $n$, and the simplicity of $N_\omega$ follows.

**Remark 3.** In the treatise by Daley and Vere-Jones [3], for example, Proposition 1 is proved in the following way: If we define $\varphi(h) := \mathbb{P}(N_\omega((0, h]) > 0)$, then by the crude stationarity, we have for any positive $h_1$ and $h_2$,
\[ \varphi(h_1 + h_2) = \mathbb{P}(N_\omega((0, h_1 + h_2]) > 0) = \mathbb{P}(N_\omega((0, h_1]) + N_\omega((h_1, h_1 + h_2]) > 0) \]
\[ \leq \mathbb{P}(N_\omega((0, h_1]) > 0) + \mathbb{P}(N_\omega((h_1, h_1 + h_2]) > 0) = \varphi(h_1) + \varphi(h_2), \]
so that $\varphi(h)$ is a sub-additive function defined on $[0, \infty)$ satisfying $\varphi(0) = 0$. To show the existence of the intensity $\lambda$, it suffices to apply the following well known lemma.
Lemma 3  Let \( g(x) \) be a sub-additive function defined on \([0, \infty)\) such that \( g(0) = 0 \). Then one has
\[
\lim_{x \to 0} g(x) x = \sup_{x > 0} \frac{g(x)}{x} \leq \infty.
\]

However, this argument does not provide the representation \( \lambda = \mathbb{E}[N^*_\omega((0, 1])] \), so that the proof of Corollary 1 requires some extra work. Our proof above is closer to that of Leadbetter [5]. See also Chung [1].

Definition 4  A crudely stationary point process \( N_\omega \) is said to be orderly when
\[
P(N_\omega((0, h]) \geq 2) = o(h) \quad (h \searrow 0).
\]

Proposition 2 (Dobrushin’s theorem)  If a crudely stationary point process \( N_\omega \) is simple and if \( \lambda < \infty \), then \( N_\omega \) is orderly.

Proof.  By Lemma 2, we can write
\[
P(N_\omega((0, h]) \geq 2) = \mathbb{E}\left[ \int_0^1 1_{\{N_\omega((x,x+h]) \geq 2\}} \, dx \right].
\]
As can be seen from the proof of Proposition 1, we have
\[
\frac{1}{h} \int_0^1 1_{\{N_\omega((x,x+h]) \geq 2\}} \, dx \leq \frac{1}{h} \int_0^1 1_{\{N_\omega((x,x+h]) \geq 2\}} \, dx
\]
\[
= \sum_{j=1}^{\nu(h)} \frac{1}{h} \left[ (0, 1] \cap (J^x_j(h) \setminus J^x_{j-1}(h)) \right] + \frac{1}{h} \left[ (0, 1] \cap (J_{\nu(h)+1}(h) \setminus J_{\nu(h)}(h)) \right]
\]
\[
\leq N^*_\omega((0,1]) + 1,
\]
and
\[
\lim_{h \searrow 0} \frac{1}{h} \int_0^1 1_{\{N_\omega((x,x+h]) \geq 2\}} \, dx = \mathbb{E}\left[ \#(j : x_j(\omega) \in (0,1], \, m_j(\omega) \geq 2) \right].
\]
Since \( \mathbb{E}[N^*_\omega((0,1])] = \lambda < \infty \), we can apply the dominated convergence theorem, to obtain
\[
\lim_{h \searrow 0} \frac{1}{h} \mathbb{P}(N_\omega((0, h]) \geq 2) = \mathbb{E}\left[ \#(j : x_j(\omega) \in (0,1], \, m_j(\omega) \geq 2) \right],
\]
which is equal to 0 if \( N_\omega \) is simple.

Remark 4.  The condition \( \lambda < \infty \) cannot be dropped. For a counter example, see Exercise 3.3.2 of [3].

Proposition 3 (Korolyuk’s theorem)  A crudely stationary, orderly point process is simple.
Proof. By Fatou’s lemma and the orderliness of $N_w$, 

$$
E\{j : x_j(\omega) \in (0,1], \, m_j(\omega) \geq 2]\} = E\left[\liminf_{h \downarrow 0} \frac{1}{h} \int_0^1 1_{\{N_w((x,x+h]) \geq 2\}} dx\right] 
$$

$$
\leq \liminf_{h \downarrow 0} \frac{1}{h} P(N_w((0,h]) \geq 2) = 0 ,
$$

so that with probability one, $N_w$ has no multiple points in $(0, 1]$. By crude stationarity, the above argument is also valid if $(0, 1]$ is replaced by $(n, n+1]$ for any integer $n$. Hence $N_w$ is simple.

**Proposition 4** For a crudely stationary point process $N_w$ with finite intensity $\lambda$, the limits 

$$
\lambda_k := \lim_{h \downarrow 0} \frac{1}{h} P(1 \leq N_w((0,h]) \leq k)
$$

exists for $k = 1, 2, \ldots$, and satisfy $\lambda_k \nearrow \lambda$ as $k \to \infty$. Moreover for $k = 1, 2, \ldots$, 

$$
\pi_k := \frac{\lambda_k - \lambda_{k-1}}{\lambda} = \lim_{h \downarrow 0} P(N_w((0,h]) = k \mid N_w(0,h]) > 0)
$$

where we set $\lambda_0 := 0$.

Proof. As before, one has 

$$
\frac{1}{h} \int_0^1 1_{\{1 \leq N_w((x,x+h]) \leq k\}} dx \leq 1 + N^*_w((0,1])
$$

and 

$$
\lim_{h \downarrow 0} \frac{1}{h} \int_0^1 1_{\{1 \leq N_w((x,x+h]) \leq k\}} dx = \{j : x_j(\omega) \in (0,1], \, m_j(\omega) \leq k\}.
$$

Since $\lambda = E[N^*_w((0,1])] < \infty$, we can apply the dominated convergence theorem and Lemma 2, to obtain 

$$
\lambda_k = \lim_{h \downarrow 0} \frac{1}{h} E\left[\int_0^1 1_{\{1 \leq N_w((x,x+h]) \leq k\}} dx\right] = E[\{j : x_j(\omega) \in (0,1], \, m_j(\omega) \leq k\}].
$$

This representation of $\lambda_k$ immediately gives 

$$
\lim_{k \to \infty} \lambda_k = E[\{j : x_j(\omega) \in (0,1]\}] = E[N^*((0,1])] = \lambda,
$$

by the monotone convergence theorem. The last statement of the proposition is obvious.

**Corollary 2** For a crudely stationary point process with finite intensity, we have 

$$
\lambda \sum_{k=1}^{\infty} k \pi_k = E[N_w((0,1])] = m.
$$
3. The Palm measure

Let us assume that the probability space \((\Omega, \mathcal{F}, P)\), on which our point process \(N_\omega\) is defined, is equipped with a measurable flow \(\{\theta_t\}_{t \in \mathbb{R}}\). Here a measurable flow \(\{\theta_t\}\) is, by definition, a family of bijections \(\theta_t : \Omega \to \Omega\) such that

(a) \(\theta_0\) is the identity mapping, and for any \(s, t \in \mathbb{R}\), \(\theta_s \circ \theta_t = \theta_{s+t}\) holds;

(b) the mapping \((t, \omega) \to \theta_t(\omega)\) from \(\mathbb{R} \times \Omega\) into \(\Omega\) is jointly measurable with respect to \(\mathcal{B}(\mathbb{R}) \times \mathcal{F}\), where \(\mathcal{B}(\mathbb{R})\) is the Borel \(\sigma\)-algebra on \(\mathbb{R}\).

Let us further assume that the relation

\[
\int_{\mathbb{R}} N_{\theta_t}\varphi(x) = \int_{\mathbb{R}} N_\omega(dx)\varphi(x-t) \tag{6}
\]

holds for any \(t \in \mathbb{R}\) and any continuous function \(\varphi\) with compact support. If the probability measure \(P\) is \(\{\theta_t\}\)-invariant in the sense \(P \circ \theta_t^{-1} = P\) for all \(t \in \mathbb{R}\), then by (6), our point process \(N_\omega\) is stationary.

**Definition 5** The Palm measure of a point process \(N_\omega\) is a measure kernel \(Q(x, d\omega)\) on \(\mathbb{R} \times \Omega\) such that for any jointly measurable, non-negative function \(f(x, \omega)\), the relation

\[
\int_{\Omega} P(d\omega) \int_{\mathbb{R}} N_\omega(dx) f(x, \omega) = \int_{\mathbb{R}} \lambda(dx) \int_{\Omega} Q(x, d\omega) f(x, \omega) \tag{7}
\]

holds, where \(\lambda(dx)\) is the mean measure of \(N_\omega\) which is defined by \(\lambda(B) = E[N_\omega(B)]\) for \(B \in \mathcal{B}(\mathbb{R})\) and which we assume to be finite for bounded Borel sets \(B\).

Now let \(u(t)\) be a probability density function on \(\mathbb{R}\). Define a new probability measure \(P_u\) by

\[
\int_{\Omega} P_u(d\omega) g(\omega) = \int_{\mathbb{R}} u(t) dt \left(\int_{\Omega} P(d\omega) g(\theta_t\omega)\right), \tag{8}
\]

where \(g(\omega)\) is an arbitrary non-negative measurable function on \(\Omega\). Then the following result holds.

**Theorem 1** For any probability density \(u(t)\) on \(\mathbb{R}\), the Palm measure \(Q_u(x, d\omega)\) exists for the point process \(N_\omega\) defined on the probability space \((\Omega, \mathcal{F}, P_u)\).

**Proof.** Let \(f(x, \omega) \geq 0\) be jointly measurable on \(\mathbb{R} \times \Omega\). Then we can rewrite the left
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hand side of (7) in the following way:

\[
\int_\Omega \mathbf{P}_s(d\omega) \int_\mathbb{R} N_\omega(dx) f(x, \omega) = \int_\mathbb{R} u(t) dt \int_\Omega \mathbf{P}(d\omega) \int_\mathbb{R} N_{\theta_t \omega}(dx) f(x, \theta_t \omega)
\]

\[
= \int_\mathbb{R} u(t) dt \int_\Omega \mathbf{P}(d\omega) \int_\mathbb{R} N_\omega(dx) f(x - t, \theta_t \omega)
\]

\[
= \int_\Omega \mathbf{P}(d\omega) \int_\mathbb{R} N_\omega(dx) \int_\mathbb{R} u(t) dt f(x - t, \theta_t \omega)
\]

\[
= \int_\Omega \mathbf{P}(d\omega) \int_\mathbb{R} N_\omega(dx) \int_\mathbb{R} u(x - s) ds f(s, \theta_{x-s} \omega)
\]

\[
= \int_\mathbb{R} ds \int_\Omega \mathbf{P}(d\omega) \int_\mathbb{R} N_\omega(dx) u(x - s) f(s, \theta_{x-s} \omega).
\] (9)

At this stage, take \( f(x, \omega) = \varphi(x) \). Then (9) reduces to

\[
\int_\mathbb{R} \varphi(s) \lambda(ds) = \int_\mathbb{R} \varphi(s) \ell_u(s) ds
\] (10)

with

\[
\ell_u(s) = \int_\Omega \mathbf{P}(d\omega) \int_\mathbb{R} N_\omega(dx) u(x - s).
\] (11)

If we define, for each \( s \in \mathbb{R} \), the measure \( Q_u(s, d\omega) \) on (\( \Omega, \mathcal{F} \)) by

\[
\int_\Omega Q_u(s, d\omega) g(\omega) = \frac{\mathbf{1}_{\mathbb{R} \in \mathbb{R}}(\ell_u(s))}{\ell_u(s)} \int_\Omega \mathbf{P}(d\omega) \int_\mathbb{R} N_\omega(dx) u(x - s) g(\theta_{x-s} \omega),
\] (12)

then (9) takes the form of (7), and the theorem is proved.

When \( \mathbf{P} \) is \( \{\theta_r\} \)-invariant, then we have \( \mathbf{P}_s = \mathbf{P} \) for any probability density \( u \) on \( \mathbb{R} \), and

\[
\ell_u(s) = \int_\Omega \mathbf{P}(d\omega) \int_\mathbb{R} N_{\theta_s \omega}(dx) u(x) = \int_\Omega \mathbf{P}(d\omega) \int_\mathbb{R} N_\omega(dx) u(x) =: \ell > 0
\]

is a constant. Moreover one can compute as

\[
\int_\Omega Q_u(s, d\omega) g(\omega) = \frac{1}{\ell} \int_\Omega \mathbf{P}(d\omega) \int_\mathbb{R} N_{\theta_s \omega}(dx) u(x) g(\theta_s \omega)
\]

\[
= \frac{1}{\ell} \int_\Omega \mathbf{P}(d\omega) \int_\mathbb{R} N_{\theta_s \omega}(dx) u(x) g(\theta_{x-s}(\theta_s \omega)) = \frac{1}{\ell} \int_\Omega \mathbf{P}(d\omega) \int_\mathbb{R} N_\omega(dx) u(x) g(\theta_{x-s} \omega).
\]

Hence if we define a measure \( \hat{\mathbf{P}}(d\omega) \) on (\( \Omega, \mathcal{F} \)) by

\[
\int_\Omega \hat{\mathbf{P}}(d\omega) g(\omega) = \int_\Omega \mathbf{P}(d\omega) \int_\mathbb{R} N_\omega(dx) u(x) g(\theta_s \omega),
\]

then we get

\[
Q_u(s, d\omega) = \frac{1}{\ell} [\hat{\mathbf{P}} \circ \theta_s](d\omega),
\]

and (7) can be written in the form
\[
\int_\Omega \mathbf{P}(d\omega) \int_\mathbb{R} N_u(dx)f(x, \omega) = \int_\mathbb{R} dx \int_\Omega \hat{\mathbf{P}}(d\omega)f(x, \theta_{-t}\omega), \tag{13}
\]
which is the defining relation of the Palm measure in the stationary case (see [6]). (13) shows in particular that the definition of \(\hat{\mathbf{P}}\) is independent of the choice of \(u\).

Our consideration of the probability measure \(P_u\) is motivated by the following observation.

**Proposition 5** The probability measure \(P\) is \(\{\theta_t\}\)-invariant if and only if the following two conditions hold:

(i) \(P_u = P\) for any probability density function \(u(t)\) on \(\mathbb{R}\);
(ii) the set \(H\) of all bounded measurable functions \(\varphi(\omega)\) on \(\Omega\) such that \(t \mapsto \varphi(\theta_t\omega)\) is continuous for all \(\omega \in \Omega\) is dense in \(L^2(\Omega, \mathbf{P})\).

**Proof.** The necessity of (i) is obvious. That (ii) also follows from the \(\{\theta_t\}\)-invariance of \(P\) is proved in [6] (see Lemma II. 3). To prove the sufficiency of (i) and (ii), fix an arbitrary \(t_0 \in \mathbb{R}\) and take a sequence of probability density \(\{u_n\}_{n}\) so that \(u_n(t)dt \rightarrow \delta_{t_0}(dt)\) weakly. Now for any \(\varphi \in H\), \(t \mapsto \varphi(\theta_t\omega)\) is continuous and bounded by \(\|\varphi\|_{\infty} := \sup_{\Omega} |\varphi(\omega)|\). Hence we can apply the dominated convergence theorem, to get

\[
\int_\Omega \mathbf{P}(d\omega) \varphi(\theta_{t_0}\omega) = \int_\Omega \mathbf{P}(d\omega) \left( \lim_{n \to \infty} \int_\mathbb{R} \varphi(\theta_t\omega)u_n(t)dt \right)
= \lim_{n \to \infty} \int_\mathbb{R} \left( \int_\Omega \mathbf{P}(d\omega) \varphi(\theta_t\omega) \right) u_n(t)dt
= \lim_{n \to \infty} \int_\Omega \mathbf{P}_u(d\omega) \varphi(\omega) = \int_\Omega \mathbf{P}(d\omega) \varphi(\omega)
\]

by condition (i). But if \(H\) is dense in \(L^2(\Omega, \mathbf{P})\), we can approximate an arbitrary bounded measurable function \(g(\omega)\) by the elements of \(H\), to obtain

\[
\int_\Omega \mathbf{P}(d\omega)g(\theta_{t_0}\omega) = \int_\Omega \mathbf{P}(d\omega)g(\omega)
\]
for any \(t_0 \in \mathbb{R}\). This sows the \(\{\theta_t\}\)-invariance of \(P\).

In most cases of application, \(\Omega\) itself is a topological space with \(\mathcal{F}\) the Baire \(\sigma\)-algebra generated by that topology and \(t \mapsto \theta_{t}\omega\) is continuous for all \(\omega \in \Omega\). In such a case, \(H\) contains the class \(C_b(\Omega)\) of all bounded continuous functions on \(\Omega\), which is dense in \(L^2(\Omega, \mathbf{P})\). Hence condition (ii) is not as restrictive as it may appear.
Basic Theorems on Point Processes

See [4] for a general treatment of stationary random measures on a topological group.

References


