Hiyoshi Review of Natural Science
Keio University No. 52, 37–47 (2012)

New Proofs of Some Basic Theorems on Stationary Point Processes

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Summary——We give new proofs of three basic theorems on stationary point processes on the real line——theorems of Khintchine, Korolyuk, and Dobrushin. Moreover we give a direct construction of the Palm measure for a class of point processes which includes stationary ones as special cases.

1. Introduction

The purpose of this note is to give new proofs, based on a same simple idea, to some basic theorems on stationary point processes on the real line **R**, as stated in standard treatises on point processes such as Daley and Vere-Jones (see §3.3 of [3]).

To begin with, let us introduce necessary definitions and notation. By M_p, we denote the set of all integer-valued Radon measures on **R**. Namely M_p is the totality of all measures $N(dx)$ on **R** such that for any bounded Borel set B, $N(B)$ is a non-negative integer. Let us call any such measure a *counting measure*. For a counting measure $N \in M_p$, let us define

$$X(t) := N((0, t]) \quad (t \geq 0), \qquad := -N((t, 0)) \quad (t < 0). \tag{1}$$

Then the function $X(t)$ is right-continuous, integer-valued, locally bounded and non-decreasing. Hence $X(t)$ is piecewise constant on **R** and the set Δ, finite or countably infinite, of its points of discontinuity has no accumulation points other than $\pm\infty$. Thus the points in Δ can be ordered as

$$\cdots < x_{-1} < x_0 \leq 0 < x_1 < x_2 < \cdots,$$

so that if we let $m_n := X(x_n) - X(x_n - 0)$, then $N(dx)$ can be represented as

$$N(dx) = \sum_n m_n \delta_{x_n}(dx), \tag{2}$$

where δ_a denotes the unit mass placed at a. Each m_n is a positive integer and is called

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the *multiplicity* of the point x_n. In general, either $N([0, \infty))$ or $N((-\infty, 0))$ can be finite, in which case either $\{x_n\}_{n>0}$ or $\{x_n\}_{n\le 0}$ is a finite sequence. If in the former [resp. latter] case $\{x_n\}_{n>0}$ [resp. $\{x_n\}_{n\le 0}$] terminates with x_ν, then we will set $x_n = \infty$ [resp. $x_n = -\infty$] for $n > \nu$ [resp. $n < \nu$]. When $m_n = 1$ for all n such that $x_n \neq \pm\infty$, the counting measure N is said to be *simple*. For each $N \in M_p$ with representation (2), let us associate a simple counting measure N^* defined by

$$N^*(dx) = \sum_n \delta_{x_n}(dx). \tag{3}$$

In order to make M_p a measurable space, we define $\mathcal{M}_p$ to be the σ-algebra of subsets of M_p generated by all mappings of the form

$$M_p \ni N \mapsto N(B) \in [0, \infty] \tag{4}$$

for all Borel sets $B \subset \mathbf{R}$. Then we see that x_n, m_n and N^* are all measurable functions of N, as the following lemma shows.

Lemma 1 *(i) The set*

$$C := \{N \in M_p : N((-\infty, 0]) = N((0, \infty)) = \infty\} = \{N \in M_p : x_n \textit{ is finite for all } n\}$$

belongs to $\mathcal{M}_p$.

(ii) For each integer n, x_n and m_n are $\mathcal{M}_p$-measurable functions of N.

(iii) The mapping $M_p \ni N \mapsto N^* \in M_p$ is $\mathcal{M}_p/\mathcal{M}_p$-measurable.

Proof. (i) The assertion is obvious from the definition of $\mathcal{M}_p$, since we can write

$$C = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \{N \in M_p : N((-n, 0]) > k,\ N((0, n]) > k\}.$$

(ii) The measurability of x_1 follows from the relation

$$\{N \in M_p : x_1 > t\} = \{N \in M_p : N((0, t]) = 0\},$$

which holds for all $t \geq 0$. Now for each $k \geq 1$, define

$$x_1^{(k)} := \sum_{j=1}^{\infty} \frac{j}{2^n} \mathbf{1}_{((j-1)/2^n,\, j/2^n]}(x_1) + \infty \cdot \mathbf{1}_{\{x_1 = \infty\}}.$$

Then we see that $x_1^{(k)}$ is measurable in N and that $x_1^{(k)} \searrow x_1$ as $k \to \infty$. By the right-continuity of $X(t) = N((0, t])$ at $t > 0$, we have, as $k \to \infty$,

$$\mathbf{1}_{\{x_1 < \infty\}} \cdot X(x_1^{(k)}) = \sum_{j=1}^{\infty} \mathbf{1}_{((j-1)/2^n,\, j/2^n]}(x_1) X\left(\frac{j}{2^k}\right) \longrightarrow X(x_1) = m_1,$$

which shows the measurability of m_1 in N.

Next let $\tilde{X}(t) := X(t) - X(t \wedge x_1)$. This is measurable in N for all $t \geq 0$, since

$$X(t \wedge x_1) = X(t)\mathbf{1}_{\{x_1 \geq t\}} + X(x_1)\mathbf{1}_{\{x_1 < t\}}.$$

If we apply the above argument to $\tilde{X}(t)$ instead of $X(t)$, we can verify the measurability of x_2 and m_2 in N, and the argument can be iterated to give the measurability of all x_n and m_n.

(iii) For each $j = 0, 1, 2, \ldots$ and $t > 0$, the sets

$$\{N \in M_p : N^*((0, t]) = j\} = \{N \in M_p : x_j \leq t < x_{j+1}\}$$

and

$$\{N \in M_p : N^*((-t, 0]) = j\} = \{N \in M_p : x_{-j} \leq t < x_{-j+1}\}$$

belong to M_p. Now for each $n \geq 1$, let $\mathcal{G}_n$ be the class of all Borel subsets B of $[-n, n]$ such that the mapping

$$M_p \ni N \mapsto N^*(B) \in [0, \infty) \tag{5}$$

is measurable. Then $\mathcal{G}_n$ is seen to be a λ-system which contains the class of intervals

$$I := \{(0, t] : 0 < t \leq t\} \cup \{(-t, 0] : 0 < t \leq n\}$$

which forms a π-system. Hence by Dynkin's π-λ theorem (see e.g. Durrett [2]), $\mathcal{G}_n$ contains all Borel subsets of $[-n, n]$. Since $n \geq 1$ is arbitrary, and since we can write $N^*(B) = \lim_{n\to\infty} N^*(B \cap [-n, n])$, the mapping (5) is measurable for all Borel subsets of $\mathbf{R}$.

Remark 1. By an argument similar to (iii), it is easy to show that M_p is generated by mappings $M_p \ni N \mapsto X(t)$ for all t, where $X(t)$ is defined in (1).

Definition 1 *A* point process N_ω *is a random variable defined on a probability space* $(\Omega, \mathcal{F}, \mathbf{P})$ *and taking values in the measurable space* $(M_p, \mathcal{M}_p)$*.*

Definition 2 *A point process* N_ω *is said to be* crudely stationary *if for any bounded interval* I *and for any* $x \in \mathbf{R}$, $N_\omega(I)$ *and* $N_\omega(I + x)$ *are identically distributed. Its* mean density *is the expectation value* $m := \mathbf{E}[N_\omega((0, 1])] \leq \infty$*.*

Definition 3 *A point process* N_ω *is said to be* stationary *if for any* $C \in \mathcal{M}_p$ *and* $x \in \mathbf{R}$, *one has the identity*

$$\mathbf{P}(N_\omega(\cdot) \in C) = \mathbf{P}(N_\omega(x + \cdot) \in C).$$

Obviously, N_ω is crudely stationary if it is stationary.

Remark 2. By another application of π-λ theorem, one can show without difficulty that N_ω is stationary if and only if for any finite family of Borel subsets $B_1, \ldots, B_n$ of $\mathbf{R}$, and of non-negative integers $k_1, \ldots, k_n$, the identity

$$\mathbf{P}(N_\omega(B_i) = k_i,\ i = 1, \ldots, n) = \mathbf{P}(N_\omega(x + B_i) = k_i,\ i = 1, \ldots, n)$$

holds for any $x \in \mathbf{R}$.

2. Basic theorems and their proofs

Our argument is based on the following lemma, which is an immediate consequence of Definition 2.

Lemma 2 *Let the point process N_ω be crudely stationary. Then for any bounded interval I and for any non-negative integer k,*

$$\mathbf{P}(N_\omega(I) = k) = \int_0^1 \mathbf{P}(N_\omega(x + I) = k)\,dx = \mathbf{E}\left[\int_0^1 \mathbf{1}_{\{N_\omega(x+I)=k\}}\,dx\right].$$

Proposition 1 (Khintchine's theorem) *For any crudely stationary point process N_ω, the limit*

$$\lambda := \lim_{h \searrow 0} \frac{1}{h} \mathbf{P}(N_\omega((0, h]) > 0)$$

exists and satisfies $\lambda \leq m$. λ is called the intensity *of the point process N_ω.*

Proof. Let N_ω be represented as (2) and define the point process N_ω^* by (3). If we set $\nu(\omega) := N_\omega^*(0, 1]$, it satisfies $x_{\nu(\omega)}(\omega) \leq 1 < x_{\nu(\omega)+1}(\omega)$. Obviously we have

$$\begin{aligned} \{x \in (0,1] : N_\omega((x, x+h]) > 0\} &= (0,1] \cap \left[\bigcup_{j=1}^{\infty} [x_j(\omega) - h, x_j(\omega))\right] \\ &= (0,1] \cap \left[\bigcup_{j=1}^{\nu(\omega)+1} J_j^\omega(h)\right] = \sum_{j=1}^{\nu(\omega)+1} \left[(0,1] \cap (J_j^\omega(h) \setminus J_{j-1}^\omega(h))\right], \end{aligned}$$

where we have set $J_j^\omega(h) := [x_j(\omega) - h, x_j(\omega))$ and $J_0 = \emptyset$. Hence

$$\begin{aligned} \frac{1}{h} \int_0^1 \mathbf{1}_{\{N_\omega((x, x+h]) > 0\}}\,dx &= \frac{1}{h} \sum_{j=1}^{\nu(\omega)+1} |(0,1] \cap (J_j^\omega(\mathrm{h}) \setminus J_{j-1}^\omega(h))| \\ &= \sum_{j=1}^{\nu(\omega)+1} \frac{1}{h}\{(1 \wedge x_j(\omega)) - (0 \vee x_{j-1}(\omega) \vee (x_j(\omega) - h)\}_+, \end{aligned}$$

where for a Borel subset B of $\mathbf{R}$, $|B|$ denotes its Lebesgue measure and for a real number a, $a_+ := a \vee 0 = \max\{a, 0\}$ denotes its positive part. Now it is easy to see that for $1 \le j \le \nu(\omega)$,

$$\frac{1}{h}\{(1 \wedge x_j(\omega)) - (0 \vee x_{j-1}(\omega) \vee (x_j(\omega) - h)\}_+ \nearrow 1$$

as $h \searrow 0$, and that for $j = \nu(\omega) + 1$,

$$\frac{1}{h}\{(1 \wedge x_{\nu(\omega)+1}(\omega)) - (0 \vee x_{\nu(\omega)}(\omega) \vee (x_{\nu(\omega)+1}(\omega) - h))\}_+$$

is bounded by 1 and tends to 0 as $h \searrow 0$. Thus we can apply the monotone convergence theorem, the dominated convergence theorem and Lemma 2, to obtain

$$\begin{aligned} \frac{1}{h}\mathbf{P}(N_\omega((0, h]) > 0) &= \mathbf{E}\left[\frac{1}{h}\sum_{j=1}^{\nu(\omega)+1} |(0,1] \cap (J_j^\omega \setminus J_{j-1}^\omega(h))|\right] \\ &\to \mathbf{E}\left[\sum_{j=1}^{\nu(\omega)} 1\right] = \mathbf{E}\left[N_\omega^*((0,1])\right], \end{aligned}$$

as $h \searrow 0$. Thus the desired limit λ exists and is equal to $\mathbf{E}[N_\omega^*((0, 1])]$. Clearly it satisfies the inequality $\lambda \le \mathbf{E}[N_\omega((0, 1])] = m$.

Corollary 1 *If N_ω is simple, then $\lambda = m$. When $m < \infty$, the converse is also true.*

Proof. N_ω is simple if and only if $N_\omega^* = N_\omega$ almost surely, which obviously implies $\lambda = m$. On the other hand, if $\lambda = m < \infty$, then

$$\mathbf{E}[N_\omega((0, 1]) - N_\omega^*((0, 1])] = m - \lambda = 0.$$

But $N_\omega((0, 1]) - N_\omega^*((0, 1]) \ge 0$ in general, so that $N_\omega((0, 1]) = N_\omega^*((0, 1])$ almost surely. The same argument is valid if the interval $(0, 1]$ is replaced by $(n, n + 1]$, so that $N_\omega((n, n + 1]) = N_\omega^*((n, n + 1])$ almost surely for all integers n, and the simplicity of N_ω follows.

Remark 3. In the treatise by Daley and Vere-Jones [3], for example, Proposition 1 is proved in the following way: If we define $\varphi(h) := \mathbf{P}(N_\omega((0, h]) > 0)$, then by the crude stationarity, we have for any positive h_1 and h_2,

$$\begin{aligned} \varphi(h_1 + h_2) &= \mathbf{P}(N_\omega((0, h_1 + h_2]) > 0) = \mathbf{P}(N_\omega((0, h_1]) + N_\omega((h_1, h_1 + h_2]) > 0) \\ &\le \mathbf{P}(N_\omega((0, h_1]) > 0) + \mathbf{P}(N_\omega((h_1, h_1 + h_2]) > 0) = \varphi(h_1) + \varphi(h_2), \end{aligned}$$

so that $\varphi(h)$ is a sub-additive function defined on $[0, \infty)$ satisfying $\varphi(0) = 0$. To show the existence of the intensity λ, it suffices to apply the following well known lemma.

Lemma 3 *Let $g(x)$ be a sub-additive function defined on $[0, \infty)$ such that $g(0) = 0$. Then one has*

$$\lim_{x \searrow 0} \frac{g(x)}{x} = \sup_{x>0} \frac{g(x)}{x} \leq \infty .$$

However, this argument does not provide the representation $\lambda = \mathbf{E}[N^*_\omega((0, 1])]$, so that the proof of Corollary 1 requires some extra work. Our proof above is closer to that of Leadbetter [5]. See also Chung [1].

Definition 4 *A crudely stationary point process N_ω is said to be* orderly *when*

$$\mathbf{P}(N_\omega((0, h]) \geq 2) = o(h) \quad (h \searrow 0).$$

Proposition 2 (Dobrushin's theorem) *If a crudely stationary point process N_ω is simple and if $\lambda < \infty$, then N_ω is orderly.*

Proof. By Lemma 2, we can write

$$\mathbf{P}(N_\omega((0, h]) \geq 2) = \mathbf{E}\left[\int_0^1 \mathbf{1}_{\{N_\omega((x, x+h]) \geq 2\}} dx\right].$$

As can be seen from the proof of Proposition 1, we have

$$\begin{aligned} & \frac{1}{h}\int_0^1 \mathbf{1}_{\{N_\omega((x,x+h]) \geq 2\}} dx \leq \frac{1}{h}\int_0^1 \mathbf{1}_{\{N_\omega((x,x+h]) > 0\}} dx \\ = & \sum_{j=1}^{\nu(\omega)} \frac{1}{h} |(0,1] \cap (J^\omega_j(h) \setminus J^\omega_{j-1}(h))| + \frac{1}{h} |(0,1] \cap (J_{\nu(\omega)+1}(h) \setminus J_{\nu(\omega)}(h)| \\ \leq & N^*_\omega((0,1]) + 1 , \end{aligned}$$

and

$$\lim_{h \searrow 0} \frac{1}{h}\int_0^1 \mathbf{1}_{\{N_\omega((x,x+h]) \geq 2\}} dx = \sharp\{j : x_j(\omega) \in (0,1],\ m_j(\omega) \geq 2\} .$$

Since $\mathbf{E}[N^*_\omega((0, 1])] = \lambda < \infty$, we can apply the dominated convergence theorem, to obtain

$$\lim_{h \searrow 0} \frac{1}{h}\mathbf{P}(N_\omega((0, h]) \geq 2) = \mathbf{E}[\sharp\{j : x_j(\omega) \in (0,1],\ m_j(\omega) \geq 2\}],$$

which is equal to 0 if N_ω is simple.

Remark 4. The condition $\lambda < \infty$ cannot be dropped. For a counter example, see Exercise 3.3.2 of [3].

Proposition 3 (Korolyuk's theorem) *A crudely stationary, orderly point process is simple.*

Proof. By Fatou's lemma and the orderliness of N_ω,

$$\begin{aligned}\mathbf{E}[\sharp\{j : x_j(\omega) \in (0,1],\ m_j(\omega) \geq 2\}] &= \mathbf{E}\left[\liminf_{h\searrow 0}\frac{1}{h}\int_0^1 \mathbf{1}_{\{N_\omega((x,x+h])\geq 2\}}\,dx\right] \\ &\leq \liminf_{h\searrow 0}\frac{1}{h}\mathbf{P}(N_\omega((0,h])\geq 2) = 0 ,\end{aligned}$$

so that with probability one, N_ω has no multiple points in (0, 1]. By crude stationarity, the above argument is also valid if (0, 1] is replaced by $(n, n+1]$ for any integer n. Hence N_ω is simple.

Proposition 4 *For a crudely stationary point process N_ω with finite intensity λ, the limits*

$$\lambda_k := \lim_{h\searrow 0}\frac{1}{h}\mathbf{P}(1 \leq N_\omega((0,h]) \leq k)$$

exists for $k = 1, 2, \ldots$, and satisfy $\lambda_k \nearrow \lambda$ as $k \to \infty$. Moreover for $k = 1, 2, \ldots$,

$$\pi_k := \frac{\lambda_k - \lambda_{k-1}}{\lambda} = \lim_{h\searrow 0}\mathbf{P}(N_\omega((0,h]) = k \mid N_\omega((0,h]) > 0) ,$$

where we set $\lambda_0 := 0$.

Proof. As before, one has

$$\frac{1}{h}\int_0^1 \mathbf{1}_{\{1\leq N_\omega((x,x+h])\leq k\}}\,dx \leq 1 + N_\omega^*((0,1]) ,$$

and

$$\lim_{h\searrow 0}\frac{1}{h}\int_0^1 \mathbf{1}_{\{1\leq N_\omega((x,x+h])\leq k\}}\,dx = \sharp\{j : x_j(\omega) \in (0,1],\ m_j(\omega) \leq k\} .$$

Since $\lambda = \mathbf{E}[N_\omega^*((0,1])] < \infty$, we can apply the dominated convergence theorem and Lemma 2, to obtain

$$\lambda_k = \lim_{h\searrow 0}\frac{1}{h}\mathbf{E}\left[\int_0^1 \mathbf{1}_{\{1\leq N_\omega((x,x+h])\leq k\}}\,dx\right] = \mathbf{E}[\sharp\{\mathrm{j} : x_j(\omega) \in (0,1],\ m_j(\omega) \leq k\}] .$$

This representation of λ_k immediately gives

$$\lim_{k\to\infty}\lambda_k = \mathbf{E}[\sharp\{j : x_j(\omega) \in (0,1]\}] = \mathbf{E}[N^*((0,1])] = \lambda,$$

by the monotone convergence theorem. The last statement of the proposition is obvious.

Corollary 2 *For a crudely stationary point process with finite intensity, we have*

$$\lambda\sum_{k=1}^{\infty} k\pi_k = \mathbf{E}[N_\omega((0,1])] = m .$$

3. The Palm measure

Let us assume that the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, on which our point process N_ω is defined, is equipped with a measurable flow $\{\theta_t\}_{t\in\mathbf{R}}$. Here a measurable flow $\{\theta_t\}$ is, by definition, a family of bijections $\theta_t : \Omega \to \Omega$ such that

(a) θ_0 is the identity mapping, and for any $s, t \in \mathbf{R}$, $\theta_s \circ \theta_t = \theta_{s+t}$ holds;
(b) the mapping $(t, \omega) \mapsto \theta_t(\omega)$ from $\mathbf{R} \times \Omega$ into Ω is jointly measurable with respect to $\mathcal{B}(\mathbf{R}) \times \mathcal{F}$, where $\mathcal{B}(\mathbf{R})$ is the Borel σ-algebra on $\mathbf{R}$.

Let us further assume that the relation

$$\int_{\mathbf{R}} N_{\theta_t\omega}(dx)\varphi(x) = \int_{\mathbf{R}} N_\omega(dx)\varphi(x-t) \tag{6}$$

holds for any $t \in \mathbf{R}$ and any continuous function φ with compact support. If the probability measure $\mathbf{P}$ is $\{\theta_t\}$-invariant in the sense $\mathbf{P} \circ \theta_t^{-1} = \mathbf{P}$ for all $t \in \mathbf{R}$, then by (6), our point process N_ω is stationary.

Definition 5 *The Palm measure of a point process $N_\omega(dx)$ is a measure kernel $Q(x, d\omega)$ on $\mathbf{R} \times \Omega$ such that for any jointly measurable, non-negative function $f(x, \omega)$, the relation*

$$\int_\Omega \mathbf{P}(d\omega)\int_{\mathbf{R}} N_\omega(dx) f(x, \omega) = \int_{\mathbf{R}} \lambda(dx)\int_\Omega Q(x, d\omega) f(x, \omega) \tag{7}$$

holds, where $\lambda(dx)$ is the mean measure of N_ω which is defined by $\lambda(B) = \mathbf{E}[N_\omega(B)]$ for $B \in \mathcal{B}(\mathbf{R})$ and which we assume to be finite for bounded Borel sets B.

Now let $u(t)$ be a probability density function on $\mathbf{R}$. Define a new probability measure $\mathbf{P}_u$ by

$$\int_\Omega \mathbf{P}_u(d\omega)g(\omega) = \int_{\mathbf{R}} u(t)dt\Big(\int_\Omega \mathbf{P}(d\omega)g(\theta_t\omega)\Big)\ , \tag{8}$$

where $g(\omega)$ is an arbitrary non-negative measurable function on Ω. Then the following result holds.

Theorem 1 *For any probability density $u(t)$ on $\mathbf{R}$, the Palm measure $Q_u(x, d\omega)$ exists for the point process N_ω defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P}_u)$.*

Proof. Let $f(x, \omega) \geq 0$ be jointly measurable on $\mathbf{R} \times \Omega$. Then we can rewrite the left hand side of (7) in the following way:

$$\begin{aligned}
\int_\Omega \mathbf{P}_u(d\omega)\int_\mathbf{R} N_\omega(dx)f(x,\omega) &= \int_\mathbf{R} u(t)dt\int_\Omega \mathbf{P}(d\omega)\int_\mathbf{R} N_{\theta_t\omega}(dx)f(x,\theta_t\omega)\\
&= \int_\mathbf{R} u(t)dt\int_\Omega \mathbf{P}(d\omega)\int_\mathbf{R} \mathrm{N}_\omega(dx)f(x-t,\theta_t\omega)\\
&= \int_\Omega \mathbf{P}(d\omega)\int_\mathbf{R} N_\omega(dx)\int_\mathbf{R} u(t)dtf(x-t,\ \theta_t\,\omega)\\
&= \int_\Omega \mathbf{P}(d\omega)\int_\mathbf{R} N_\omega(dx)\int_\mathbf{R} u(x-s)dsf(\mathrm{s},\theta_{x-s}\,\omega)\\
&= \int_\mathbf{R} ds\int_\Omega \mathbf{P}(d\omega)\int_\mathbf{R} N_\omega(dx)u(x-s)f(s,\theta_{x-s}\,\omega)\,. \qquad (9)
\end{aligned}$$

At this stage, take $f(x,\ \omega) = \varphi(x)$. Then (9) reduces to

$$\int_\mathbf{R} \varphi(s)\lambda(ds) = \int_\mathbf{R} \varphi(s)\ell_u(s)ds \qquad (10)$$

with

$$\ell_u(s) = \int_\Omega \mathbf{P}(d\omega)\int_\mathbf{R} N_\omega(dx)u(x-s)\ . \qquad (11)$$

If we define, for each $s\in\mathbf{R}$, the measure $Q_u(s,\ d\omega)$ on $(\Omega,\ \mathcal{F})$ by

$$\int_\Omega Q_u(s,d\omega)\,g(\omega) = \frac{\mathbf{1}_{(0,\infty)}(\ell_u(s))}{\ell_u(s)}\int_\Omega \mathbf{P}(d\omega)\int_\mathbf{R} N_\omega(dx)\,u(x-s)\,g(\theta_{x-s}\omega)\ , \qquad (12)$$

then (9) takes the form of (7), and the theorem is proved.

When $\mathbf{P}$ is $\{\theta_t\}$-invariant, then we have $\mathbf{P}_u = \mathbf{P}$ for any probability density u on $\mathbf{R}$, and

$$\ell_u(s) = \int_\Omega \mathbf{P}(d\omega)\int_\mathbf{R} N_{\theta_s\omega}(dx)\,u(x) = \int_\Omega \mathbf{P}(d\omega)\int_\mathbf{R} N_\omega(dx)\,u(x) =: \ell > 0$$

is a constant. Moreover one can compute as

$$\begin{aligned}
&\int_\Omega Q_u(s,d\omega)\,g(\omega) = \frac{1}{\ell}\int_\Omega \mathbf{P}(d\omega)\int_\mathbf{R} N_{\theta_s\omega}(dx)u(x)\,g(\theta_x\,\omega)\\
&= \frac{1}{\ell}\int_\Omega \mathbf{P}(d\omega)\int_\mathbf{R} N_{\theta_s\omega}(dx)\,u(x)\,g(\theta_{x-s}(\theta_s\,\omega)) = \frac{1}{\ell}\int_\Omega \mathbf{P}(d\omega)\int_\mathbf{R} N_\omega(dx)\,u(x)\,g(\theta_{x-s}\omega)\ .
\end{aligned}$$

Hence if we define a measure $\hat{\mathbf{P}}(d\omega)$ on $(\Omega,\ \mathcal{F})$ by

$$\int_\Omega \hat{\mathbf{P}}(d\omega)g(\omega) = \int_\Omega \mathbf{P}(d\omega)\int_\mathbf{R} N_\omega(dx)\,u(x)\,g(\theta_x\,\omega)\ ,$$

then we get

$$Q_u(s,d\omega) = \frac{1}{\ell}(\hat{\mathbf{P}}\circ\theta_s)(d\omega)\ ,$$

and (7) can be written in the form

$$\int_\Omega \mathbf{P}(d\omega)\int_\mathbf{R} N_\omega(dx)f(x,\omega) = \int_\mathbf{R} dx\int_\Omega \hat{\mathbf{P}}(d\omega)f(x,\theta_{-x}\omega)\ , \tag{13}$$

which is the defining relation of the Palm measure in the stationary case (see [6]). (13) shows in particular that the definition of $\hat{\mathbf{P}}$ is independent of the choice of u.

Our consideration of the probability measure $\mathbf{P}_u$ is motivated by the following observation.

Proposition 5 *The probability measure* $\mathbf{P}$ *is* $\{\theta_t\}$*-invariant if and only if the following two conditions hold:*

(i) $\mathbf{P}_u = \mathbf{P}$ *for any probability density function* $u(t)$ *on* $\mathbf{R}$*;*
(ii) *the set* H *of all bounded measurable functions* $\varphi(\omega)$ *on* Ω *such that* $t \mapsto \varphi(\theta_t\omega)$ *is continuous for all* $\omega \in \Omega$ *is dense in* $L^2(\Omega, \mathbf{P})$*.*

Proof. The necessity of (i) is obvious. That (ii) also follows from the $\{\theta_t\}$-invariance of $\mathbf{P}$ is proved in [6] (see Lemma II. 3). To prove the sufficiency of (i) and (ii), fix an arbitrary $t_0 \in \mathbf{R}$ and take a sequence of probability density $\{u_n\}_n$ so that $u_n(t)dt \to \delta_{t_0}(dt)$ weakly. Now for any $\varphi \in H$, $t \mapsto \varphi(\theta_t\omega)$ is continuous and bounded by $\|\varphi\|\infty := \sup_\Omega |\varphi(\omega)|$. Hence we can apply the dominated convergence theorem, to get

$$\begin{aligned}
\int_\Omega \mathbf{P}(d\omega)\varphi(\theta_{t_0}\omega) &= \int_\Omega \mathbf{P}(d\omega)\left(\lim_{n\to\infty}\int_\mathbf{R}\varphi(\theta_t\omega)u_n(t)dt\right)\\
&= \lim_{n\to\infty}\int_\mathbf{R}\left(\int_\Omega \mathbf{P}(d\omega)\varphi(\theta_t\omega)\right)u_n(t)dt\\
&= \lim_{n\to\infty}\int_\Omega \mathbf{P}_{u_n}(d\omega)\,\varphi(\omega) = \int_\Omega \mathbf{P}(d\omega)\,\varphi(\omega)
\end{aligned}$$

by condition (i). But if H is dense in $L^2(\Omega, \mathbf{P})$, we can approximate an arbitrary bounded measurable function $g(\omega)$ by the elements of H, to obtain

$$\int_\Omega \mathbf{P}(d\omega)g(\theta_{t_0}\omega) = \int_\Omega \mathbf{P}(d\omega)g(\omega)$$

for any $t_0 \in \mathbf{R}$. This sows the $\{\theta_t\}$-invariance of $\mathbf{P}$.

In most cases of application, Ω itself is a topological space with $\mathcal{F}$ the Baire σ-algebra generated by that topology and $t \mapsto \theta_t\omega$ is continuous for all $\omega \in \Omega$. In such a case, H contains the class $C_b(\Omega)$ of all bounded continuous functions on Ω, which is dense in $L^2(\Omega, \mathbf{P})$. Hence condition (ii) is not as restrictive as it may appear.

See [4] for a general treatment of stationary random measures on a topological group.

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