Title	An experimental study of the exponents in the power-law behaviors of rolling disks
Sub Title	
Author	横山, 英祐(Yokoyama, Eisuke) 伊藤, ゆい(Ito, Yui) 下村, 裕(Shimomura, Yutaka)
Publisher	慶應義塾大学日吉紀要刊行委員会
Publication year	2010
Jtitle	慶應義塾大学日吉紀要. 自然科学 (The Hiyoshi review of the natural science). No.48 (2010. 9) ,p.1- 9
JaLC DOI	
Abstract	The dependence of the exponents in the power-law behavior of a rolling disk on its mass and on its aspect ratio, which is defined as the ratio of the diameter to the thickness of the disk, is experimentally studied with the help of a high-speed video system. The exponents overall do not depend so much either on the mass or on the aspect ratio of the disk. However, for relatively thick disks whose aspect ratios are smaller than 5, we find a tendency for the magnitudes of the exponents both of the inclinations angle and of the precession rate to increase as the aspect ratio increases, which means a thicker disk halts more abruptly than a thinner one. These results suggest that the air viscosity does not essentially contribute to the energy dissipation.
Notes	原著論文
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=AN10079809-20100930-0001

Hiyoshi Review of Natural Science
Keio University No. 48, 1–9 (2010)

An Experimental Study of the Exponents in the Power-Law Behaviors of Rolling Disks

Eisuke YOKOYAMA*, Yui ITO*, and Yutaka SHIMOMURA**

Summary——The dependence of the exponents in the power-law behavior of a rolling disk on its mass and on its aspect ratio, which is defined as the ratio of the diameter to the thickness of the disk, is experimentally studied with the help of a high-speed video system. The exponents overall do not depend so much either on the mass or on the aspect ratio of the disk. However, for relatively thick disks whose aspect ratios are smaller than 5, we find a tendency for the magnitudes of the exponents both of the inclinations angle and of the precession rate to increase as the aspect ratio increases, which means a thicker disk halts more abruptly than a thinner one. These results suggest that the air viscosity does not essentially contribute to the energy dissipation.

Key words: rolling disks; Euler's disk; exponents; power law; finite-tme singularity

1. Introduction

When we happen to drop a coin on a hard ground, we hear the familiar sound that becomes higher and higher as it comes to rest. This kind of sound is generally produced by a circular disk rolling on a horizontal surface. Until recently, this kind of rolling motion has drawn little attention even though its physical mechanism is not well understood. However, a commercial toy named the Euler's disk[1)] revived more interests in the rolling disk.

The Euler's disk is a carefully crafted disk, which is a 3" wide chrome-plated, cast iron cylinder steel disk and spun on a concave mirror base. The Euler's disk is designed to maximize the duration of a motion and sonic hum, and the surface that the disk spins on is slightly concave so that the amount of time needed for the disk to come to rest is well over sixty seconds.

It was a paper[2)] by Moffatt that invoked many theoretical interests in the Euler's disk.

* 横山英祐，伊藤ゆい，慶應義塾大学法学部卒業生．A graduate of Faculty of Law, Keio University, who took as an undergraduate the class of "Seminar in Natural Sciences" by Y. Shimomura in the academic year 2009-2010.
** 下村裕，自然科学研究教育センター，慶應義塾大学日吉物理学教室（〒 223–8521　横浜市港北区日吉 4–1–1）．Research and Education Center for Natural Sciences and Department of Physics, Keio University, Hiyoshi, Yokohama 223-8521, Japan [Received Mar. 31, 2010]

For a circular disk shown in Fig.1, he took into account the friction due to the air viscosity between the rolling disk and the surface to theoretically show the finite-time singularity of the precession rate Ω, which means the divergence of Ω in the limit of the abrupt halt of the disk. It was pointed out[3] later by Engh, Nelson, and Roach that the air viscosity has little effect on the final whirling motions. However, Moffatt's important finding was the condition for a rate of energy dissipation to lead to a finite-time singularity: under the adiabatic approximation, a finite-time singularity will occur for a rate of energy dissipation proportional to a power of the inclination angle θ, whose exponent is less than one.[4]

Following papers[5)-11)] studied other mechanisms of energy dissipation such as a rolling friction, a slipping friction, or vibrations of a deformable disk, which could produce the familiar sound and lead to a loss of contact with the surface.

Among them, the references 6) and 9) measured with the help of high-speed video imaging the exponents in the power-law behaviors of rolling disks, which are expressed by

$$\theta = C_\theta (t_0 - t)^{a_\theta} , \tag{1}$$

$$\Omega = C_\Omega (t_0 - t)^{a_\Omega} , \tag{2}$$

$$n = C_n (t_0 - t)^{a_n} , \tag{3}$$

where n the component of the angular velocity about the axis of symmetry, t the time, t_0 the time for the disk to halt, C_θ, C_Ω, C_n the constants, and a_θ, a_Ω, a_n are the exponents. They reported the results for different cases with several disks and surfaces, and supported the claim by Engh, Nelson, and Roach.[3] However, they did not so much focus on the dependence of the exponents either on the mass or on the aspect ratio of the disk.

In the present paper, with the help of two high-speed cameras, which take two images from different angles, and an image-processing software (Move-tr/3D), the temporal evolutions of θ, Ω, n are measured. In order to seek for the origin of energy dissipation, we study the dependence of the exponents a_θ, a_Ω, a_n on the mass m of the disk and on the aspect ratio r, which is defined as the ratio $2a/b$ of the diameter $2a$ to the thickness b of the disk. In section 2, the experimental method is explained, and in section 3 the results are presented, and finally in section 4, the main conclusions are summarized and some discussions about the results are given.

2. Experimental method

In the present study, we study the dependence of the exponents a_θ, a_Ω, a_n on the mass m and the aspect ratio $r=2a/b$ of the disk. The temporal evolutions of θ, Ω, n are

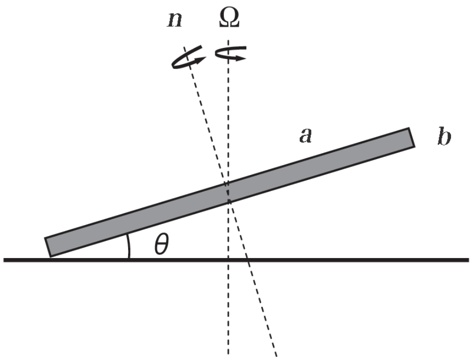

Fig. 1. A circular disk rolling on a horizontal surface.

measured to determine each exponent, which is done three times for a disk and the average of the three results is presented as the exponent in the following. We use eleven different disks, all of which are made of brass, so that the total number of our experiments is 33. The diameter of the disk ranges 30mm to 80mm, the thickness does 1mm to 20mm, the mass does 5.88g to 845g, and the aspect ratio does 1.5 to 80, as shown in Table 1 in section 3.

We put three markers (one on the center, two on the edges of the upper surface of the disk making a 90-degree angle) on each of the disks, so that the image-processing software (Move-tr/3D) can recognize the configuration of the disks. The two images taken from separate video systems are synchronized. We set the shutter speed at 1/2000 second, and the filming speed at 300 frames per second. After we collect the images, Move-tr/3D calculates the three-dimensional configuration of the disks. From the temporal series of the three-dimensional positions of the three markers, their velocities, which are their time-derivatives, are also numerically estimated by using the central difference method. From these data, we evaluate the Euler angles (θ, φ, ψ) and their time-derivatives ($\dot{\theta}$, $\dot{\varphi}$, $\dot{\psi}$), from which we derive (θ, Ω, n), by the method given in Appendix.

3. Experimental Results

Thirty-three sets of the temporal evolutions of θ, Ω, n are obtained by the method explained in section 2. Fig. 2 shows the time developments of θ, Ω, n in the rolling motion of the disk No. 2 in Table 1 whose diameter is 70mm and thickness is 20mm, as a typical case.

In order to determine the exponents a_θ, a_Ω, a_n, we fit the obtained data in the power-

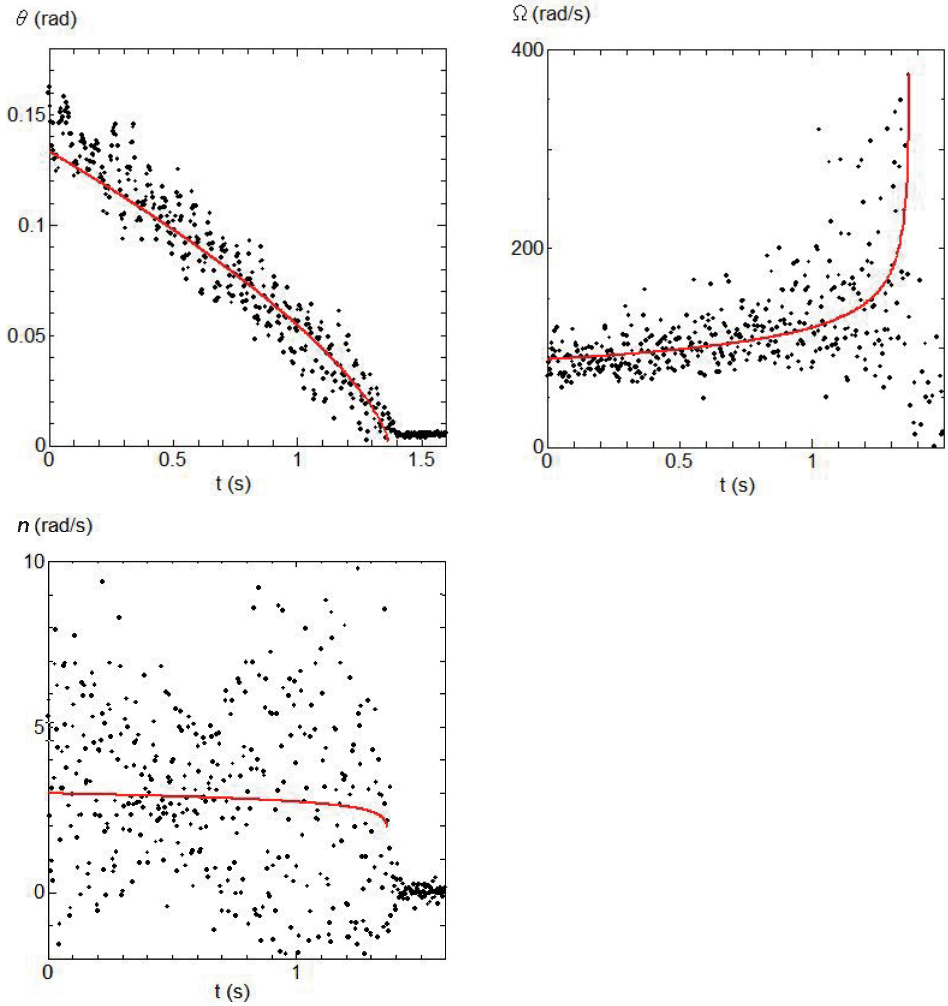

Fig. 2. Time development of θ, Ω and n in the rolling motion of the disk No. 2 in Table 1.

law formulae (1), (2), and (3) by using the least square method, after converting the original data into their logarithm. The component n of the angular velocity about the axis of symmetry often becomes negative by an error, since it is a small quantity calculated by the subtraction of a large number from a almost-equal large number. So, we ignore the data of n that have negative values, assuming (3). The precession rate Ω rarely becomes negative, but when it gets negative by an error, the datum is taken into account by using its absolute value. On the other hand, the inclination angle θ measured in the experiments is always positive in accordance with (1). The red curves in Fig. 2 are the fitting functions obtained by these procedures. As described in section 2, we adopt the average of three results for each disk to get its exponents a_θ, a_Ω, a_n,

No.	$2a$ (mm)	b (mm)	m (g)	$r = 2a/b$	a_θ	a_Ω	a_n
1	80	20	845.00	4.0	0.72	-0.23	-0.04
*2	70	20	672.00	3.5	0.64	-0.24	0.09
3	60	20	480.98	3.0	0.62	-0.20	0.15
4	50	20	333.86	2.5	0.56	-0.20	0.19
5	40	20	213.65	2.0	0.56	-0.15	0.13
6	30	20	122.11	1.5	0.52	-0.17	0.04
7	50	10	168.22	5.0	0.68	-0.26	-0.26
8	40	10	106.20	4.0	0.57	-0.20	0.18
**9	80	1	41.90	80.0	0.62	-0.28	-0.16
**10	50	1	16.50	50.0	0.62	-0.26	-0.09
**11	30	1	5.88	30.0	0.59	-0.19	-0.18

Table 1. The circular disks used in the experiment and the measured exponents a_θ, a_Ω, a_n in their power-law behaviors. The diameter, the thickness, the mass, and the aspect ratio of a disk are denoted by $2a$, b, m, and $r=2a/b$, respectively.
* The temporal evolution of the disk No. 2 is shown in Fig. 2.
** The exponents for the disks No. 9, 10, and 11 are excluded in Fig. 5.

which are presented in Table 1.

Fig. 3 shows the dependence of the exponent a, which represents one of a_θ, a_Ω, a_n, on the mass m. It is observed that even though the mass varies, the exponent a is almost constant. So, there is no obvious dependence of the exponent on the mass.

Fig. 4 shows the dependence of the exponent a on the aspect ratio r. As in Fig. 3, the exponent a is roughly constant, and there is no clear dependence to be seen. However, in this figure, the relation between the exponent a and the aspect ratio r smaller than 5 is hard to observe.

Fig. 5 zooms in the part of Fig. 4 for r smaller than 5, excluding the results for r larger than 5 (corresponding to cases for the disks No. 9, 10, and 11 in Table 1). We notice a tendency that the magnitudes both of a_θ and of a_Ω become larger as r increases. This may suggest that there are linear dependences of the exponents a_θ and a_Ω on the aspect ratio r, as shown by the red lines, though the accuracy of a_n is not enough for definite comments about it. If the linear dependence is extrapolated to the region at larger r, the exponent a_θ, for example, should be around 2.0 at r = 30, 3.0 at r = 50, and 4.5 at r = 80. However, these extrapolations clearly do not agree with Fig. 4, where a_θ is around 0.6 all at r = 30, 50, and 80.

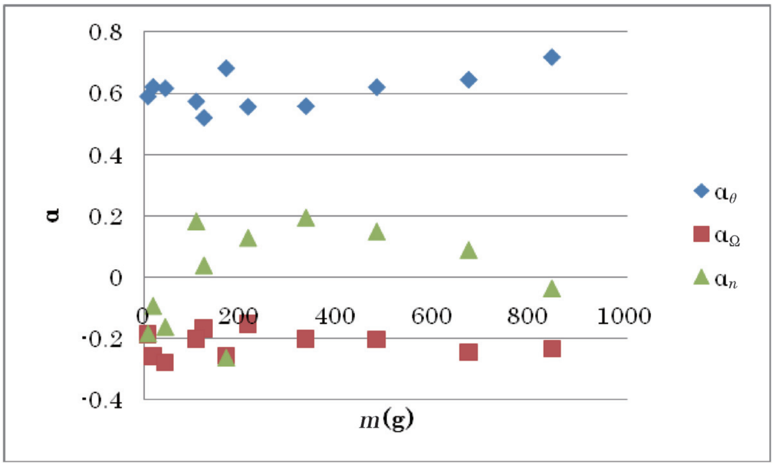

Fig. 3. The dependence of the exponent a on the mass m.

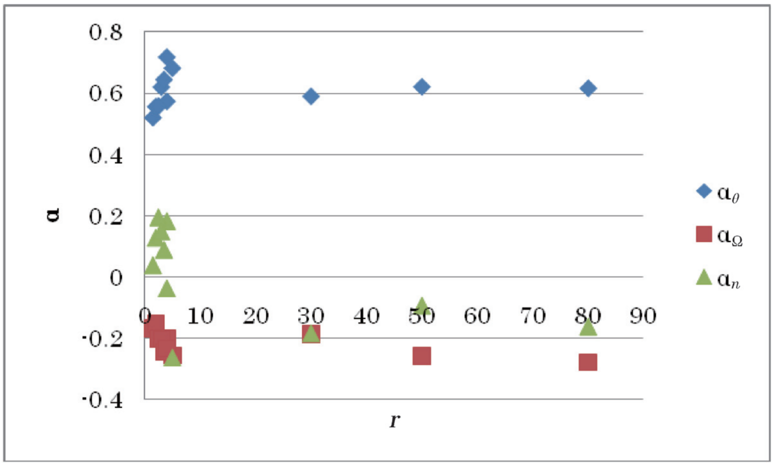

Fig. 4. The dependence of the exponent a on the aspect ratio r.

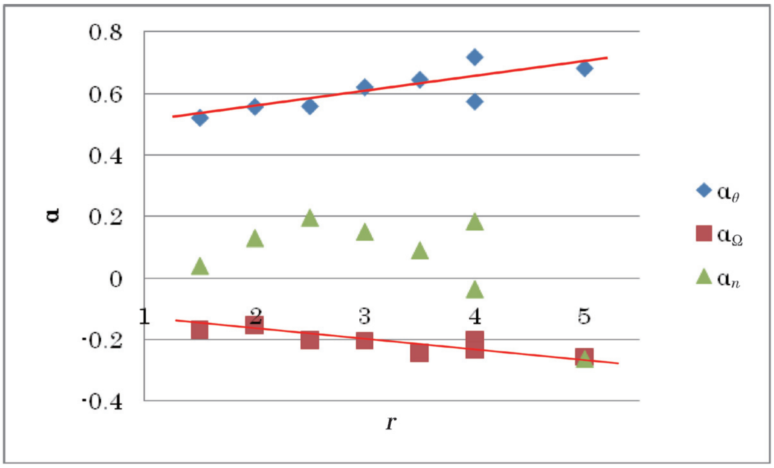

Fig. 5. The dependence of the exponent a on the aspect ratio r smaller than 5.

4. Conclusions and Discussions

As mentioned in section 1, the different mechanisms of energy dissipation have been pointed out to theoretically explain the finite-time singularity of the precession rate Ω. We experimentally study the dependence of the exponent a on the mass of the disk m and its aspect ratio r, in order to find the origin of the energy dissipation. The exponents do not overall vary so much for various disks, but we notice the exponents seem dependent on the aspect ratio below 5, where the absolute values of the exponents a_θ and a_Ω increase as the aspect ratio increases. This means a thicker disk halts more abruptly than a thinner one. The physical explanation for this feature is not clear at the moment, and left for future works.

However, the results of our experiments suggest that the origin of the energy dissipation is not the air viscosity by the reasons given in the followings.

Firstly, the motion of the disk is affected by the inertia due to the mass of the disk, whereas the air viscosity is not. So, if the air viscosity is a vital factor in the energy dissipation, the exponents representing the dynamics would be dependent on the mass, which is not the case in Fig. 3 where the exponents of the disks of a different mass do not vary so much. On the other hand, the friction between the disk and the surface is affected by the mass of the disk. It is possible to think that the effects of the mass on the motion of the disk and on the friction could cancel out to result in the almost constant expontents shown in Fig. 3.

Secondly, as shown in Fig. 5, the absolute values of the exponents a_θ and a_Ω for the disks No. 1 to 6 in Table 1, all of which have the same thickness 20mm, increase as the aspect ratio increases up to 4. Here, according to the theory by Moffatt,[2] the magnitudes of the exponents become smaller as the energy dissipation becomes larger. Therefore, this means the energy dissipation of a disk with a smaller diameter is larger. This contradicts the guess that the air viscosity is the main cause of energy dissipation, since the air viscosity would produce less dissipation of energy for a disk with a smaller diameter.

Thirdly, the fact that both the exponents a_θ and a_Ω of the three disks No. 9, 10, and 11 in Table 1 are almost constant again contradicts the view that the air viscosity is the main factor of the energy dissipation. The three disks are all 1mm thick, and their diameters are 30mm, 50mm, and 80mm. If the energy dissipation is mainly caused by the air viscosity, such big difference in the diameters should affect the exponents, which is, however, not actually observed in Fig. 4.

Acknowledgments

We are grateful to the Keio Gijyuku Academic Development Fund, by which this work was partially supported.

Appendix

Here, we explain how to calculate the Euler angles (θ, φ, ψ) and their time derivatives ($\dot{\theta}$, $\dot{\varphi}$, $\dot{\psi}$) from the positions and the velocities of the two markers.

Let (X, Y, Z) and (ξ, η, ζ) be the orthogonal coordinates fixed to the laboratory and the disk, respectively. The origin of the latter is the center point of the upper circular surface of the disk, whose position is given by (X_0, Y_0, Z_0) in the former. If we introduce a new coordinates (x, y, z) = ($X-X_0$, $Y-Y_0$, $Z-Z_0$), then the origin of (x, y, z) coincides with that of (ξ, η, ζ). It is well-known that they are related to each other by the following transformation:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}, \tag{A.1}$$

where the matrix A is given by

$$A = \begin{pmatrix} \cos\theta\cos\varphi\cos\psi - \sin\varphi\sin\psi & -\cos\theta\cos\varphi\sin\psi - \sin\varphi\cos\psi & \sin\theta\cos\varphi \\ \cos\theta\sin\varphi\cos\psi + \cos\varphi\sin\psi & -\cos\theta\sin\varphi\sin\psi + \cos\varphi\cos\psi & \sin\theta\sin\varphi \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{pmatrix}. \tag{A.2}$$

The two markers are located on the perimeter of the upper circular surface of the disk so that their position vectors make a right angle. We normalize length with the radius of the disk to express the coordinates of the two markers as (1, 0, 0) and (0, 1, 0) in (ξ, η, ζ). If we denote the corresponding coordinates in (x, y, z) as (x_1, y_1, z_1) and (x_2, y_2, z_2), respectively, the relation (A.1) gives

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{A.3}$$

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{A.4}$$

From the above relations (A.3) and (A.4), we derive the following expressions for the Euler angles (θ, φ, ψ) :

$$\tan\theta = \sqrt{\frac{z_1^2 + z_2^2}{1-\left(z_1^2 + z_2^2\right)}}\,, \qquad \left(0 < \theta < \frac{\pi}{2}\right), \tag{A.5}$$

$$\tan\theta = \frac{y_1 z_1 + y_2 z_2}{x_1 z_1 + x_2 z_2}\,, \tag{A.6}$$

$$\tan\psi = -\frac{z_2}{z_1}\,. \tag{A.7}$$

We differentiate (A.5), (A.6), and (A.7) to obtain their time derivatives ($\dot{\theta}$, $\dot{\varphi}$, $\dot{\psi}$) as follows:

$$\dot{\theta} = \frac{z_1\dot{z}_1 + z_2\dot{z}_2}{\sqrt{\left(z_1^2 + z_2^2\right)\left\{1-\left(z_1^2 + z_2^2\right)\right\}}}\,, \tag{A.8}$$

$$\dot{\varphi} = \frac{\left(x_1 z_1 + x_2 z_2\right)\left(\dot{y}_1 z_1 + y_1\dot{z}_1 + \dot{y}_2 z_2 + y_2\dot{z}_2\right) - \left(y_1 z_1 + y_2 z_2\right)\left(\dot{x}_1 z_1 + x_1\dot{z}_1 + \dot{x}_2 z_2 + x_2\dot{z}_2\right)}{\left(x_1 z_1 + x_2 z_2\right)^2 + \left(y_1 z_1 + y_2 z_2\right)^2}\,, \tag{A.9}$$

$$\dot{\psi} = \frac{\dot{z}_1 z_2 - z_1\dot{z}_2}{z_1^2 + z_2^2}\,. \tag{A.10}$$

Since Ω is defined as $\dot{\varphi}$, the component n of the angular velocity about the axis of symmetry is calculated from (A.9) and (A.10) by

$$n = \dot{\varphi}\cos\theta + \dot{\psi}\,. \tag{A.11}$$

References

1) http://www.eulersdisk.com/, March (2010).

2) H. K. Moffatt: "Euler's disk and its finite-time singularity", Nature, **404** (2000) 833–834.

3) G. Engh, P. Nelson, and J. Roach: "Numismatic gyrations", Nature, **408** (2000) 540.

4) H. K. Moffatt: "Reply to "Numismatic gyrations"", Nature, **408** (2000) 540.

5) A. A. Stanislavsky and K. Weron: "Nonlinear oscillations in the rolling motion of Euler's disk", Physica D, **156** (2001) 247–259.

6) K. Easwar, F. Rouyer, and N. Menon: "Speeding to a stop: The finite-time singularity of a spinning disk", Phys. Rev. E, **66**, 045102(R) (2002) 1–3.

7) P. Kessler, and O. M. O'Reilly: "The ringing of Euler's disk", Regul. Chaotic Dyn., **7** (2002) 50–60.

8) D. Petrie, J. L. Hunt, and C. G. Gray: "Does the Euler Disk slip during its motion?", Am. J. Phys., **70** (2002) 1025–1028.

9) H. Caps, S. Dorbolo, S. Ponte, H. Croisier, and N. Vandewalle: "Rolling and slipping motion of Euler's disk", Phys. Rev. E, **69**, 056610 (2004) 1–6.

10) C. Le Saux, R. I. Leine, and C. Glocker: "Dynamics of a Rolling Disk in the Presence of Dry Friction", J. Nonlinear Sci., **15** (2005) 27–61.

11) R. Villanueva and M. Epstein: "Vibrations of Euler's disk", Phys. Rev. E, **71**, 066609 (2005) 1–7.