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Complexity of Graph and its J Program

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Summary — We obtain a new approach to calculate the number of spanning trees for any simple undirected graphs on the basis of the singular value of the incidence matrix. By this result we obtain J program for computing the complexity of graphs.

Key words : J program , singular value , complexity of graph

1 INTRODUCTION

$G=(V,E)$ is a simple undirected graph with a vertex set $V=\{v_1,v_2,......,v_n\}$ and an edge set $E=\{e_1,e_2,......,e_m\}$. $K(G)$ is the comlexity of graph G i.e. the number of spanning tree of G. By giving an arbrtrary orientation to each edge of G,we define the $(0.1.-1)$ −incidence matrix $E=(e_{ij})$ of G in the following way

$$e_{ij}=\begin{cases}1 & \text{if } v_i \text{ is the initial vertex of } e_j\\ -1 & \text{if } v_i \text{ is the terminal vertex of } e_j\\ 0 & \text{otherwise}\end{cases}$$

E_i is the matrix obtained from E by removing the i-th row. It is clear that

$$\text{rank } E = \text{rank } E_i = n-1$$

for $i \in \{1,2,...,n\}$.

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Also we denote A_i the matrix obtained from the adjacent matrix A of G by removing the i-th row and i-th column. It means A_i is the adjacency matrix of $G-v_i$. By the same definition as A_i, D_i is the degree matrix $D=(\delta_{ij}d(v_i))$ of G,where $d(v_i)$ is the degree of vertex v_i. We put

$$L = D - A$$

and call L the combinatrial Laplacian of G. Matrix-tree Theorem states that

$$\kappa(G) = \det L_{ij}$$

where L_{ij} is the (i,j) cofactor of L and $i,j \in \{1,2,....,n\}$.

Moreover between the $(0,1,-1)$-incidence matrix and the combinatrial Laplacian, there is the relation

$$E^t E = L.$$

A,D is the adjacency matrix and the degree matrix of G respectively, then

$$E_i{}^t E_i = L_{ii} = D_i - A_i$$

where L_{ii} is the (i,i) cofactor of L and $i \in \{1,2,....,n\}$.

Consequently, $E^t E$ and $E_i{}^t E_i$ are independent of the orientation given to G and we have

$$\kappa(G) = \det E_i{}^t E_i = \det(D_i - A_i)$$

2 SINGULAR VALUE

Now,we introduce the singular value Sing (G) of graph G. Let E be a $(0,1,-1)$-incidence matrix of G, then $E_i{}^t E_i$ is a regular and real synmetrix matrix. E_i is positive definite and $E_i{}^t E_i$ is diagonalizable. We give the following definition in the same manner as the definition of the spectrum of a graph.

DEFINITION

E is the $(0,1,-1)$-incidence matrix of G and $\lambda_1, \lambda_2, \ldots, \lambda_s$ with $\lambda_1 > \lambda_2, \ldots > \lambda_s > 0$ are the eigenvalues of $E_i{}^t E_i$ and $m_1, m_2, \ldots, m_s$ are their multiplicities.

We call $u_1 = \sqrt{\lambda_1}, u_2 = \sqrt{\lambda_2}, \ldots, u_s = \sqrt{\lambda_s}$ as the singular values of E_i and denote the singular values of G with respect to E_i by

$$Sing_i(G) = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_s \\ m_1 & m_2 & \cdots & m_s \end{pmatrix}$$

In particular, if $Sing_i(G)$ is unique independently of i, we call it as the singular value of G and denote it by $Sing(G)$.

For a regular graph G and any vertex v of G, $G \cdot v$'s are isomorphic, therefore there exists $Sing(G)$. For the $(0,1,-1)$-incidence matrix E of G, $E_i{}^t E_i$ is diagonalizable and

$$\det E_i{}^t E_i = \prod_{k=1}^{s} (u_k{}^2)^{m_k}$$

Then we obtain the following new method to calculate the complexity of a graph.

THEOREM Let G be a graph and let

$$Sing(G) = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_s \\ m_1 & m_2 & \cdots & m_s \end{pmatrix}$$

then,

$$\kappa(G) = \prod_{k=1}^{s} (u_k{}^2)^{m_k}$$

3 EXAMPLES

By this theorem , we calculate the singular values of several special families of graphs and write down their complexities, which equal to the well known results.

1. For the complete graph with n vertices K_n,

$$Sing(K_n) = \begin{pmatrix} \sqrt{n} & 1 \\ n-2 & 1 \end{pmatrix}$$

So we have

$$\kappa(K_n) = n^{n-2}$$

2. For the circuit graph of length n C_n,

$$\mathrm{Sing}(C_n) = \begin{pmatrix} \sqrt{2-2\cos\frac{(n-1)\pi}{n}} & \sqrt{2-2\cos\frac{(n-2)\pi}{n}} & \cdots & \sqrt{2-2\cos\frac{\pi}{n}} \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

So we have

$$\kappa(C_n) = \prod_{k=1}^{n-1}(2-2\cos\frac{k\pi}{n})$$

3. For the wheel graph with n vertices W_n,

$$\mathrm{Sing}\ (W_n) = \begin{cases} \begin{pmatrix} \sqrt{3-2\cos\frac{2(l-1)\pi}{2l-1}} & \sqrt{3-2\cos\frac{2(l-2)\pi}{2l-1}} & \cdots & \sqrt{3-2\cos\frac{2\pi}{2l-1}} & 1 \\ 2 & 2 & \cdots & 2 & 1 \end{pmatrix} & \text{if } n = 2l, \\ \begin{pmatrix} 5 & \sqrt{3-2\cos\frac{(l-1)\pi}{l}} & \sqrt{3-2\cos\frac{(l-2)\pi}{l}} & \cdots & \sqrt{3-2\cos\frac{\pi}{l}} & 1 \\ 1 & 2 & 2 & \cdots & 2 & 1 \end{pmatrix} & \text{if } n = 2l+1. \end{cases}$$

So we have

$$\kappa(W_n) = \begin{cases} \prod_{k=1}^{l-1}\left(3-2\cos\frac{2k\pi}{2l-1}\right) & \text{if } n = 2l, \\ 5\prod_{k=1}^{l-1}\left(3-2\cos\frac{k\pi}{l}\right) & \text{if } n = 2l+1. \end{cases}$$

4. For the complete bipartite graph $K_{r,s}(r \geq s)$

$$\mathrm{Sing}\ (K_{r,s}) = \begin{cases} \begin{pmatrix} 1 \\ s \end{pmatrix} & \text{if } r = 1, \\ \begin{pmatrix} \sqrt{\frac{r+s+\sqrt{(r+s)^2-4s}}{2}} & \sqrt{r} & \sqrt{s} & \sqrt{\frac{r+s-\sqrt{(r+s)^2-4s}}{2}} \\ 1 & s-1 & r-2 & 1 \end{pmatrix} & \text{if } r \geq 2. \end{cases}$$

So we have

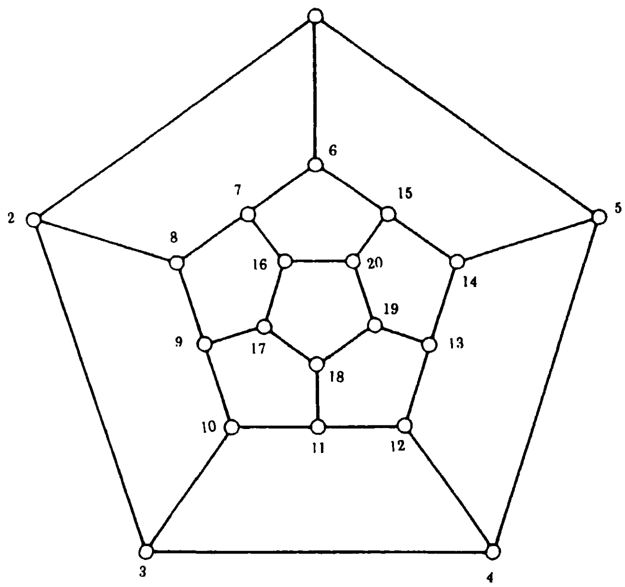
$$\kappa(K_{r,s}) = r^{s-1}s^{r-1}$$

4 COMPLEXITY of G(20,30)

By this theorem we obtain J program for computing the complexity of graphs.(cf.Appendix)

[Examples]

$$G=(V,E), |V|=20,\ |E|=30$$



We label the number 1,2,……,20 to all vertices of graph G and indicate edges by two vertices number. We input the number series corresponding to all edges of G, no particular order observed. The important point of this program is that input is a simple number series and not a matrix. Then we obtain $Sing(G)$ and $\chi(G)$ of graph G(20,30) as output.

$X=$. 1 2 2 3 3 4 4 5 1 5 1 6 2 8 3 10 4 12 5 14
6 7 7 8 8 9 9 10 10 11 11 12 12 13 13 14 14 15 6 15 7 16
9 17 11 18 13 19 15 20 16 20 16 17 17 18 18 19 19 20

$\chi(G) = 5{,}184{,}000$

```
File: a:\jfw\temp\7.jx
   0!:0<'a:\jfw\temp\jikkou.js'
   a{EG12}
 1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
_1  1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
 0 _1  1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
 0  0 _1  1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
 0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
 0  0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  0  0  0
_1  0  0  0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  0
 0 _1  0  0  0  0  0  0  1  0  0  0  0  0  0  0  0  0  0
 0  0 _1  0  0  0  0  0  0  0  1  0  0  0  0  0  0  0  0
 0  0  0 _1  0  0  0  0  0  0  0  0  1  0  0  0  0  0  0
 0  0  0  0 _1  1  0  0  0  0  0  0  0  0  0  0  0  0  0
 0  0  0  0  0 _1  1  0  0  0  0  0  0  0  0  0  0  0  0
 0  0  0  0  0  0 _1  1  0  0  0  0  0  0  0  0  0  0  0
 0  0  0  0  0  0  0 _1  1  0  0  0  0  0  0  0  0  0  0
 0  0  0  0  0  0  0  0 _1  1  0  0  0  0  0  0  0  0  0
 0  0  0  0  0  0  0  0  0 _1  1  0  0  0  0  0  0  0  0
 0  0  0  0  0  0  0  0  0  0 _1  1  0  0  0  0  0  0  0
 0  0  0  0  0  0  0  0  0  0  0 _1  1  0  0  0  0  0  0
 0  0  0  0  0  0  0  0  0  0  0  0 _1  1  0  0  0  0  0
 0  0  0  0 _1  0  0  0  0  0  0  0  0  1  0  0  0  0  0
 0  0  0  0  0 _1  0  0  0  0  0  0  0  0  1  0  0  0  0
 0  0  0  0  0  0  0 _1  0  0  0  0  0  0  0  1  0  0  0
 0  0  0  0  0  0  0  0  0 _1  0  0  0  0  0  0  1  0  0
 0  0  0  0  0  0  0  0  0  0  0 _1  0  0  0  0  0  1  0
 0  0  0  0  0  0  0  0  0  0  0  0  0 _1  0  0  0  0  1
 0  0  0  0  0  0  0  0  0  0  0  0  0  0 _1  0  0  0  1
 0  0  0  0  0  0  0  0  0  0  0  0  0  0 _1  1  0  0  0
 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0 _1  1  0  0
 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0 _1  1  0
 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0 _1  1
E
2.28825 2.28825 2.28785 2.23607 2.23607 2.23607 2.02639 1.73205 1.73205 1.73205 1.59298 1.41421 1.41421 1.41421 1.41421 1.0547
c
    5.23607     5.23607     5.14314           5           5           5     4.10625           3           3           3     2.53758
 _0.2158509   0.1419907  0.06390311 _0.03155372   0.1897746  _0.2722152  _0.4288822  _0.3337145  _0.2868612 _0.07965767   0.2412281
  0.3640726   0.0280088 _0.06847656 _0.04263723  0.03497197   0.4374968   0.2372254   _0.238238   0.2176861   0.2141387  0.05577387
 _0.3582888  0.07044466 _0.06847656  _0.2192229  _0.2060927  _0.3223531   0.2372254   0.1727904   0.2215997  _0.2665278  0.05577387
   0.230793   0.1157638  0.06390311   0.2934139 _0.01865382   0.1570715  _0.4288822   0.3991621  _0.1524246   0.1320467   0.2412281
_0.01514206  _0.2577545  0.06390311  _0.2818601  _0.1711207   0.1151437  _0.4288822 _0.06544767   0.4392858 _0.05238905   0.2412281
 _0.2082926   0.3012917 _0.06847656   0.1876692   0.3958673    0.050138   0.2372254   0.1609241  0.06526147   0.3461855  0.05577387
  0.1181375  _0.3455096 _0.06847656   0.1057447  _0.4145211   0.1049335   0.2372254    0.238238  _0.2176861  _0.2141387  0.05577387
    0.15778   0.3293008   0.1513281  _0.3676048   0.2434004 0.008210205 _0.07077387   0.1727904   0.2215997  _0.2665278  _0.2712112
 _0.2401513  _0.2750649   0.1513281   0.3360811 _0.05362579  _0.2804254 _0.07077387   0.1609241  0.06526147   0.3461855  _0.2712112
 0.01514206   0.2577545  _0.4071683  _0.2618601  _0.1711207   0.1151437 _0.08815797  0.06544767  _0.4392858  0.05238905  0.09002492
  0.2082926  _0.3012917   0.1513281   0.1876692   0.3958673    0.050138 _0.07077387  _0.1609241 _0.06526147  _0.3461855  _0.2712112
 _0.1181375   0.3455096   0.1513281   0.1057447  _0.4145211   0.1049335 _0.07077387   _0.238238   0.2176861   0.2141387  _0.2712112
   _0.15778  _0.3293008 _0.06847656  _0.3676048   0.2434006 0.008210206   0.2372254  _0.1727904  _0.2215997   0.2665278  0.05577387
  0.2401513   0.2750649 _0.06847656   0.3360511 _0.05362579  _0.2804254   0.2372254  _0.1609241 _0.06526147  _0.3461855  0.05577387
  0.3582888 _0.07044466   0.1513281  _0.2192229  _0.2060927  _0.3223531 _0.07077387  _0.1727904  _0.2215997   0.2665278  _0.2712112
  _0.230793  _0.1157638  _0.4071683   0.2934139 _0.01868382   0.1570715 _0.08815797  _0.3991621   0.1524246  _0.1320467  0.09002492
    4.2e_11    _4.76e_10   0.5659612      _1e_12           0       3e_12   0.2390725      _3e_12    1.26e_10           0   0.5840514
  0.2158509  _0.1419907  _0.4071683 _0.03155372   0.1897746  _0.2722152 _0.08815797   0.3337145   0.2868812  0.07965767  0.09002492
 _0.3640726  _0.0280088   0.1513281 _0.04263723  0.03497197   0.4374968 _0.07077387    0.238238  _0.2176861  _0.2141387  _0.2712112

0.874032 0.874032 0.3172501

            2            2            2            2     1.11238    0.763932    0.763932  0.1006476
   _0.2423072   _0.3968477 0.0006213406   0.07759506  _0.2578396 _0.01640588  _0.2576772  0.1436578
 _0.004593238   _0.2044077   _0.2017293    0.3737966  _0.2433513   0.2048126  _0.3022997  0.2082574
    0.4140367  _0.07843702    0.1186334    0.1748438  _0.2433513   0.3579391 _0.07220084  0.2082574
    0.1405383    0.2273162    0.3507321    0.1666899  _0.2578396   0.2313579   0.1146307  0.1436578
    0.1017689    0.1695314   _0.3513536   _0.2442849  _0.2578396   _0.214952   0.1430465  0.1436578
    0.2780907   _0.1013456  _0.03036937   _0.3656428  _0.2433513  _0.3642056 _0.02622308  0.2082574
   _0.2377139     _0.19244    0.2023507   _0.2962015  _0.2433513  _0.2414973  _0.2738839  0.2082574
   _0.2734875    0.3057533    0.2320987 _0.008153964  0.04183646  _0.1593929  _0.3288228  0.2518963
   _0.1763218     0.270877   _0.3209841    0.1213577  0.04183646   0.1164418  _0.3460847  0.2518963
    0.1017689    0.1695314   _0.3513535   _0.2442849   0.2804861    0.214952  _0.1430465  0.2701824
    0.2780907   _0.1013456  _0.03036937   _0.3656426  0.04183646   0.3642056  0.02622308  0.2518963
   _0.2377139     _0.19244    0.2023507   _0.2962015  0.04183646   0.2414873   0.2738839  0.2518963
   _0.2734975    0.3057833    0.2320987 _0.008153964  _0.2433513   0.1593929   0.3288228  0.2082574
   _0.1763218     0.270877   _0.3209841    0.1213677  _0.2433513  _0.1164418   0.3460847  0.2082574
    0.4140367  _0.07843702    0.1186334    0.1748438  0.04183646  _0.3579391  0.07220084  0.2518963
    0.1405383    0.2273162    0.3507321    0.1666899   0.2804861  _0.2313579  _0.1146307  0.2701824
            0      4.6e_11            0        1e_12   0.4457779     1.5e_11      _6e_12  0.2795615
   _0.2423072   _0.3968477 0.0006213406   0.07759506   0.2804861  0.01640588   0.2576772  0.2701824
 _0.004593238   _0.2044077   _0.2017293    0.3737966  0.04183646  _0.2048126   0.3022997  0.2518963
```

19 eigen values and their eigen vectors

Appendix : J Program for Computing Complexity

J is a general-purpose programming language available as shareware on a wide variety of computers(http://www.jsoftware.com). The characteristics of J is that we can handle vectors, matrices, and other arrays as single entities like a pocket calculator. J has many kinds of convenient functions for instance, matrix operations including transpose or inverse matrix, multiple regression, outer products, operations with complex numbers, Taylor expansions, approximations by polynomials, decomposition into prime numbers and GCD and LCM. We can easily get J freeware from the following address ftp://take.soc.ae.keio.ac.jp/take/j/exper/fd1/jfw.exe.

Subroutines : (NB. Denotes Comments)

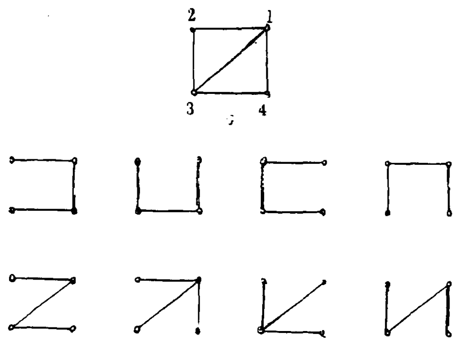
```
NB. Write to Screen
   print=.(1!:2)&2
NB. Matrix Product or Inner Product
   mp=.+/ .*
NB. Select eigen vectors corresponding to nonzeo eigen values
   nsel=.1e_5&<@(0&{)@{.#"1 }.
NB. Convert a vector of locations to a matrix form
   mat=.3 :'1(<"1 (<:y.),.(n#2)#i.n)}((>./y.),n=.-:#y.)$0'
NB. Replace the first "1" in each column by "_1"
   minus=.3 :'_1(<"1 t,.i.#t=.(|:y.)i."1(1))}y.'
NB. Jacobi's method for eigenvalues and vectors
NB. form: [tolerance] jacobi mat
NB. default tolerance 1e_5
NB. returns: eigenvalues;eigenvectors
   jacobi=: 3 : 0
1e_5 jacobi y.
:
r=. y.
ir=. i.#r
q=. id=. =ir
diag=. (<0 1)&|:
len=. +/&.(*:"_)
imax=. i.>./
utm=. ,</~ir
mp=. +/ .*
ndx=. ,{;~ir
perm=. 0 0&{ ; ] ; |. ; 1 1&{
sign=. _1: ^ <&0
count=. 0
while.
  big=. imax utm * |,r
  ind=. big { ndx
  p=. -ind { r
  (x. < |p) *. big > 0
```

```
do.
  u=. -: -/ (>ind) { diag r
  v=. len p,u
  cos=. %: (v+|u) % +:v
  sin=. (sign u) * p % +: v * cos
  s=. ((cos,-sin),sin,cos) (perm >ind) } id
  r=. s mp r mp |:s
  q=. q mp |:s
  count=. >:countW
end.
r=. diag r
inx=. ¥:r
(inx{r);inx{"1 q
)
```

Main Program for Calculating Complexity of a Graph

```
  Complex=.3 : 0
a=:}.minus print Mtrx=:mat y.
b=.a mp (|:a)
c=:>jacobi b
n=.{:$print Vec=:nsel c
print*/print Eig=:n{.{.{.c
)
```

An Example :




```
NB. data
  x=.1 2 2 3 3 4 1 4 1 3
NB. Execution form
  Complex x
NB. Graph matrix
1 0 0 1 1
1 1 0 0 0
0 1 1 0 1
0 0 1 1 0
NB. Eigen vectors
_0.4082494  _0.7071062 0.5773503
 0.8164966 _1.24831e_6 0.5773503
_0.4082472   0.7071074 0.5773503
NB. Eigen values
4 2 1
NB. Complexity
8
```

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