慶應義塾大学学術情報リポジトリ
Keio Associated Repository of Academic resouces

Title	A note on Feferman and Takeuti II : philosophical perspectives on inductive definitions
Sub Title	
Author	秋吉, 亮太(Akiyoshi, Ryota)
Publisher	三田哲學會
Publication year	2025
Jtitle	哲學 (Philosophy). No.155 (2025. 3) ,p.1- 16
JaLC DOI	
Abstract	It is widely acknowledged that both of Feferman and Takeuti maintained a sustained interest in theories of inductive definitions. Feferman, inspired by Kreisel, proposed a constructive foundation for subsystems of analysis based on intuitionistic theories of inductive definitions. Notably, one of Feferman's motivations was to provide more perspicious methods for proving Takeuti's results regarding "Takeuti's conjecture". This proposal subsequently became a driving force behind proof-theoretic results of these theories obtained by Schütte's school in 1970s. On the other hand, as pointed out by Kreisel in his review, Takeuti implicitly used a particular form of inductive definitions in order to define ordering relations between ordinal diagrams. Furthermore, he critically discussed Feferman's project based on theories of inductive definitions. In this paper, we explore the differences between Feferman's and Takeuti's views on theories of inductive definitions. It will be argued that Takeuti's critiques of these theories are not convincing. Instead, their differences arise from their fundamentally distinct perspectives: Feferman's approach, influenced by a "Hilbertian" view, regarded intuitionistic theories of inductive definitions as more "secure" than subsystems of analysis, while Takeuti did not hold this strong presumption regarding the security of constructive approaches.
Notes	投稿論文
Genre	Journal Article
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=AN00150430-00000155-0001

慶應義塾大学学術情報リポジトリ(KOARA)に掲載されているコンテンツの著作権は、それぞれの著作者、学会または出版社/発行者に帰属し、その権利は著作権法によって保護されています。引用にあたっては、著作権法を遵守してご利用ください。

The copyrights of content available on the KeiO Associated Repository of Academic resources (KOARA) belong to the respective authors, academic societies, or publishers/issuers, and these rights are protected by the Japanese Copyright Act. When quoting the content, please follow the Japanese copyright act.

A Note on Feferman and Takeuti II: Philosophical Perspectives on Inductive Definitions

Ryota Akiyoshi*

It is widely acknowledged that both of Feferman and Takeuti maintained a sustained interest in theories of inductive definitions. Feferman, inspired by Kreisel, proposed a constructive foundation for subsystems of analysis based on intuitionistic theories of inductive definitions. Notably, one of Feferman's motivations was to provide more perspicious methods for proving Takeuti's results regarding "Takeuti's conjecture". This proposal subsequently became a driving force behind proof-theoretic results of these theories obtained by Schütte's school in 1970s.

On the other hand, as pointed out by Kreisel in his review, Takeuti implicitly used a particular form of inductive definitions in order to define ordering relations between ordinal diagrams. Furthermore, he critically discussed Feferman's project based on theories of inductive definitions.

In this paper, we explore the differences between Feferman's and Takeuti's views on theories of inductive definitions. It will be argued that Takeuti's critiques of these theories are not convincing. Instead, their differences arise from their fundamentally distinct perspectives: Feferman's approach, influenced by a "Hilbertian" view, regarded intuitionistic theories of inductive definitions as more "secure" than subsystems of analysis, while Takeuti did not hold this strong presumption regarding the security of constructive approaches.

* Email: georg.logic@gmail.com. The author is partially supported by KAKENHI 24K03340. The author expresses his gratitude to Andrew Arana for insightful discussions on these topics and thanks the anonymous referee for careful readings, many helpful comments, and suggestions.

1 Introduction

It is widely acknowledged that both of Feferman and Takeuti maintained a sustained interest in theories of inductive definitions. Feferman, inspired by Kreisel, proposed a constructive foundation for subsystems of analysis based on intuitionistic theories of inductive definitions inspired by Kreisel. Notably, one of Feferman's motivations was to provide more perspicious methods for proving Takeuti's proof-theoretic results related to "Takeuti's conjecture". Consequently, this proposal became a driving force behind proof-theoretic results for these theories, as developed by logicians in Schütte's school in 1970s (cf. [9]).

Let us quote Feferman's preface in [9, p.9];

> The first significant proof-theoretic results obtained for impredicative systems (after Spector's work) were in Takeuti 1967 for Π^1_1-*CA*(+*BI*) [24], later extended to Δ^1_2 -*CA* by Takeuti, Yasugi 1973 [30].

After explaining that Takeuti's approach follows in the tradition of Gentzen's consistency proof of arithmetic, Feferman comments that Takeuti's ordinals called "ordinal diagrams" are not (naturally) based on systems of ordinal functions. Therefore, he concludes that it is necessary to find another way to prove the same results [9, p.9];

> [...] the details of the arguments for Takeuti's method were extremely difficult to follow, and a more perspicuous treatment was much to be hoped for and sought.

On the other hand, as Kreisel pointed out in his review [14], Takeuti implicitly used a particular form of inductive definitions to define ordering relations between ordinal diagrams introduced in [20]. Furthermore, he critically discussed Feferman's project based on theories of inductive definitions [25].

In this paper, we explore the differences between Feferman's and Takeuti's views on theories of inductive definitions.[1] It will be argued that Takeuti's critiques of these theories are not convincing. Instead, their differences arise from their fundamentally distinct perspectives: Feferman's approach, influenced by a "Hilbertian" view, regarded intuitionistic theories of inductive definitions as more "secure than subsystems of analysis", while Takeuti did not hold this strong presumption regarding the security of constructive approaches.

This note is structured as follows. In Section 2, Feferman's proposal based on theories of inductive definitions is described. Next, Takeuti's discussions on Feferman's assessment of Takeuti's results are explained and examined in Section 3. Finally, it will be shown that their differences arise from the fact that Takeuti's view on these theories differs significantly from Feferman's.

2 Feferman's Proposal Based on Theories of Inductive Definitions

In this section, we explain Feferman's foundational proposal based on theories of inductive definitions. He has described this proposal in several papers, most clearly in the preface to a book he co-edited, titled *Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies* [9]. We will use this Feferman's preface as a clue to describe the project.

To explain how theories of inductive definitions came to be studied, Feferman first points out that they were introduced by Kreisel [9, p.6];

> The study of formal theories featuring inductive definitions in both single and iterated form was initiated by Kreisel 1963 [13]. The immediate stimulus was the question of constructive justification of Spector's 1961 consistency proof for analysis [18].

Broadly speaking, the hope was that the study of theories of inductive definitions would provide a justification for the principle of bar recursion, which Spector

[1] Our previous paper [1] examined distinctions between Feferman's and Takeuti's approaches to predicativity. In the present study, we identify a similar reasoning underlying their philosophical perspectives on inductive definitions, further illuminating a consistent philosophical divergence in their foundational approaches.

introduced for the sake of the consistency proof of analysis. However, these theories proved too weak to justify bar recursion. As a result, Spector's proof was not entirely convincing, and furthermore, it did not yield any ordinal-theoretic information about analysis. Consequently, Feferman noted that considerable efforts in the 1960s were devoted to extending ordinally informative proof theory to impredicative subsystems of analysis, marking 1967–1968 as a pivotal period in this endeavor.

Let us explain the theories of inductive definitions very briefly. The formal theory ID_1 is based on the following axioms;

1. $A(P_A, x) \to P_A(x)$,
2. $\forall x(A(B,x) \to B(x)) \to \forall x(P_A(x) \to B(x))$ for each B,

   where $A(P_A, x)$ is arithmetical in P and contains it only positively.

This theory can be seen as formalizing the process of constructing a mathematical object from below. Roughly speaking, the first axiom states that the set P_A is a fixed point in the sense that if we apply the inductive process to it, then we would obtain the same set. The second one states that the set P_A is the smallest one in the sense that if some set B is closed under the inductive process, then all elements in P_A are also contained in B.

One important example of this theory that should be noted is the case of the accessibility relation when

$$A(P_A, x) \equiv A_0(x) \wedge \forall y[\langle y, x\rangle \in R \to P(x)].$$

For example, the class $\mathcal{O}$ of Church–Kleene constructive ordinals is given by the accessibility part of the following relation; (i) $0 \in \mathcal{O}$, (ii) if $x \in \mathcal{O}$, then $x+1 \in \mathcal{O}$, and (iii) if a recursive function enumerates elements of $\mathcal{O}$, then $limit(f) \in \mathcal{O}$ (where $limit(f)$ is the limit of the set $\{f(n) \mid n \in \omega\}$).

Iterating these definitions, we obtain stronger theories $ID_2, ID_3, \ldots$, but without going into details here, we will explain the importance of these theories in the foundations of mathematics. According to Feferman, the accessibility inductive definition holds a privileged position in the foundations of mathematics because there is a direct picture of how the members of such sets are generated. Under this picture, the axioms can be recognized as correct.

On the other hand, there are many formal theories in the foundations of mathematics dealing with objects for which such direct pictures are not available. For example, consider the comprehension axiom in second-order arithmetic;

$$\exists X \forall x (x \in X \longleftrightarrow A(x))$$

where $\exists X$ ranges over sets of natural numbers and $A(x)$ is an arbitrary formula. A number of interesting theories can be obtained by imposing certain restrictions on the logical symbols occurring in $A(x)$ in some way. For instance, if $A(x)$ is restricted into the form $\forall Y B(Y,x)$ known as Π^1_1-formula where B contains no second-order quantifier, then a version of Π^1_1-comprehension axiom is obtained. This axiom is impredicative in the sense that it defines a set by referring to all sets, including the set being defined.

Under these circumstances, Feferman asked if such a principle could be reduced in some way to a theory of inductive definition. Furthermore, the base theory to which the principle is reduced should be intuitionistic due to the constructive nature of accessibility inductive definitions, where there is a direct picture of how the members of such sets are generated in those theories.

To explain this project more precisely, let us use the framework of Feferman's "What rests on what? The proof-theoretic analysis of mathematics" [10], which was published later. Let T_1 and T_2 be formal theories based on the languages L_1 and L_2, respectively. Moreover, let Φ be a primitive recursive class of formulas contained in both L_1 and L_2 containing every closed equation $t_1 = t_2$. Now, a formal system T_1 is defined to be *proof-theoretically reducible to T_2 by f, conservatively for* Φ, and write $f : T_1 \leq T_2$ for Φ if f is a partial recursive function such that (i) if $Proof_{T_1}(x,y)$ holds (meaning "y is a proof of x in T_1") for $x \in \Phi$, then $f(y)$ is well-defined with $Proof_{T_2}(x, f(y))$, and (ii) the formalization of (i) is formally provable in T_2. More roughly, every provable sentence A in T_1 is already provable in T_2 for each $A \in \Phi$. It should be noted here that it is necessary to find a concrete "algorithm" f to transform a proof of A in T_1 into another one in T_2.

Feferman provides several examples that satisfy these conditions, one of which is an example of intuitionistic theories of inductive definitions. Let Δ^1_2-CA denote the comprehension axiom restricted to Δ^1_2-formulas, which lies between Π^1_1-CA and

Π^1_2-*CA*, and let $ID^i_{<\varepsilon_0}$ denote the theory of ε_0-times iterated inductive definition based on intuitionistic logic. The main result, established in a series of steps due to works of Pohlers, Feferman and Sieg [9], is the following;

$$\Delta^1_2 - CA \leq ID^i_{<\varepsilon_0}.$$

To provide a historical context for this kind of result, it is important to outline the historical background. In 1958, Takeuti made a significant breakthrough by proving the consistency of the subsystem of second-order logic with the strength of Π^1_1-CA_0. Subsequently, he extended this result to Π^1_1-CA_0 with bar induction [21, 24]. In the course of his works, Takeuti introduced novel systems of ordinals known as "ordinal diagrams" to prove the termination of reduction steps [20, 22]. It is worth noting that while indispensable for his proofs, they did not resonate intuitively with Western logicians. Let us quote Feferman's passages again from [9, p.9];

> The first significant proof-theoretic results obtained for impredicative systems (after Spector's work) were in Takeuti 1967 for Π^1_1-*CA*(+*BI*), later extended to Δ^1_2 -*CA* by Takeuti, Yasugi 1973. [...] the details of the arguments for Takeuti's method were extremely difficult to follow, and a more perspicuous treatment was much to be hoped for and sought.

It should be noted here that there is a discrepancy in Feferman's understanding of the period in which Takeuti groundbreaking results were obtained. Takeuti published his landmark results on the consistency of Π^1_1-CA_0 in 1958 [21], not in 1967 [24]. However, it would be not difficult to imagine that Feferman may have had trouble accessing the 1958 paper at the time, as it was published in a journal of the Mathematical Society of Japan.

Feferman's call laid the groundwork for significant developments in proof theory after the 1970s. Buchholz introduced a new infinitary rule the "$\Omega_{\mu+1}$-rule", and simplified the ordinal notation systems that Feferman had investigated [6, 7, 8]. Additionally, Pohlers introduced another approach known as "local predicativity" in [15]. [2,3]

3 Takeuti's Responses to Feferman's Assessment of Takeuti's Results

In this section, we examine Takeuti's responses to Feferman's assessment of Takeuti's contributions in proof theory, and argue that Takeuti's criticism is not convincing. To this end, we first present Takeuti's responses to Feferman's critical view of Takeuti's results in proof-theory. Yasugi, who was Takeuti's student and collaborator, offers a notable recollection of the controversy [32][4];

> I met Kreisel, who was called the king of proof theory, at a seminar in Ithaca and at the International Conference on Proof Theory in Buffalo (1968). He was a man who always took the podium to deliver a sermon. During a break in this conference, the "Takeuti–Feferman controversy" took place. It was memorable as a head-on debate between two directions of proof theory.

Yasugi does not clearly specify which Buffalo conference she is referring to, but it seems clear from the context that she is referring to the well-known international conference later published as *Intuitionism and Proof Theory: Proceedings of the Summer Conference at Buffalo N.Y. 1968* [12]. According to Yasugi's account, Takeuti continued to advocate for the consistency proof within finitistic standpoint, and that it was important for Feferman to explore the mathematical consequences of a formal system considered and the mathematical relationships among such systems. Unfortunately, as a result, not much productive discussion took place.

---

2 This development led Jäger's proposal to shift from proof theory of second-order arithmetic to that of Kripke–Platek set theory in his Ph.D. thesis. Further developments in proof theory influenced by the Feferman–Schütte school can be found in references such as [9, 17].

3 It is also worthwhile to mention the groundbreaking work of Arai and Rathjen in the 1990's, who tackled the long-standing open problem of ordinal analysis for Π^1_2-comprehension and related principles [5, 16]. Their contributions significantly advanced the field, marked a culmination of efforts to clarify, and establish the ordinal-theoretic foundations of these complex systems.

4 In this paper, translations from Japanese texts are by the author unless otherwise indicated.

To better understanding of this interesting controversy, let us turn to another of Takeuti's writings [25], which offers a counterargument to Feferman's assessment of Takeuti's consistency results. In the chapter of finitistic standpoint, Takeuti explains how to prove the well-foundedness of ordinals up to ε_0 within this framework. Although he does not provide a detailed proof, he asserts that the proof is a natural extension of the proofs for the cases of natural numbers and ordered pairs of them. To the author's best knowledge, the proof, if carried out in detail, would closely resemble the detailed argument in his *Proof Theory* [27, 29].[5]

Takeuti then proceeds to discuss the well-foundedness of ordinal diagrams in relation to his proof of consistency for Π^1_1-CA. After explaining what is Π^1_1-CA, he acknowledges that the proof of well-foundedness of ordinal diagrams is quite intricate and challenging. He mentions that the proofs published so far only suggest that such proof can be carried out within a finite standpoint, and that set-theoretic principles known as "inductive definitions" are actually used in his previous paper. Although he does not specify the exact proof he is referring to, it is likely that he is discussing the proof from his seminal paper introducing ordinal diagrams [20].

It would be well worthwhile to note here that it was Kreisel who pointed out the use of set-theoretic principles in Takeuti's proof [14]. Before delving into Takeuti's reaction to Feferman, let us give some remarks in this review. After criticizing Takeuti's claims in his paper [23], Kreisel continues

> As to the constructive character of the author's proof, this is not evident without further analysis, because his proof of well foundedness of o.d. uses the obviously impredicative notion of accessibility. However, the reviewer has been able to formalize the proofs of transfinite induction (t.i.) in the intuitionistic theory of generalized inductive definitions, by taking one of the properties of the notion of accessibility as inductive definition of the species $A_i(x)$ [...], and deriving the others needed for the proof of t.i. by means of the corresponding principle of proof by induction.

[5] In fact, the proof there is known for its density and complexity. For a detailed study of this proof, see [3, 31].

Let us make some remarks about this passage. First, the proof that Kreisel is discussing here is probably the one presented in a different paper [20], since the proof is not presented in detail in the paper reviewed [23]. Second, it was Arai who took up Kreisel's idea and extended it in great detail [4].[6]

Going back to Takeuti, according to him, Feferman adopted some theory of inductive definitions as a fundamental principle and proved Takeuti's results more easily, and Feferman believes these are elegant improvements of Takeuti's results. In contrast, Takeuti criticizes these as reversals of the original results because inductive definitions in some specific form shed light on the solution, but inductive definitions conceived in a general form are set-theoretic and therefore of little significance for the consistency proof.

After this rebuttal, Takeuti offers his recollections of the debate with Feferman at an international conference in 1968. According to Takeuti, he and Feferman had a short discussion at the official table at that meeting, and then had a lengthy discussion privately, but could not come to a satisfactory conclusion. Here, Takeuti does not clearly state which international conference it was, but in light of Yasugi's memories [32], one can infer that it was at the Buffalo conference that was later published as [12].

Feferman did not offer any kind of reply to Takeuti's objections, but it is not too difficult to derive possible objections from his works. The point of Takeuti's objection was that Feferman's argument relies on a general principle and is therefore not appropriate for a proof of consistency. To develop this point a bit further, it would be useful to consider a model of the axioms of the theory of inductive definition called ID_1;

1. $A(P_A,x) \to P_A(x)$,
2. $\forall x(A(B,x) \to B(x)) \to \forall x(P_A(x) \to B(x))$ for each B,

   where $A(P_A,x)$ is arithmetical in P and contains it only positively.

As Takeuti says, it is customary to appeal to set-theoretic principles to construct a model that satisfies these axioms in general. More precisely, the fixed point $\mathscr{P}_A$ by

[6] To the author's best knowledge, Arai carried out Kreisel's idea and extended it because Arai proved the well-foundeness of ordinal diagrams not only for finite orders, but for the infinite orders in the corresponding theories of inductive definitions.

an "monotone" operator A defined for all subsets of natural numbers is given by

$$\mathscr{P}_A := \bigcap\{X|A(X) \subset X\}$$

where X ranges over subsets of the set of natural numbers. Then, it can be shown that $\mathscr{P}_A$ satisfies the two axioms above.

However, it is critically important to note that this argument considers the axioms in their general form. Let us recall here that Feferman paid attention to the special case of the definition, that is, the case of accessibility;

$$A(P_A, x) \equiv A_0(x) \wedge \forall y[\langle y, x\rangle \in R \to P(x)].$$

The philosophical reason for this is that there is a direct picture of how the members of such sets are generated so that the two axioms can be recognized as correct. Furthermore, Feferman considers the following proof-theoretic reduction as important;

$$\Delta^1_2 - CA \leq ID^i_{<\varepsilon_0}.$$

Though Feferman did not mention it in [10], the results in the monograph [9] implies the following;

$$\Delta^1_2 - CA \leq ID_{<\varepsilon_0}(acc)^i.$$

Thus, when Feferman is discussing the proof-theoretic reduction as above, it turns out that the theory being reduced is the theory of iterated inductive definition in a specific form. Furthermore, the form is such that there is a direct picture of how the members of such sets are generated. Therefore, we conclude that Takeuti's critical objection to Feferman is not convincing.

4 A True Aspect of Feferman-Takeuti Controversy

In this section, we briefly argue that the conflict between Feferman and Takeuti actually stems from a more fundamental difference in philosophical positions in the foundations of mathematics.

Feferman reflects on the proof-theoretic results achived by Arai and Rathjen in the 1990s, noting that the development from the Hilbert program has advanced to such an extent that its achievements are no longer clearly explained [11, p.34]; "Here a basic problem is that the line of development from the Hilbert program has become so attenuated that one can no longer say in any clear-cut way what is accomplished. The leading work in that direction is that of Arai and Rathjen on the subsystem of analysis based on Π^1_2 comprehension. The Gentzen–Schütte–Takeuti aim was to prove consistency by constructive means." Feferman acknowledges that it remains somewhat elusive to pinpoint the precise implications of their work. In particular, he highlights an ongoing challenge: finding a constructive formal theory to which Π^1_2 comprehension can be reduced remains an open question.

This brings us to a critical point when dealing with exceptionally strong theories; if the theory under consideration is proof-theoretically very powerful, then its foundational base theory must also be quite strong, often involving a transfinite induction up to extremely large ordinals. As a result, it becomes less obvious why such a theory should be regarded as more constructive than the theory being analyzed.

This observation is particularly relevant to Takeuti's results. It is noteworthy that Feferman sought more perspicuous methods to prove Takeuti's results, underscoring the complex nature of their implications.

As discussed in our previous paper [2], the trends we observe can be traced back to Hilbert, who originated this line of thought. He used a term "sicher" (which means "secure" or "certain") to express the significance of proof theory. This continuity highlights the inherent link between proof theory and the confidence — a persistent and fundamental theme throughout its development. To understand the difference between Takeuti's and Hilbert's perspectives on the foundations of mathematics, it should be instructive to examine Takeuti's early paper titled "From the Viewpoint of Formalism II" in 1956 [19, p.299]. As noted in [2], there is a passage indicating a fundamental difference between Takeuti's conception of formalism and Hilbert's one. In essence, Takeuti's standpoint of formalism does not aim to serve as the ultimate and irreducible foundation for mathematics. Instead, it functions as a flexible framework that can be examined and refined through the ongoing development and exploration of his consistency program. This perspective signifies a departure from Hilbert's view, emphasizing the dynamic and evolving nature of Takeuti's formalism, which is open to further inquiry and refinement rather than being considered the most reliable and stable foundation.

Before closing this paper, we will briefly revisit Takeuti's unique view of the foundations of mathematics, which we explored previously in [2]. To this end, it is important to focus on Takeuti's paper titled "About Mathematics" [26]. In this paper, Takeuti articulates his perspective on mathematical foundations and references philosophical terms and ideas, including concepts from Nishida–Suetsuna's philosophy, such as "intuition" and "self-reflection".

A significant term introduced in the paper is "infinite mind", a word coined by Gödel.[7] This term is crucial to understanding Takeuti's exploration of mathematical foundations and his philosophical perspective. According to Takeuti, "an infinite mind" refers to an abstract, idealized capacity that transcends human cognitive limitations. This infinite mind can examine and consider infinitary objects one by one. In this context, they include mathematical objects such as infinite sets or functions defined over such a set. In contrast, "a finite mind" describes the cognitive capacity of humans or beings with limited cognitive abilities. Human cognition is inherently finite, constrained by time, resources, and mental capacity. Consequently, a finite mind cannot individually assess or verify an infinite number of cases.

Furthermore, Takeuti's distinction between the infinite mind and the finite mind highlights the challenges posed by infinitary mathematics. It underscores the need to develop rigorous methods and foundational frameworks that bridge the gap between our finite cognitive abilities and the infinite nature of mathematical concepts and objects. This perspective invites reflection on the nature of mathematical cognition and its implications for the foundations of mathematics.

The following passages support our reading [26, p. 173]:

> the following problems are very important in foundations of mathematics:
> 1) To formulate function of infinite mind completely. [...] To formulate the function of infinite mind in a better way, or to find new axioms of set theory.
> 2) To justify the world of our infinite mind in the world of a finite mind. We want to confirm rationality of modern mathematics since the world of our

[7] Cf. [28, p.255].

mind is finite, and the world of modern mathematics is just conjecture about our infinite mind. For this, we need to sufficiently develop the mathematics of the world of finite mind. [...]
3) To formulate the function of the finite mind completely. [...] (Translated by the author and Andrew Arana).

The second and third clauses are particularly important for the purposes of this paper. The second clause outlines the activities inherent in proof theory, specifically the undertaking of consistency proofs for a formalized mathematics. Given the inherent limitations of the human mind, it becomes imperative to ascertain the rationality of modern mathematics, including areas such as set theory and analysis. The term "rationality," as derived from the translation of "合理性"(Gorisei), conveys the idea that the conceptual framework of the infinite mind must harmonize with the finite mind's understanding of the mathematical world. In other words, there exists a mechanism within the realm of the infinite mind that either aligns with or diverges from the machinery of the finite mind. It is crucial, however, that the former does not produce contradictions with the latter.

It is worth noting that Takeuti does not employ words with strongly connotative terms such as "justify". His aim is not to validate modern mathematics based on the most reliable or absolute position, as Hilbert might have sought. Instead, he focuses on elucidating the relationship between the infinite and finite minds. The utilization of the term "rationality" ("合理性"(Gorisei)) suggests the relative nature of the connection between the infinite mind and finite mind.

5 Conclusion

As discussed so far, Feferman called for the invention and development of more perspicuous methods to prove and refine Takeuti's consistency proof of Π^1_1-CA and related systems because he believed that "the details of the arguments for Takeuti's method were extremely difficult to follow". Consequently, he and other logicians in Schütte's school achieved results that reduced various impredicative principles to the corresponding theories of inductive definitions based on intuitionistic logic.

According to Takeuti, Feferman simplified the proof of Takeuti's results by adopting a theory of inductive definitions as a fundamental principle, viewing these as elegant improvements. In contrast, Takeuti criticized Feferman for using the theory in its general form, arguing that it was set-theoretic and therefore not suitable for a consistency proof. However, as shown in this paper, this criticism is not convincing. Feferman focused on a special case of definitions, specifically the case of accessibility:

$$A(P_A, x) \equiv A_0(x) \wedge \forall y[\langle y, x\rangle \in R \rightarrow P(x)].$$

The philosophical reason for this approach is that there is a direct picture of how the members of such sets are generated, allowing the two axioms to be recognized as correct.

Finally, we have argued that the conflict between Feferman and Takeuti reflects a deeper philosophical divergence regarding consistency proofs and proof-theoretic reductions. For Feferman, clarifying why a theory is considered more constructive compared to the theory being analyzed is always challenging, as it assumes that a more constructive theory must be more reliable. Conversely, Takeuti views formalism as open to further inquiry and refinement, rather than as a definitive and stable foundation. Takeuti's perspective allows for the exploration of various conceptual frameworks, such as the infinite mind, finite mind, and weak mind, and their interrelations. This dynamic and relativistic view of the foundations of mathematics contrasts with Feferman's more static approach.

References

[1] Ryota Akiyoshi. A note on Feferman and Takeuti: Their thoughts on predicativity and finitism. *Tetsugaku (Mita Philosophy)*, 153: 53–79, 2024.

[2] Ryota Akiyoshi and Andy Arana. Takeuti's proof-theory in the context of the kyoto school. *Jahrbuch für Philosophie das Tetsugaku-Ronso* 哲学論叢, 46: 1–17, 2019.

[3] Ryota Akiyoshi and Andy Arana. Takeuti No Seisosei Syomei Saihou (Takeuti's well-ordering proofs revisited). *Tetsugaku (Mita Philosophy)*, 146: 83–110, 2021.

[4] Toshiyasu Arai. An accessibility proof of ordinal diagrams in intuitionistic theories for iterated inductive definitions. *Tsukuba Journal of Mathematics*, 8(2): 209–218, 1984.

[5] Toshiyasu Arai. A sneak preview of proof theory of ordinals. *Annals of the Japan Association for Philosophy of Science*, 20: 29–47, 2012.

[6] Wilfried Buchholz. *Eine Erweiterung der Schnitteliminationsmethode*. Habilitationsschrift, Ludwig-Maximilians-Universität München, 1977.

[7] Wilfried Buchholz. The $\Omega_{\mu+1}$-rule. In Wilfried Buchholz, Solomon Feferman, Wolfram Pohlers, and Wilfried Sieg, editors, *Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies*, volume 897 of *Lecture Notes in Mathematics*, pages 188–233. Springer, Berlin, 1981.

[8] Wilfried Buchholz. Ordinal analysis of ID_ν. In Wilfried Buchholz, Solomon Feferman, Wolfram Pohlers, and Wilfried Sieg, editors, *Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies*, volume 897 of *Lecture Notes in Mathematics*, pages 234–260. Springer, Berlin, 1981.

[9] Wilfried Buchholz, Solomon Feferman, Wolfram Pohlers, and Wilfried Sieg. *Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies*, volume 897 of *Lecture Notes in Mathematics*. Springer, Berlin, 1981.

[10] Solomon Feferman. *In the Light of Logic*. Oxford University Press, Oxford, 1998.

[11] Solomon Feferman. Kurt Schütte's way. In Reinhard Kahle and Michael Rathjen, editors, *The Legacy of Kurt Schütte*, pages 21–35. Springer, Cham, 2020.

[12] Akiko Kino, John Myhill, and Richard E. Vesley, editors. *Intuitionism and Proof Theory: Proceedings of the Summer Conference at Buffalo N.Y., 1968*, volume 60 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, Amsterdam, 1970.

[13] Georg Kreisel. Generalized inductive definitions. In *Stanford Report on the Foundations of Analysis*. Stanford, 1963. mimeographed, Ch. III.

[14] Georg Kreisel. Review of On the Fundamental Conjecture of GLC VI. *Zentralblatt für Mathematik*, 106: 237–238, 1964.

[15] Wolfram Pohlers. Proof-theoretical analysis of ID_ν by the method of local predicativity. In Wilfried Buchholz, Solomon Feferman, Wolfram Pohlers, and Wilfried Sieg, editors, *Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies*, volume 897 of *Lecture Notes in Mathematics*, pages 261–357. Springer, Berlin, 1981.

[16] Michael Rathjen. Recent advances in ordinal analysis: Π^1_2-CA and related systems. *Bulletin of Symbolic Logic*, 1: 468–485, 1995.

[17] Michael Rathjen. The realm of ordinal analysis. In S. Barry Cooper and John K. Truss, editors, *Sets and Proofs: Invited Papers from Logic Colloquium '97, European Meeting of the Association for Symbolic Logic, Leeds, July 1997*, volume 258 of *London Mathematical Society Lecture Note Series*, pages 219–279. Cambridge University Press, Cambridge, 1999.

[18] Clifford Spector. Provably recursive functionals of analysis: a consistency proof

of analysis by an extension of principles in current intuitionistic mathematics. In F. D. E. Dekker, editor, *Recursive Function Theory: Proceedings of Symposia in Pure Mathematics*, pages 1–27, Providence, Rhode Island, 1962. American Mathematical Society.

[19] Gaisi Takeuti. Keishikisyugi No Tachiba Kara II (From the point of view of formalism II). *Kagakukisoron Kenkyu (Annals of the Japan Association for Philosophy of Science)*, 2(3): 295–299, 1956.

[20] Gaisi Takeuti. Ordinal diagrams. *Journal of the Mathematical Society of Japan*, 9(4): 386–394, 1957.

[21] Gaisi Takeuti. On the fundamental conjecture of GLC V. *Journal of the Mathematical Society of Japan*, 10(2): 121–134, 1958.

[22] Gaisi Takeuti. Ordinal diagrams II. *Journal of the Mathematical Society of Japan*, 12(4): 385–391, 1960.

[23] Gaisi Takeuti. On the fundamental conjecture of GLC VI. *Journal of the Mathematical Society of Japan*, 37(8): 440–443, 1961.

[24] Gaisi Takeuti. Consistency proofs of subsystems of classical analysis. *The Annals of Mathematics*, 86(2): 299–348, 1967.

[25] Gaisi Takeuti. *Suugaku Kisoron No Sekai (The World of Foundations of Mathematics)*. Nihon Hyouron Sya, 1972.

[26] Gaisi Takeuti. Suugaku Ni Tsuite (About mathematics). *Kagakukisoron Kenkyu (Annals of the Japan Association for Philosophy of Science)*, (4): 170–174, 1972.

[27] Gaisi Takeuti. *Proof Theory*, volume 81 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, Amsterdam, 1st edition, 1975.

[28] Gaisi Takeuti. Proof theory and set theory. *Synthese*, 62(2): 255–263, 1985.

[29] Gaisi Takeuti. *Proof Theory*, volume 81 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, Amsterdam, 2nd edition, 1987. Reissued by Dover Publications, 2013.

[30] Gaisi Takeuti and Mariko Yasugi. The ordinals of the systems of second order arithmetic with the provability Δ^1_2-comprehension axiom and with the Δ^1_2-comprehension axiom respectively. *Japanese Journal of Mathematics*, 41: 1–67, 1973.

[31] Aaron Thomas-Bolduc and Eamon Darnell. Takeuti's well-ordering proof: An accessible recontruction. *The Australasian Journal of Logic*, 19(1): 1–31, 2022.

[32] Mariko Yasugi. Syoumeiron No Omoide (Memories of proof theory: Focusing on consistency proofs). *Sugaku (Mathematics)*, 49: 292–296, 1997.