Title	Foundations of mathematics a la Kyoto school : from Nishida to Takeuti via Suetsuna
Sub Title	
Author	秋吉, 亮太(Akiyoshi, Ryota)
Publisher	三田哲學會
Publication year	2023
Jtitle	哲學 (Philosophy). No.151 (2023. 3) ,p.1- 36
JaLC DOI	
Abstract	In this article, we attempt to describe a conceptual thread among three interesting figures in the fields of philosophy, mathematics, and logic: Nishida, Suetsuna, and Takeuti. First, we briefly explain Nishida's thoughts in this field. Next, we review the mathematical philosophy of the well-known number theorist Suetsuna (1898–1970) influenced by Nishida's thoughts. Finally, We focus on Takeuti (1926–2017)'s mathematical philosophy, who is the most well-known Japanese logician in the field of proof-theory. By reviewing his thoughts and explaining them in terms of Nishida's philosophy, we indicate that an important source of Takeuti's philosophical ideas behind his foundations of mathematics should be Nishida's philosophy.
Notes	
Genre	Journal Article
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=AN00150430-00000151-0001

Foundations of Mathematics a la Kyoto School:

From Nishida to Takeuti via Suetsuna

Ryota Akiyoshi*, **

In this article, we attempt to describe a conceptual thread among three interesting figures in the fields of philosophy, mathematics, and logic: Nishida, Suetsuna, and Takeuti. First, we briefly explain Nishida's thoughts in this field. Next, we review the mathematical philosophy of the well-known number theorist Suetsuna (1898–1970) influenced by Nishida's thoughts. Finally, We focus on Takeuti (1926–2017)'s mathematical philosophy, who is the most well-known Japanese logician in the field of proof-theory. By reviewing his thoughts and explaining them in terms of Nishida's philosophy, we indicate that an important source of Takeuti's philosophical ideas behind his foundations of mathematics should be Nishida's philosophy.

1 Introduction

In this article, we attempt to describe a conceptual thread among three interesting figures in the fields of philosophy, mathematics, and logic: Nishida, Suetsuna, and Takeuti. One might wonder if the content of this paper might be highly non-usual or peculiar because Gaisi Takeuti has been chiefly known as a mathematician rather than as a philosopher. However, as we will see below, it is natural to explain Takeuti's mathematical philosophy in this context.

As is well-known, Kitaro Nishida (1870–1945) is the founder of the Kyoto school, who was the most influential Japanese philosopher of the twentieth century.

---

* Keio University, Kyoto University, Waseda Institute for Advanced Study.

** The author is partially supported by KAKENHI 19K00022. We would like to thank the anonymous referee for careful readings, many helpful comments, and suggestions.

In particular, he established the discipline of philosophy as practiced in the West by criticizing philosophers such as Kant, Hegel, and James, and provided a possible direction for the practice of philosophy in the East. His book *Zen no Kenkyu* (善の研究, *Inquiry into the Good*) is especially famous for arguing that the most basic element in philosophy must be Junsui Keiken (純粋経験，pure experience). This idea was developed during his entire career as a philosopher and has been examined by many successors such as Satomi Takahashi (高橋里美) and Hajime Tanabe (田辺元) of the Kyoto school. Their works would give the impression that Nishida's main contribution should be only in the field of philosophy of religion so that this should have a sharp contrast with philosophers in the West such as Plato, Aristotles, Decartes, Leibniz, and Kant since Nishida's philosophy is based on some key notions in Buddhism such as Zen.

However, if we read Nishida's other texts in more detail, it turns out that this is not the case; for example, his early paper "Ronri no Rikai to Suuri no Rikai" (論理の理解と数理の理解，"Understanding in Logic and in Mathematics") originally published in 1912 (republished in NKI[1]) contains his reflections on Frege-Russell's logicism, Poincaré's philosophical thinking about infinity, and Dedekind's self-representative system. Another evidence supporting this point could be his later paper "Suugaku no Testugaku-teki Kisozuke" (数学の哲学的基礎づけ，Philosophical Foundations of Mathematics) published in his last year (republished in NKXI), which deals with some key questions in the foundations of mathematics, such as the conceptual priority between ordinal and cardinal numbers; moreover, he attempts to give detailed philosophical analyses of concrete mathematical notions, such as set, group, and ring.

Though Nishida's successors were philosophers without much background of mathematics, it was Joichi Suetsuna (1898–1970) who was an internationally well-known number theorist and adopted Nishida's philosophy to answer key questions in the foundations of mathematics at that time. Suetsuna met Nishida for the first time in 1943 and quickly learned Nishida's philosophy. In his paper "Ronti to Ningen Sonzai" (論理と人間存在, Logic and Human existence) published in 1946 [21], Suetsuna compares Western logical philosophy and Eastern logical philosophy. After reviewing philosophical thoughts by Plato, Aristotle, Hegel, Leibniz, Dignāga (陳那), and Xuanzang (玄奘), Suetsuna says that it is high time to build our own culture in Japan. Here, we should remind readers of this paper that some intellectual people in Japan were looking for a new direction to take after the World War II. According to Suetsuna, Nishida's philosophy should be a very reliable guide for developing our own culture. As we will see below, Suetsuna adopted Nishida's philosophy to answer questions in the foundations of mathematics, such as the conceptual order and the priority of ordinal and cardinal numbers.

Gaisi Takeuti (1926–2017) is the most well-known Japanese logician for his works in many fields of logic, such as proof-theory, set-theory, and quantum logic. In particular, he proposed "Takeuti's conjecture" in 1953 and obtained several partial but stunning solutions for it in the 1950–60s. He was invited by Kurt Gödel into the Institute for Advanced Study in Princeton in 1959–60, 66–68, and 71–72. Though he was chiefly known as a mathematician, he wrote many papers in English and Japanese to explain his philosophical ideas about the foundations of mathematics. As shown in [1], Takeuti used several key notions in Nishida's philosophy such as "active intuition" and "self-reflection", which indicates that his view about the foundations of mathematics was influenced by it. The source of such notions for him was Suetsuna, as pointed out by himself[2], which had been used throughout his career. The surprising novelty in his thinking is that his view of the foundations of mathematics is influenced by Japanese philosophy and his mathematical work would be obtained based on such a view. This point contrasts sharply with any philosophers in Kyoto school who worked on mathematical philosophy and with mathematical logicians who had a keen interest in it. Perhaps a rare, comparable example should be Kurt Gödel.

This paper is structured in the following way. First, we briefly explain Nishida's thoughts in this field in Section 2. Next, we review the well-known number theorist Suetsuna's mathematical philosophy in Section 3. As we will see, he started to develop his mathematical philosophy after meeting with Nishida. Finally, we focus on Takeuti's mathematical philosophy in Section 4, who is the most well-known Japanese logician in the field of proof-theory. By reviewing his thoughts and explaining them in terms of Nishida's philosophy, we indicate a previously unknown thread from Nishida to Takeuti via Suetsuna.

2 Nishida's mathematical philosophy

In this section, we explain some key elements of Nishida's mathematical philosophy. His interests in logic and mathematics were retained throughout his life. It would be instructive to note that Nishida was a mathematically talented student when he was a student of Hojo, who was a professor of mathematics of Dai-shi Kōtō Gakkō (第四高等学校, the Fourth Higher School).

In view of the contents of the next sections, we focus on two articles. The first one is "Ronri no Rikai to Suuri no Rikai" (論理の理解と数理の理解, "Understanding in Logic and in Mathematics") originally published in 1912 (republished in NKI), which contains his thoughts about Frege-Russell's logicism, Poincaré's philosophical thinking about infinity, and Dedekind's self-representative system. In this paper, he mentions a key notion in his philosophy, that is, Jikaku (自覚, self-awareness). The importance of this paper is observed by the fact that Nishida himself notes at the end of it that his next book *Jikaku ni okeru Chokkan to Hansei* (自覚に於ける直観と反省, *Intuition and Reflection in Self-Consciousness*) is a kind of successor of this paper. The second paper focused on this article is "Suugaku no Testugaku-teki Kisozuke" (数学の哲学的基礎づけ, Philosophical Foundations of Mathematics) published in his last year (republished in NKXI) in which he addresses several problems in the foundations of mathematics. For example, he deals with the issue of the conceptual priority between an ordinal number and a cardinal number, and with the identity problem regarding a set and its elements in the set theory. An important fact in the paper is that he used not only the notion of Jikaku but also another very important one of "Jikomujun-teki Jikodouitsu" 「自己矛盾的自己同一」(absolute inconsistent self-identity). As we will see in the next section, Suetsuna's mathematical philosophy can be regarded as a further development of this kind of direction.

In the paper "Ronri no Rikai to Suuri no Rikai", Nishida characterizes an important essence of logical thinking in the following way:

> I would think that all logical thinkings are internal developments of a something general, that is, a kind of creative power. Here, "general" does not mean the standard meaning of something abstract in general, but a kind of internal creative power. [NKI, p.251]

He also remarks that such a thing cannot be an object of thinking. If we use the notion of self-awareness here, then it would mean that such something general cannot be an object of the act of self-awareness. He also notes that the essence of logical thinking is to make it possible to infer a particular proposition from general ones.

After discussing logical thinking, he continues to clarify the notion of mathematical thinking. For this he first explains the context of mathematical philosophy regarding numbers. For example, he mentions Frege-Russell's logic which aimed at deriving all mathematics from pure logic. In more detail, it must be shown that all definitions of arithmetic such as $0, 1 \ldots$ are definable in terms of his (higher-order) logic, and all axioms of Peano arithmetic are provable from such definitions and logical inferences. However, Poincaré opposed this project and claimed that some principles, such as mathematical induction, cannot be logical because such a principle makes it possible to infer a general proposition from particular ones. Nishida writes that this principle incorporates a kind of power of creation (intuition) by which we realize the fact that it is possible to iterate one process as an object indefinitely. Nishida explains relevant parts of Rickert's "Das Eine, die Einheit und die Eins" published in the previous year and asks a key question "How is it possible to obtain the notion of number from purely logical concepts?".

Here, it would be interesting to see Nishida's remarks about Rickert and Poincaré. According to Rickert, "das Eine" is a fundamental object of pure logical thinking and it is something completely unspecified; moreover, another object called "das Anderes" is logically required. Finally, "die Einheit des Einen und des Anderen" is considered to obtain a kind of unity of das Eine and das Anderes. These are purely logical objects, and all other things are non-logical. To obtain the equation $1 = 1$, we must be able to freely change the places of das Eine and das Anderes by dropping the dichotomy between them. Rickert points out that the notion of "homogenes Medium" is necessary for obtaining such a free change. This "homogenes Medium" is non-logical, therefore the concept of number cannot be purely logical. Nishida sees the similarity between this notion "Homogenität" and Poincaré's basic entity, which makes it possible to iterate a process as an object indefinitely and lies behind mathematical induction.

Nishida compares mathematical thinking with logical one and says the following:

> Something really dynamic generally incorporates the development of infinity inside it. "Productive imagination" (productive Einbildungskraft) is rather the nature of thinking, hence we could say that mathematics expresses this point in a clearer way than the standard formal logic. [NKI, pp. 259–260]

Now, the question "How is it possible to obtain the notion of number from purely logical concepts?" is rephrased in the view of Nishida's remark above as follows: How is it possible to obtain the notion of Homogenität and from purely logical concepts and how to iterate such a thing if there is? According to Nishida, Hegel's argument about "das Unendliche" provides a clue to this problem. An infinity has its own motivation (or internal creative power) for its dialectical development[3]. Finally, Nishida discusses Dedekind's essay "Was sind und was sollen die Zahlen?" (1888), in which a system is called infinite if it is "similar" (isomorphic) to its proper part. Dedekind appeals to the notion of "my own realm of thoughts" (meine Gedankenwelt) to prove the existence of an infinite system by considering self-reflection on such a thought. Inspired by this, Nishida says that infinite series can be derived from infinite productivity in our thinking by self-reflection. In his term, we read:

> In other words, a purely logical object, that is, the consciousness of something without any content, that is, the self-representation system of abstract thinking,

> forms a series of numbers. [NKI, p.266]

Let us summarize Nishida's point in his "Ronri no Rikai to Suuri no Rikai" whose main question is how to obtain infinity from purely logical concepts. His answer is that it can be derived from infinite productivity in our thinking through self-reflection, and the infinite series obtained by this can be regarded as a production of our thinking.

Next, we focus on Nishida's later paper "Philosophical foundations of mathematics" (NKXI [16]) in which several examples from the set theory, elementary number theory, and modern algebra such as group, ring, and field are analyzed in his philosophical framework. In these analyses, he explicitly uses important keywords such as "Jikomujun-teki Jikodouitsu", "Jikaku" (自覚, self-awareness), and "Kouiteki Chokkan" (行為的直観, active intuition).

He starts by describing his view of a number. Because this point is important for the next section, we quote his phrase.

> A number reflects itself in it and a symbolic form of the self-expression of a concrete reality, that is, it is a symbolic form of my Basho-logic[4]. Formal logic is just an abstract form of it. [NKXI, 238]

Here the word "formal logic" means a formalized system of valid inferences. Hence, according to Nishida, the method of formalization of mathematics is not sufficient for the foundations of mathematics, and it is necessary to give a proper foundation using his own notions such as Jikomujun-teki Jikodouitsu, Jikaku, and Kouiteki Chokkan. After the quotation above, he explains the general principle of Basho-logic.

> Each individual expresses one as a whole together with that each individual becomes the self-expression of one as a whole. In this system of Mujun-teki Jikodouitsu of many with one, the relationship as a whole and one among individual many, that is, the whole, moves from something determined to

something to determine. [NKXI, 238]

Nishida calls this movement as self-limitation of form (形の自己限定, Katachi no Jiko Gentei). Moreover, this is called Jikaku-teki. A typical example of this movement is Dedekind's the self-representation system of abstract thinking mentioned above. Although many examples are analysesed in that paper, we focus only on several examples in the paper because his main ideas should be clear for readers.

His first concrete example is from set theory[5]. A basic relation between sets A and B is called "equivalence" (A is reflected in B in his terminology) if and only if there is a bijective map from A to B. From a modern point of view, this is a standard definition of equivalence from the viewpoint of cardinality. Moreover, Nishida states that a set A is included in B if there is such an equivalence between A and the (proper) part of B. In a modern term, the cardinality of A is called smaller than the cardinality B. Consider two different sets A and B now. If the two sets A and B are equivalent, then they are regarded as "one as a whole of individual many" because they are the same in the sense of being equivalent and they are still different sets. This is one example of his Jikomujun-teki Jikodouitsu. He asks the origin of this relation and says that it should go back to the notion of Jikaku. This word reminds us of the discussion in the previous part of this section about "Ronri no Rikai to Suuri no Rikai".

Nishida clearly believes that this kind of reflection is central to the foundation of set theory. This opinion is to be found in his claim that Hausdorff's view of a set is clear enough from his point of view. According to Nishida, Hausdorff claims that the ability of our thinking to grasp individual elements (as many) as a set (as one) gives a foundation of the notion of set. The author of this paper is not sure which paper by Hausdorff Nishida had in his mind, but his intention is that the foundations of number must be based on (or explained by) his notions such as Jikomujun-teki Jikodouitsu.

In this paper, Nishida uses the notions of time and space which are two opposite elements in his Basho or Gutaiteki Ippansya (具体的一般者, something concrete and general). Nishida characterizes time as Ta no Jikohitei-teki Ichi (多の自己否定的一, one of self-denying many) and space as Ichi no Jikohitei-teki Ta (一の自己否定的多, many of self-denying one). He explained several notions using these terms. For example, let us consider the principle of arithmetic addition. He characterizes it as Kuukan-teki Tatiba ni okeru Ko to Zen tono Jiko Douitu-teki Katei (空間的立場における個と全との矛盾的自己同一的過程, the process of inconsistent self-identity of individual and whole in space). Since he does not give a concrete example of this, we explain this in the following way. Let us consider a simple equation, $1+1=2$. To obtain this, we first need to consider 1, and add (or count) another 1. However, we need to represent them as a whole to see 2. Nishida uses the words “count” and “time” for the second step and another one “space” for the third step. The point here is that the notion of space is necessary to represent the results obtained by counting them in time. Hence, Nishida characterizes this as the process of the inconsistent self-identity of individuals and the whole in space.

The explanation above for the case of addition could be natural if we adopt Nishida’s framework, but it is indeed very different from the modern debate on the foundations of mathematics such as Hilbert’s finitism. This point should become clearer if we examine his view of the operation of multiplication. He characterizes it as Jikan-teki Tatiba niokeru Ko to Zen tono Jiko Douitu-teki Katei (時間的立場における個と全との矛盾的自己同一的過程, the process of inconsistent self-identity of individual and whole in time)[6]. According to another explanation, it is the result of a self-development of a whole as a negative individual (全の個否定的自己表現). Again, we explain this using our own simple example, $2\times 2=4$ since Nishida does not provide such an example. To see the meaning of this equation, we first need to have 2 as an individual. Then, we should place 2 twice side-by-side in space. By counting these, we obtain 4 (in time). In the last crucial step, we need to appeal to the notion of time; hence, the operation of the multiplication is characterized as the process of inconsistent self-identity of individual and whole in time.

This view of multiplication may seem peculiar from the standard viewpoint of the foundations of mathematics. Now we briefly compare Nishida with a major figure in this context, that is, Hilbert’s formalism. When Hilbert considered the foundations of mathematics, he aimed to prove the consistency of formalized mathematics in a finitistic standpoint where only finite objects and basic operations defined on them are permitted. According to Hilbert, finitary reasoning is the fundamental basis for all scientific thought. The following passage is from "Die Grundlegen der elementaren Zahlenlehre":

> This is the fundamental mode of thought that I hold necessary for mathematics and for all scientific thought, understanding, and communication, without which mental activity is not possible at all. ([11, p.486], [15, p. 267] in English).

As you see, Hilbert was clearly influenced by Kant and used the notion of intuition for its basis. In this standpoint, a number such as 2 means strokes $||$. Then, the operation of the addition is explained by adding more stokes. For example, $2+1$ indicates the result of adding one more stroke next to $||$. The operation of the multiplication is explained in a similar way[7]. In contrast, Nishida explains the operation of the multiplication differently from the operation of the addition.

Using his own view of addition and multiplication, Nishida explains more examples from algebra, such as group, ring, and field. In this part, he often repeats and explains the definition of group, ring, identity element, etc. For example, he regards identity elements in groups as the focus of the power of self-reflection or self-restriction. Sometimes, he uses the word "active intuition" to denote this kind of power. He thinks that this kind of philosophical explanation is necessary for the foundations of mathematics, and the axiomatic system of mathematics is just an abstract form of these. In some sense, this kind of direction would be similar to intuitionists such as Poincaré and Brouwer, since they claimed that mathematics must be founded on mathematical intuition. However, according to Nishida, the notion of intuition in his philosophy is much more general than theirs.

Before closing this section, we give another remark on Nishida's thoughts in the context of the foundations of mathematics. Nishida wrote another paper "Ronri to Suuri" (論理と数理, Logic and Mathematics) on this topic in 1944 in which he presented his view about Brouwer, who was a major figure in the development of the foundations of mathematics. His position is called intuitionism, in which all mathematics must be based on intuition of time. Nishida quotes a passage on p. 86 of Brouwer's paper "Intuitionism and Formalism" published in 1913 [3] :

> Finally this basal intuition of mathematics, in which connected and separate, the continuous and the discrete are united, gives rise immediately to the intuition of the linear continuum, i.e., of the "between", which is not exhaustible by the interposition of new units and, therefore, can never be thought of as mere collection of units.

Nishida finds in this passage his notion of Jikomujun-teki Jikodouitsu (self-inconsistent identity) and says that the smallest infinite ordinal ω would be the limit of this position and claim from his philosophical position that a number is not only an ordinal one, but cardinal one as well in view of Kouiteki Chokkan (active intuition). The first remark is that Nishida's claim that ω is the limit of intuitionism is incorrect because Brouwer developed a general intuitionistic set theory in which a much bigger ordinal is definable. For Brouwer, a limit ordinal is definable in his standpoint if it is defined as the limit of some operations that are permitted in it[8]. However, The author of this paper believes that this would be due to the limitation of the resource that Nishida could obtain. The point that ω is regarded as Jikomujun-teki Jikodouitsu clearly influenced Suetsuna, whose mathematical philosophy is the topic of the next section. Indeed, as we will see below, Suetsuna generalized this point into much larger infinite ordinals.

3 Suetsuna's mathematical philosophy

Joichi Suetsuna is a well-known mathematician mainly working on number theory. He was born in Oita (Kyushu) on November 28,1898 and studied number theory in the direction of Teiji Takagi (1875–1960), who was a student of Hilbert in Göttingen. After graduating from the Department of Mathematics at Tokyo University in 1922, he became a lecturer at Kyushu University in the same year.

Later, he became a professor of mathematics at Tokyo University in 1935 and retired in 1959.

Not only his works in number theory are internationally well-known, but his book *Kaisekiteki Seisuu Ron* (解析的整数論, *Analytic Number Theory*) is regarded as an excellent textbook in this field. After his passing away, the collected works *Suetsuna Joichi Tyosaku Syuu* (末綱如一著作集, *Suetsuna Joichi Collected Works*) were published in 1989. Moreover, he wrote several articles and books on the topics of foundations of mathematics and Buddhist philosophy such as *Ronri To Suuri* (論理と数理, *Logic and Mathematics*, 1947), *Suugaku To Suugakushi* (数学と数学史, *Mathematics and the History of Mathematics*, 1944), *Suugaku no Kiso* (数学の基礎, *The Foundations of Mathematics*) in 1952, and even a book on Kegon in 1957 under the title *Kegonsyu No Sekai* (華厳宗の世界, *The World of Kegon*). As we see below, his view of the foundations of mathematics had been developed under the strong influence of Nishida.

Suetsuna met Nishida for the first time in 1943, as pointed out by Takahashi [25]. According to him, Torataro Shimomura in Kyoto School played an important role in the first meeting for them. During this period, Suetsuna had a strong interest in the foundations of mathematics. Simomura reported that he quickly learned the main ideas of Nishida's philosophy. If we remember that his first paper "Yuugen no Tachiba to Kyokugen Gainen" (有限の立場と極限概念, the finitist standpoint and the notion of limit) [20] and the book *Suugaku To Suugakushi* (数学と数学史, *Mathematics and the History of Mathematics*) were published in 1944, it is highly plausible that Suetsuna began to study Nishida's philosophy after the meeting. Their interaction continued until 1944, one year before Nishida's passing away. After this end of the interaction with Nishida, Suetsuna started another interaction with Daisetz Suzuki, which was an important starting point for his study of Kegon.

We can find Nishida's crucial influence on Suetsuna in general in his paper "Ronti to Ningen Sonzai" (論理と人間存在, Logic and Human existence) published in 1946 [21]. In this paper, Suetsuna compares Western philosophers with Eastern philosophers. For example, he explains the thoughts by Plato, Aristotle, Hegel, and Leibniz in the former. He also explains thoughts by Dignāga (陳那), Xuanzang (玄奘) in the latter. After reviewing these philosophical thoughts, Suetsuna says that it is time to build our own culture in Japan. Here, we should remind readers of this paper that some intellectual people were looking for a new direction to take since the end of the World War II. According to Suetsuna, Nishida's philosophy should be a very reliable guide for developing their own cultures. He even says "We stick to this and develop it so that we have to aim to build a new culture of Japan. There is no doubt here".

In what follows, we explain Suetsuna's basic idea of his mathematical philosophy and the crucial influence of Nishida. First, we focus on his paper *Suugaku No Kiso* (数学の基礎, *The Foundations of Mathematics*) published in 1944 [19] after his first meeting with Nishida in 1943 because this paper already contains essential ideas in his mathematical philosophy. He begins with explaining the situation of the foundations of mathematics, such as the Burali-Forti paradox in set theory and mentions two types of solutions to this issue; (1) Brouwer's intuitionism, and (2) Hilbert's formalism. In the former approach, the basis of mathematics is the intuition to grasp a mathematical object. For example, the set of natural numbers is permitted in this standpoint because it is regarded as potentially infinite constructed by the successor operation $x \mapsto x+1$, which is permitted from this standpoint. On the other hand, we must give up the law of the excluded middle $\forall x A(x) \vee \neg \forall x A(x)$ in general because we cannot check it when x ranges over an infinite set. Therefore, some important parts of mathematics should be abandoned in this approach. In the latter approach, to save the entirety of mathematics, it is sufficient to prove the consistency of formalized mathematics. However, the problem of the consistency of analysis (the theory of real numbers) is quite difficult and remains open.

After summarizing the controversy in the foundations of mathematics, Suetsuna points out that the basis of our recognition is based on active intuition, which is a key word in Nishida's philosophy.

> There exists an active intuition of us as a recognition subject underlying our recognition, as claimed by Nishida repeatedly. [19, p.9][9]

He refers to Nishida's paper "Ronri To Suuri" (論理と数理, *Logic and Mathematics*) published in 1944[10] here and claims that the problem of the foundations of mathematics remains unsolved unless we clarify the role of active intuition on which mathematics is based.

In particular, we grasp infinity by the role of active intuition. He says

> Active intuition in Nishida's philosophy is creative and formative, and it makes it possible to grasp infinity. [19, pp. 9–10]

Suetsuna takes a set of natural numbers and one of the real numbers as examples. In the former case, there is no problem in grasping it because it is countable. The point of being countable is that we can grasp each element of the set as a natural number so that each element can be regarded as "determined as an individual". On the other hand, how about the latter case? According to Suetsuna, the notion of a set is extended considerably into uncountable cases so that several paradoxes are discovered. Hence, we should examine the foundation of the theory of real numbers. Having the collection of real numbers in his mind, he calls the whole「全体」(Zentai) as "Basho"「場所」(Place) if some individuals form a whole and each one is independent and particular so that many as one and one as many form "Zettai-teki Jikomujun-teki Jikodouitsu"「絶対的自己矛盾的自己同一」(absolute inconsistent self-identity)[11].

According to Suetsuna, many issues in the foundations of mathematics are explained in terms of Nishida's words, such as Zettai-teki Jikomujun-teki Jikodouitsu. For example, he explains the notion of the natural numbers, which has been considered as the basis of all other mathematics. To explain it, the successor operation $x \mapsto x+1$ is crucial. Supposing 1 as given, the number 2 is given by this operation. We continue this operation further to obtain: $2, 3, 4, \ldots$. This process is clearly based on the intuition of time, as it is needed for counting something. However, Suetsuna says that this is not enough to make a new number, say 2 from the original one, say 1. We need to intuit the result of adding 1 to the previous number, say 1 in the framework of space. He says that these contradictory functions of intuition are included in active intuition. Moreover, he continues that here is one example of Zettai-teki Jikomujun-teki Jikodouitsu between an ordinal number and a cardinal number. He also points out that Brouwer's notion of intuition is just one aspect of active intuition.

After considering the theory of natural numbers, Suetsuna continues to analyze other issues in the foundations of mathematics. In particular, he says "The most important issue when we consider natural numbers from the viewpoint of mathematics is that all of them form an infinite set". In the formalist foundation, we show only the consistency of a formalized system of natural numbers[12]. However, he thinks that this is not enough from a philosophical viewpoint. In this context, he mentions Skolem's result in [18] that any axiomatic system has a model in which its domain has a larger cardinality than that of a set of natural numbers if it has an infinite model. Therefore, the consistency of an axiomatic system is not enough for characterizing a model of it in a unique way[13].

Furthermore, Suetsuna explains how to define integers, rational numbers, real numbers from natural numbers in the standard way, and points out many examples of active intuition. For example, the fraction numbers can be considered as follows. Let us consider a collection whose number of elements is n. Because the collection as a whole can be considered as 1, each element is regarded as $1/n$. Hence, we can consider the totality of all fraction numbers, which is regarded as Basho. Here, we remark Suetsuna's careful terminology. He correctly says that such a totality is also a set (「集合」) meaning that the existence of this is provable in a standard set-theory. Moreover, he says that the natural number n and the fraction number $n/1$ are Jikomujun-teki Jikodouitsu. In addition, he says that they are the same if they are considered from the viewpoint of space, that is, as quantity, although they are distinct if they are considered from the viewpoint of time, that is, from the viewpoint of the process of concept formation. As we explained in the previous section, Nishida provides a similar analysis of the cases of the set theory and the theory of natural numbers. Here, we note that Suetsuna provides a more detailed analysis of mathematical objects than Nishida, and the former's arguments can be considered as further developments of the latter's basic ideas. Moreover, Suetsuna explains the theory of real numbers as follows: To explain real numbers, we need to explain infinite decimals because the former can be defined in terms of the latter. If we consider the sequence $s_1, s_2, \ldots$ converging to a certain limit, we can intuit the limit by the infinite process of the sequence. Finally, this limit is said to be Jikomujun-teki Jikodouitsu with the corresponding convergent sequence.

Before going to another paper, it might be interesting to see Suetsuna's criticism of Brouwer's theory of choice sequences, which is a key method for developing intuitionistic analysis. A choice sequence is an infinite sequence of rational numbers, such as $s_1, s_2, \ldots$. The notions of convergence and of limit are defined in the standard way. To develop the theory of choice sequence, the notion of "free" choice sequence must be introduced because there are so many such sequences, that is, some choice sequences must be lawless in the sense that there is no rule to determine its value s_m. Suetsuna says that Brouwer's notion of free choice sequences is contradictory, but this opinion could not be quite correct from a modern viewpoint. More precisely, there is no satisfying axiomatization of that theory even today, while several mathematical models using sheaf or category theory are known[14]. Here, it would be interesting to note that Suetsuna does not mention a principle called continuity axiom in his writings, which is essential for developing the theory of choice sequences. To explain this principle in detail is beyond the scope of this paper, but we remark that it is key to understanding intuitionistic analysis and it says that only the information of finite segments of a sequence $s_1, s_2, \ldots$ is sufficient for knowing a property about the whole sequence. In the author's opinion, this is quite natural since it was found after several logicians' serious efforts in the 1950–60s[15].

In another paper "Yuugen no Tachiba to Kyokugen Gainen" (有限の立場と極限概念, the finitist standpoint and the notion of limit) published in 1944 [20], he applies his schema to transfinite ordinals up to ε_0, which was used by Gentzen to prove the consistency of arithmetic (1936), and claims that it provides a foundation for this kind of result.

Before going to Suetsuna's argument for this ordinal, let us explain it in a standard way. If we consider a sequence of natural numbers,

$$0, 1, 2, \ldots$$

We can consider the limit of this sequence, that is, ω. Then, we can consider another sequence

$$\omega+1, \omega+2, \omega+3, \ldots$$

and the limit $\omega+\omega$. This ordinal is denoted by $\omega \times 2$. If we continue this argument, we obtain $\omega \times \omega$, that is, ω^2. Moreover, we continue this argument to obtain ω^ω. If we generalize this argument more, then we get a sequence

$$\omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \ldots$$

The (least) limit of this ordinal is called ε_0.

Suetsuna explains the process of obtaining the limit using the notions of active intuition and Jikomujun-teki Jikodouitsu. For example, a sequence of natural numbers $a_0, a_1, \ldots$ is incomplete from the viewpoint of time because we can continue it forever. Hence, we need to grasp it as a whole to obtain ω using the viewpoint of space. Active intuition including time and space makes it possible to grasp the sequence as a whole, and the sequence and the limit are identical in the sense of Jikomujun-teki Jikodouitsu.

Here, we remark on this argument in the context of the foundations of mathematics. As Suetsuna says, his schema provides a foundation for the ordinals up to ε_0; hence, one of Gentzen's theory of ordinals in his consistency proof. However, while Gentzen used ordinal *notations* up to ε_0, Suetsuna's schema applies to ordinals in set theory. Gentzen's ordinals are very different from such ordinals because ordinals in finite standpoint must be *finite*; hence, ordinals are defined as notations. For example, the ordering relation $<_s$ between ordinals in set theory is not decidable, but the ordering relation $<_f$ between them in a finite standpoint must be decidable. This is based on the idea that only finite objects and finite operations are permitted from a finite standpoint. In the case of ε_0, this difference seems unessential, but if we consider larger ordinals such as ordinal diagrams invented by Takeuti, the situation can be completely different. Therefore, it would be fruitful to examine Suetsuna's interesting idea for the case of larger ordinals.

Next, we focus on Suetsuna's book *Suugaku no Kiso*, published in 1952 [23]. This book consists of seven chapters, which are based on his papers published before and his lecture on calculus at the Department of Mathematics at Tokyo

University. We focus on the first chapter "The tasks of the foundations of mathematics." Since this book is based on his paper, the content of the chapter overlaps with those of his papers, in particular with one of *Suugaku no Kiso* explained above. In addition, the content of the chapter is more general than that of other papers, such as *Suugaku no Kiso*. Hence, we explain some basic things in this chapter and peculiar ones to it.

Before going to the modern debate on the foundations of mathematics, Suetsuna quickly reviews the history of mathematics from medieval mathematics to modern mathematics. He mentions several mathematicians in Europe, such as Vieta, Decartes, Newton, Euler, Gauss, and Cauchy. After explaining the arithmetization of analysis, Cantor's set theory and its difficulties, such as Burari-Forti's paradox and Russell's paradox, are explained. Frege–Russell's logic is also mentioned in this context[16].

Suetsuna says that there are two basic positions in the foundations of mathematics: Hilbert's formalism and Brouwer's intuitionism. Regarding formalism, he says that Hilbert's *Grundlagen der Geometrie* [10] aimed to complete the Euclidian method to formalize geometry. After formalizing geometry using symbolic logic as a fundamental tool, he showed that the consistency of geometry is reducible into one of analysis. Hilbert's program aimed to prove the consistency of mathematics, such as analysis or the theory of natural numbers in finite standpoint. After Gödel's theorems which showed that it is impossible to carry out the program in its original form, Gentzen finally proved the consistency of arithmetic in 1936 [8]. As to the formalist foundation, Suetsuna remarks his criticism that a formalist cannot explain why mathematics is applicable to natural sciences such as physics or chemistry.

Next, Suetsuna explains basic ideas in Brouwer's intuitionism and highly evaluates Brouwer's emphasis on the notion of intuition in mathematics. He also explains Brouwer's criticism of the law of the excluded middle $A \vee \neg A$ and his theory of choice sequences to develop analysis in this standpoint. According to Suetsuna, formalism is unsatisfactory because it ignores a subjective aspect of mathematics and intuitionism correctly points of its importance in mathematics. After briefly explaining Brouwer's important theories of choice sequences for developing analysis, he points out that intuitionism does not provide a foundation for classical mathematics in a proper way. Suetsuna seems to think that the range of intuitionistic mathematics is not enough for covering classical one since the former does not permit the law of the excluded middle. In the author's opinion, this criticism is not very satisfactory because intuitionistic mathematics is not just a subsystem of classical one[17].

Another important point in this paper is his claim that mathematical understanding must be based on our Koui (行為, action). He mentions Nishida's philosophy in this context and says that understanding in general must be based on the Mujun-teki Jikodouitsu (with active intuition) of time and space. Moreover, the understanding has two different aspects: Hataraku (働く, working) and Seeing (見る, seeing). We would like to point out first that both concepts clearly come from Nishida's philosophy. According to Suetsuna, the notion of time is on the side of the former, and the notion of space is on the latter. Moreover, time has functions such as "individualization," "subjectification," "meaning-holding," while space has functions such as "generalization", "objectification", "concept-fixation". Here is not the place to investigate or examine these notions, but let us briefly explain the points. For example, the number as ordinal is explained in the former framework because we count $1, 2, 3$ in time, and this action is done by some individual, say the author of this paper. Also, since this experience is my own, it is subjective. On the other hand, when we see the limit of a sequence $1, 2, 3, \ldots$, we need the notion of space in order to represent them once. Therefore, infinite ordinal and cardinal should be explained in terms of the notions of time and space.

Before closing this section, we provide a remark on the topic of the next section. Suetsuna gives several attempts to solve difficulties. In the foundations of mathematics, however, he usually uses relatively elementary examples from the field. Therefore, the author of this paper is not very sure whether Suetsuna's mathematical results are essentially connected with or based on his philosophical reflections.

4 Takeuti's mathematical philosophy

In this section, we describe Takeuti's mathematical philosophy and indicates some connection between it and the philosophy of Kyoto School. He was born in Kanazawa in 1926 and studied mathematics at University Tokyo until 1947. After this, he became a professor at Tokyo University of Education in 1962. He had been a professor at University of Illinois, Urbana Champaign from 1966 until 1996. It was Kurt Gödel who invited him into Institute for Advanced Study in Princeton in 1959–60, 66–68, and 71–72.

4.1 Brief explanation of Takeuti's proof-theory

Before explaining Takeuti's works, let us explain the situation of the foundations of mathematics when he started his career. For doing that, we first need to explain Hilbert's program very briefly. When Hilbert worked on the foundation of mathematics, the main issue was how to justify the use of infinitary concepts in mathematics. He aimed to justify all mathematics by taking the following two steps:

1. To formalize mathematics by the method of symbolic logic,
2. To prove the consistency of it.

Here, the method used for the consistency of formalized mathematics must be carefully examined because it would become a circular argument if some infinitary concepts to be justified are used in the proof. Hilbert's insight is that if we formalize a field of mathematics, then inferences occurring in it can be expressed as finite strings of finite symbols:

$$\frac{A \quad A \to B}{B}$$

Moreover, Hilbert identified a part of mathematical reasoning that he judged to be secure that he called *finitary* mathematics, maintaining that it is "the fundamental mode of thought that I hold to be necessary for mathematics and for all scientific thought, understanding, and communication, and without which mental activity is not possible at all"[18]. This passage shows that such finitary mathematics is so basic in our mathematical practice; thus, it must be secure. Hence, the consistency proof of a field of mathematics must be proved from this standpoint. In 1931, Gödel proved his striking theorem called "incompleteness theorem," which showed that this project is unrealizable, but it was Gentzen who succeeded in proving the consistency of arithmetic with mathematical induction in 1936 [8]. Here, a reader might have the impression that Gödel's result seems to contradict Gentzen's. However, this is not the case since Gödel's result showed that Hilbert's program was unrealizable in its original form and his standpoint must be extended. For these reasons, Gentzen found a way to extend Hilbert's perspective.

Returning to Takeuti, his name is notably known as "Takeuti's conjecture" [26]. First, we briefly explain this and its philosophical interests. When Takeuti started his career in the 1940s, his main interest was already the foundation of mathematics. He asked his supervisor Shokichi Iyanaga (1906–2006) about a suitable reference. Although Iyanaga suggested that Takeuti should read Jacques Herbrand's paper, he could not find it in the university library. Following Iyanaga's advice, Takeuti started reading Gerhard Gentzen's monumental works in proof-theory. After this work, the main task was to prove the consistency of analysis. To develop analysis, we must deal with a set of natural numbers. For this reason, we need higher-order quantifiers $\forall X$ ("For all sets $X, \ldots$") and $\exists X$ ("There exists a set X such that...") in addition to the standard quantifiers $\forall x$ and $\exists x$ ranging over natural numbers. Takeuti succeeded in extending Gentzen's results to this kind of stronger subsystems of analysis in the 1950–1960s.

Takeuti has been known for a long time as a mathematical logician, but he also wrote many papers in Japanese where he presented his philosophical thoughts. As we will see below, his proof-theoretic project as mathematics is based on his philosophical standpoint. For this purpose, let us sketch Gentzen's method of cut-elimination to prove the consistency of a formalized theory T of mathematics ([8, 9]).

Suppose that d is any given derivation in T. We must prove the cut-elimination theorem stating that all cut rules are eliminated from d from which the consistency of it follows. There are two things to prove:

1. There is a transformation method $r(\cdots)$ on proofs, which eliminates a cut-rule considered.
2. If h is any derivation in T, then $r(h)$ is "simpler" than h.

Both steps must be considered simultaneously, and we need a natural number or sometimes an infinite ordinal number to express the complexity of h in Step 2. If we write this number as $o(h)$, then we must show that

$$o(h) > o(r(h))$$

where $>$ is an ordering relation that is defined on natural numbers or ordinals. Then, starting with a given derivation d, we can obtain the (natural or ordinal) sequence:

$$o(h) > o(r(h)) > o(o(r(h))), \dots$$

Hence, we can conclude that the step of eliminating a cut terminates if there is no infinite strictly decreasing sequence in such (natural or ordinal) numbers. Indeed, this point must be exactly beyond Hilbert's finitism, and its admissibility must be carefully examined. In other words, it must be examined what kind of philosophical position can be accepted as finitistic. This issue has been discussed in the literature and occurs more vividly in Takeuti's case because he proved the consistency of much stronger systems than Gentzen's one ([29, 31]).

4.2 Takeuti's philosophical passages

Now, we can ask the important question of this section: what is the essence of Takeuti's finitism or what is the major philosophical idea behind Takeuti's proof-theoretic program? As we will see, his philosophical ideas are based on, or at least going back to Nishida's ones.

First, we focus on Takeuti's following passage, which is somewhat surprising if we have Hilbert's conception of formalism in our mind.

> Objectification of thinking in mathematics via symbol has investigated proposition (set), logical concepts, proofs in mathematics in formalism. What is a significance of it? Or, does it have a significant influence on mathematics, similar to the context of the set? To answer these questions, it seems that formalism is too immature. This new gate of the new field is too difficult. Is formalist too premature baby? For me, the consistency problems seem a touchstone for formalism and the foundation of formalism rather than the foundation of mathematics. I believe the bright future of rocky formalism.

> [28, p.299]

This passage shows that Takeuti's conception of formalism is very different from Hilbert's one for the following reason: According to Hilbert, his finite standpoint constitutes the most common basis for scientific thinking. Hence, finitism is regarded as the most basic part of our thinking on which other mathematics are based. In other words, the final standpoint should be *the reliable foundation* of mathematics. Hilbert's use of the word *sicher* supports our reading of this point. On the contrary, Takeuti says that the problem of his consistency program would be a clue to the investigation of the conception of formalism. Hence, his formalism is not the ultimate basis for the foundation of mathematics, but it must be *developed and examined* through developments of this consistency program.

To see Kyoto school's influence on Takeuti's thought, we focus on another passage in the same paper.

> In the above example, I have frequently used the word "illusion," but indeed I hope the readers to replace it by "intuition". In this sense, I call the intuition for grasping a function or a set in general as active intentional intuition.
> Indeed, when we try to grasp a function or set, this active intention works. What would be thought of by this active intention, the object of this active intention, or the intention itself is called a function or set. [28, p.299]

The word "active intuition" is obviously taken from Nishida's philosophy, though Takeuti does not refer to Nishida's paper. He explicitly notes that that word is from Suetsuna[19]. If we read this passage carefully, his notion of intuition is very close to Nishida's notion in the following sense. In the standard terminology, intuition means some capacity or epistemological function of a subject to grasp an object in the world in a passive manner. Here, Takeuti's intuition means not only an active epistemological function of a subject, but the mathematical object itself grasped (or made) by intuition. This is a particular feature of Nishida's notion of intuition.

Now, we focus on the paper "About mathematics" published in 1972, where Takeuti explains his philosophical thinking in great detail. In this paper, he presents his view of his proof-theoretic project in the foundations of mathematics. Moreover, he mentions key words in Nishida's philosophy such as "intuition" and "self-reflection". A key word in this paper is "infinite mind", which was coined by Gödel[20]. First, let us quote the following passage.

> Usually, it is essential for modern mathematics to assume an infinite mind and conjecture what it does. By infinite mind, I mean that I can investigate infinitely many things by checking one by one. For example, the law of the excluded middle holds for it because it can check whether $A(x)$ or $\neg A(x)$ holds one by one. Similarly, it can be seen that $\{x \mid A(x)\}$ because it can check whether $A(x)$ holds one by one. Conversely, the human mind is clearly finite. [32, p. 172]

According to Takeuti, modern mathematics is based on an infinite mind, and its main task is to conjecture what it does. By infinite mind, Takeuti means a kind of ability to check infinitary objects one by one, while a finite mind means our human being's ability. A typical branch of mathematics based on infinite mind is set-theory, and finite mind cannot check whether $A(x)$ holds for all x one by one if x ranges over an infinite domain, such as the set of natural numbers[21].

One might think that infinite mind is an analogy to one of God in the Western world, but this is not quite correct since Takeuti thinks that infinite mind is essentially connected with finite mind.

Let us explain Takeuti's view of important problems in the foundations of mathematics. After quoting the passage, we attempt to explain the intention of it.

> the following problems are very important in foundations of mathematics:
> 1) To formulate function of infinite mind completely. [...] To formulate the function of infinite mind in a better way, or to find new axioms of set theory.

> 2) To justify the world of our infinite mind in the world of a finite mind. We want to confirm rationality of modern mathematics since the world of our mind is finite, and the world of modern mathematics is just conjecture about our infinite mind. For this, we need to sufficiently develop the mathematics of the world of finite mind. [...]
> 3) To formulate the function of the finite mind completely. [...] [32, p. 173]

In the first clause, Takeuti states that formulating the function of infinite mind is to find new axioms of set theory. This is because set theory is based on infinite mind and axioms in it formulates the function of infinite mind.

The second and third clauses are very important for the purpose of this paper. The second clause describes the activity of the proof theory, such as the consistency proof of the formalized mathematics. Since our mind is finite, we need to confirm the rationality of modern mathematics (such as set theory or analysis). The word "rationality" is translated from「合理性」, which means that the world of infinite mind fits with the world of finite mind. In other words, this thinking can be stated as follows: There is machinery in the world of infinite mind that fits with or is not rational from the machinery of a finite mind. At least, the former must not be contradictory to the latter.

Moreover, for Takeuti, the investigations of set-theory and proof-theory are related to each other, and both developments should be interconnected in an essential way:

> From these considerations, it turns out that the problem of the function of mind, whether finite or infinite, is an important common task in foundations mathematics. [32, p. 173]

Given our understanding of Takeuti's thinking as explained above, one might ask the following question: Did Takeuti consider it as possible to tackle problems in the foundations of mathematics by investigating some "essential" aspects of minds such as infinite and finite minds? Is there an essential element among the functions of minds that occur in the context of foundations of mathematics? Though he describes only the outline in that paper, he uses the word "self-reflection" in the following passage:

> For example, consider a finite mind. If the function of the finite mind is completely finite, then our mathematics remains finite. However, the finite mind is potentially infinite. In other words, we can indicate infinitary objects such as $0, 1, 2, 3, \ldots$ [...] Why? This is because our mind has the ability to self-reflect, that is, the ability to observe what we are doing and to know what we are doing. [32, p. 173]

To understand this passage, let us consider the number 1 as a stroke $|$. Because adding one stroke is clearly possible, we can consider 2 as $||$. Iterating this process, we consider any finite number occurring in the sequence $0, 1, 2, \ldots$. Now, following Takeuti, we reflect on what we are doing. It is not so easy to characterize the limit of the ability of self-reflection, but Takeuti clearly considers the possibility that we can observe the process "adding a stroke beginning with the empty sequence." Therefore, we can consider the set of natural numbers as potentially infinite.

Interestingly, Takeuti claims the same thing for infinite mind as well. It is not very clear what kind of picture for an infinite mind is considered by Takeuti in the passage, but as another quotation below shows, Takeuti thought this kind of approach would be useful for "a fundamental problem of set-theory" that is, for carrying out a program to obtain a new axiom of set-theory.

> In the case of infinite mind, this problem appears implicitly and explicitly when I consider the paper on the formalization principle or a fundamental problem of set theory. [32, p. 173]

Takeuti also mentioned another type of "weak" infinite mind, which is called a constructible mind or ω mind. We will investigate this point in the future, but it is meaningful to point out that he was very interested in set-theory from ordinal theoretic point of view already in the 1950s; he showed in 1954 that if the theory of ordinal numbers is consistent, then the set theory of Fraenkel-Neumann is also consistent [22]. This result indicates a reduction of the set theory into the theory of ordinals. Presumably, for him, this was a result of analyzing set theory from the viewpoint of a weak infinite mind[23].

4.3 Explaining Takeuti's Works in Nishida's Terms

In this section, we would like to sketch how to explain Takeuti's mathematical works in terms of Nishida's philosophy. Here is a natural question: what kind of concept in Nishida's philosophy should be natural to interpret the bearer of finitism in Takeuti's text? Our answer is very simple, that is, the concept of self in Nishida is suitable for that purpose since such a bearer is an agent, that is, a kind of self.

In order to carry out this idea, we first need Nishida's concepts in his philosophy, which is well-known for being complicated and difficult. Since Deguchi developed his own interpretation of Nishida's philosophy in the style of analytic philosophy, we rely on his work [7].

As emphasized by Deguchi [7], Nishida's idea of self is *weaker* than typical ideas of it in modern Western philosophy such as the Cartesian ego, but it is *not* just a simple negation of such a notion of self. Here, we should remind that his aim was to develop the theory of self based on East Asian Buddhist tradition (Z10. 472–4). In the West, according to Nishida, the relationship between the world and the self is always dualistic in the sense that the separation between them must be strict. Contrary to this, Nishida's theory of self is *non-dualistic* since it aims to overcome many dichotomies such as one between subject and object.

In Nishida's philosophy, the notion of world (or one of absolute) must be prior to one of self. We should therefore think of self from the world, rather than *not* vice versa. In the passage of Z10. 510, Nishida says that Kant's philosophy still belongs to subjectivism in that sense. Nishida calls the position to transfer of ontological priority from self to something else as anti-subjectivism. Moreover, the Eastern self *returns* to the world while the Western one confronts to it (Z10. 472–3). Therefore, Nishida's theory of self is non-dualistic and reversionary anti-subjective.

Now, let us briefly explain one of the most important key terms in Nishida's late philosophy, that is, *active intuition*. Here, we remark that this notion is important for interpreting Takeuti's texts as pointed out by our previous paper [1]. One feature of active intuition is that it is not *passive, but active*. Also, it is somatic rather than ecstatic. Nishida's principal example of it is a carpenter's building of a house (Z9. 151). Such an activity manifests intuitive knowledge since the carpenter must intuit the nature of hammers and so on. Though Nishida says that active intuition is also social, we put this point aside in this paper.

According to Deguchi's reconstruction [7], there are three key items in late Nishida's philosophy, that is, a self, and materials, and the world. Active intuition explained above is an event of the world in which narrow self and materials can participate where a narrow self is a self without anything outside our body. If we use the example of a carpenter, the narrow self is that carpenter himself and the materials can be a hammer and the house to be built. The world can enlist in active intuition interacting with a narrow self. The world is the totality of all entities and is also a creator producing all entities as well as itself (Z10. 23). There is a one-many relation between the world and narrow selfs since the world creates a self. Here, we note that the relationship between the producer and something produced is interactive since the self expresses or reflects the world. For example, a house expresses its producer (the carpenter) in the sense that it indicates by itself his skills and visions. Moreover, Nishida says a paradoxical phrase "a narrow self is the world itself"[24].

Furthermore, there is a parallel one-many relation between a narrow self and materials. This interpretation is explained again by the example of a carpenter; a carpenter uses some construction tools and materials in order to build a house. During such a construction process, he would extend his body by such tools and would have deeper understanding about a house to be built, tools, and himself. For example, he would have some experience that he regards the extended parts of his body by tools as if they are just one part of his body. Moreover, such a carpenter experiences himself as conditioned by those materials. Hence, a relationship between a narrow self and materials can be interactive.

Our main idea in this paper is to interpret Takeuti's word "mind" as Nishida's (narrow) self. For this purpose, we summarize our idea as follows:

Nishida	Takeuti
self	mind
materials	mathematical objects, symbols

Let us interpret Takeuti's word "mind" in terms of Nishida's narrow self. We quote the following passage again from 「数学について」(About mathematics) published in 1972 [32]:

> Usually, it is essential for modern mathematics that it supposes an infinite mind and conjectures what it does. By infinite mind, I mean a mind which can investigate infinitely many things by checking one by one. [...] On the contrary, a human mind is clearly a finite mind. (Cf. [32], p. 172)

An infinite mind, then, can solve the foundational problems of the formalist project: among other capacities, it can confirm the consistency of formal theories. He views modern mathematics, encompassing set theory, as an investigation by finite human minds of the capacities of an infinite mind.

Let us explain this using our idea. A working mathematician is a self in Nishida's terminology and is a mind in Takeuti's one. In this activity, he would conjecture about a proposition in his field, say set-theory and tries to solve it. Here, he is supposing another infinite mind corresponding to set-theory while his mind itself is finite.

To solve a problem, he might use his tool, say analysis or algebra and would improve it if necessary. In this way, he interacts with materials in Nishida's terminology and mathematical objects (or symbols) in Takeuti's case. Also, his understanding or view about mathematical objects could be reflected in them as his tool. In an analogy with a carpenter, he would extend his mind by such tools and would have deeper understanding about mathematical objects.

Here, let use a familiar example from set-theory; let *ZF* be a well-known axiomatic set-theory called Zermelo-Fraenkel set theory and *GCH* a proposition called the generalized continuum hypothesis, which was shown to be independent from *ZF* by Gödel and Cohen. If our goal is $ZF \nvdash \neg GCH$, then it is enough to show the consistency of *ZF* augmented with *GCH*, that is, we need to build a model for it. As done by Gödel, we might try to construct it by considering a constructive model of it. For this purpose, Gödel introduces a nice tool, that is, the constructive hierarchy *L*. This model is shown to be a model for *ZF* augmented with *GCH* and reflects Gödel's mind and deep understanding about sets.

Before closing this section, let us briefly explain our reading using Gentzen's consistency proof of *PA* as another example. In this picture, there are two minds, that is, a mind having exactly the (infinitary) mathematical power of *PA* and another finite mind exactly having the power of a Hilbertian finite mind. To prove consistency of the infinite mind (*PA*), the finite mind investigates the infinite mind closely. During this process, it notices that some extra principle (or a tool) must be added to it, that is, the transfinite induction up to ε_0 (materials). Finally, the consistency result would mean that the finite mind with the transfinite induction is reflected into another infinite mind corresponding to *PA*. We plan to investigate Takeuti's other passages under this interpretation in the future works since this paper is already lengthy enough.

5 Conclusion

In this paper, we have reviewed the thoughts of three major Japanese thinkers in the foundations of mathematics: (1) Kitaro Nishida, (2) Joichi Suetsuna, and (3) Gaisi Takeuti. Let us summarize what we have seen in the previous sections and provide some remarks.

Nishida was the person who established the discipline of philosophy as practiced in the West and provided a new basis for philosophy in the East, based on Buddhism. In the literature, his philosophy has been examined and discussed in the context of philosophy of religion but not in the context of the philosophy of mathematics. Contrary to a common perspective, some core ideas in it have their origin in his careful discussions about Dedekind and Royce's works. In particular, Dedekind's self-representative system should be an important resource of his idea of Jikaku (self-awareness or self-reflection). In his later works, he analyzed several concrete examples of mathematics to solve paradoxical philosophical questions about sets, numbers, and mathematical operations such as addition and multiplication.

Suetsuna took over Nishida's ideas in the philosophy of mathematics and applied it to the most advanced topics in the foundations of mathematics at that time. In particular, he adopted Nishida's notion of "active intuition" to answer the question of how to form infinitary sets and numbers. Moreover, he used the notion of "Zettai-teki Jikomujun-teki Jikodouitsu" (absolute inconsistent self-identity) to explain a paradoxical feature of mathematical objects, such as the set of all natural numbers and a limit ordinal, say ε_0. It is meaningful to note here that he dealt only with relatively elementary mathematical examples, though he was an internationally well-known number theorists at the time. Moreover, his mathematical results and philosophical thinking are seemingly separate in the sense that his mathematics was not developed essentially based on his philosophy.

From this point of view, Takeuti seems to be an original thinker. He opened the new gate of proof-theory after Hilbert and Gentzen by proposing Takeuti's conjecture. In the 1950–60s, he obtained several stunning results for his conjecture and invented a new system of infinitary ordinals called "ordinal diagrams." For him, proof-theory (or formalism in his word) was not a mature field yet, so it must be developed in the way that mathematical results must be examined philosophically when they are obtained. Moreover, Takeuti had an original view of the foundations of mathematics and attempted to explain it using the term "mind" coined by Gödel. In this view, the most "reliable" basis for the foundations of mathematics was not intended to be obtained. Rather, by proof-theoretic studies, the relationships between different kinds of minds are investigated. In our paper, we aimed to interpret this picture using Nishida's philosophical term based on Deguchi's reconstruction.

Before closing this paper, we want to point out that Takeuti should be more original than Suetsuna and other philosophers such as Torataro Shimomura in the sense that Takeuti carried out his mathematical project based on his philosophical view, while this is not applicable to either Suetsuna or Shimomura. It would be very interesting to explain Takeuti's other passages under this interpretation in future work.

Footnotes

[1] "NKI" means the first volume of his collected works [16]. We use similar notations to mean other volumes as well. The translations of texts in this paper are basically done by the author (in the cases of Nishida and Suetsuna) or by the author and Andrew Arana (in the case of Takeuti).

[2] See [28, p.299].

[3] Nishida mentions Royce's criticism that Hegel's definition is imprecise and regards the notion of infinity by mathematicians such as Cantor or Dedekind as the most clear one.

[4] The word "basho" is usually translated as "place" or "topos" in English.

[5] NKXI, pp. 240–241.

[6] We note that there is a difference between this characterization of multiplication and another one of addition in the previous paragraph; the notion of time is important here while one of space is important there.

[7] Indeed, the key person in the next section, that is, Suetsuna writes that he sometimes cannot agree with Nishida, though he developed his own mathematical philosophy under the strong influence of Nishida. The examination of his thoughts in the views of others is beyond the scope of this paper, but it would be an interesting topic in the future.

[8] For the development of such a theory, we refer to Brouwer's papers [4, 5].

[9] In this paper we refer to the republished version [22].

[10] This paper was republished in NKXI.

[11] Indeed, Suetsuna uses the word "Jikomujun-teki Jikodouitsu" instead of "Zettai-teki Jikomujun-teki Jikodouitsu." However, it seems clear that what he means is Nishida's "Zettai-teki Jikomujun-teki Jikodouitsu."

[12] Here, he refers to Gentzen's seminal paper in 1936 to prove the consistency of arithmetic with mathematical induction called Peano arithmetic [8].

[13] This criticism should be important for the topic of this paper since Takeuti was the person who aimed to proved the consistency of analysis; hence, it seems that Suetsuna's point could also be applied to Takeuti as far as his philosophy is understood in the context of formalism.

[14] For such a theory, we refer to [12].

[15] Brouwer simply supposed the continuity axiom in his arguments. For example, see [6, 34]. Logicians such as Howard, Kleene or Kreisel formulated and investigated it later [13, 14].

[16] Interestingly, Suetsuna does not mention Frege in the text. It is possible that Frege's work was unknown or inaccessible in Japan at that time.

[17] It would be meaningful to point out here that Suetsuna does not mention Brouwer's uniform continuity theorem in 1927 [6], saying that all functions defined on a closed

interval are uniformly continuous. This contradicts classical mathematics because we can prove the existence of a discontinuous function in it. The author of this paper thinks that Brouwer's papers were at least very difficult to understand in detail even for a very good mathematician such as Suetsuna since their contents were quite new and peculiar, and several basic principles in intuitionism, such as the axiom of continuity, were implicitly supposed in them. After serious efforts to clarify such principles after 1950s, they are formulated explicitly now. For a modern explanation of the basic principles of intuitionism, see [34].

[18] Cf. [11, p.486] ([15, p. 267] in English).

[19] Here is the passage

Here, the word "active intentional intuition" is a coined word under Suetsuna's influence (Suetsuna scolded me that "intentional" is not suitable). Suetsuna claims that the whole of natural numbers or real numbers is grasped by active intuition. His active intention would be a part of my active intentional intuition, but it seems to hold in his case that his notion is constructive and the other point is that it can be grasped, not only as an intention but as a fact [...].

Indeed, the friendship between Takeuti and Suetsuna is observed in the fact that he dedicated the paper "On the recursive functions of ordinal numbers" [30] to Suetsuna.

[20] Cf. [33].

[21] However, if the method to check whether $A(x)$ holds is given by a concrete method such as mathematical induction, then finite mind can see it. This point is important if we remind the fact that the final standpoint by Takeuti or even by Hilbert includes mathematical induction on decidable predicate.

[22] The paper is "Construction of the set theory from the theory of ordinal numbers" [27]. By "the set theory of Fraenkel-Neumann," he means his G^1LC without bound functions.

[23] Later, this became a starting point for the Kripke–Platek set theory, which has played a major role in the history of proof-theory in Germany, such as Jäger, Rathjen, after the 1980s. For works of this school, see Rathjen's excellent paper [17]. Here, we mention Arai's ordinal analysis for "theories of ordinals" which can be seen as a direct continuation of Takeuti's line. In these works, Arai used only finite proof-figures, in contrast to German logicians, and did not analyze extensions of Kripke–Platek set theory, but theories more directly dealing with ordinals. For an exposition of his works, we refer to [2].

[24] For an investigation of this paradoxical identity, we refer to [7].

References

[1] Ryota Akiyoshi and Andy Arana. Takeuti's proof-theory in the context of the kyoto school. *Jahrbuch für Philosophie das Tetsugaku-Ronso* 哲学論叢, 46:1–17, 2019.

[2] Toshiyasu Arai. A sneak preview of proof theory of ordinals. *Annals of the Japan Association for Philosophy of Science*, 20:29–47, 2012.

[3] Luitzen Egbertus Jan Brouwer. Intuitionism and formalism. *The Bulletin of the American Mathematical Society*, 20(2):81–96, 1913.

[4] Luitzen Egbertus Jan Brouwer. Zur Begründung der intuitionistischen Mathematik I. *Mathematische Annalen*, 93:244–257, 1925.

[5] Luitzen Egbertus Jan Brouwer. Zur Begründung der intuitionistischen Mathematik II. *Mathematische Annalen*, 95:453–472, 1925.

[6] Luitzen Egbertus Jan Brouwer. Über Definitionsbereiche von Funktionen. *Mathematische Annalen*, 97:60–75, 1927. English translation with introduction by Charles Parsons in [35].

[7] Yasuo Deguchi. Late Nishida and Dialetheism. In *Daigokai Nichuu Tetsugaku Forum "Shisaku To Taiwa Ni Yoru Nichuu Kankei No Shinka: Kyokai Wo Kakyo Suru Tetsugaku No Yakuwari" Yokoshu (Proceedings of the 5th Philosophy Forum of Japan and China "Deepening Japan–China Relation by Thinking and Dialogue: Role of Philosophy Bridging over the Boundaries")*, pages 359–385. Institute of Philosophy of Chinese Academy of Social Sciences, The Philosophical Association of Japan, Institute of Humanities, Human and Social Sciences, Ritsumeikan University, 2017.

[8] Gerhard Gentzen. Die Widerspruchsfreiheit der reinen Zahlentheorie. *Mathematische Annalen*, 112:494–565, 1936. English translation in [24].

[9] Gerhard Gentzen. Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie. *Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften, Neue Folge*, 4:19–44, 1938. English translation in [24].

[10] David Hilbert. *Grundlagen der Geometrie*. Teubner, Leipzig, 1899.

[11] David Hilbert. Die Grundlagen der Elementaren Zahlentheorie. *Mathematische Annalen*, 104:485–94, 1931. English translation by William Ewald in [15].

[12] Van Der Hoeven, Gerrit, and Ieke Moerdijk. Sheaf models for choice sequences. *Annals of Pure and Applied Logic*, 27:63–107, 1984.

[13] William A. Howard and Georg Kreisel. Transfinite induction and bar induction of types zero and one, and the role of continuity in intuitionistic analysis. *Journal of Symbolic Logic*, 31:325–358, 1966.

[14] Georg Kreisel and Anne S. Troelstra. Formal systems for some branches of intuitionistic analysis. *Annals of Mathematical Logic*, 1:229–387, 1970.

[15] Paolo Mancosu, editor. *From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920s*. Oxford University Press, Oxford, 1998.

[16] Kitaro Nishida. *Nishida Kitaro Zenshu (Complete Works of Nishida Kitaro)*. Iwanami Syoten, 1987-89. 4th edition.

[17] Michael Rathjen. The realm of ordinal analysis. In S. Barry Cooper and John K. Truss, editors, *Sets and Proofs: Invited Papers from Logic Colloquium '97, European Meeting of the Association for Symbolic Logic, Leeds, July 1997*, volume 258 of *London Mathematical Society Lecture Note Series*, pages 219–279. Cambridge University Press, Cambridge, 1999.

[18] Thoralf Albert Skolem. Über die nicht-charakterisierbarkeit der zahlenreihe mittels endlich oder abzählbar unendlich vieler aussagen mit ausschliesslich zahlenvariablen. *Fundamenta Mathematicae*, 23:150–161, 1934.

[19] Joichi Suetsuna. Suugaku no kiso (the foundations of mathematics). *Kagaku (Science)*, 10, 1944. republished in [22].

[20] Joichi Suetsuna. Yuugen no tachiba to kyokugen gainen (the finitist standpoint and the notion of limit). *Kagaku (Science)*, 1, 1944. republished in [22].

[21] Joichi Suetsuna. Ronri to ningen sonzai (logic and human existence). *Riso (Ideal)*, 8, 1946. republished in [22].

[22] Joichi Suetsuna. *Ronri to Suuri (Logic and Mathematics)*. Kyoyo Bunko. Koubundo, 1947.

[23] Joichi Suetsuna. *Suugaku no Kiso (The Foundations of Mathematics)*. Iwanami Syoten, 1952.

[24] M. E. Szabo, editor. *The Collected Papers of Gerhard Gentzen*, volume 55 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, Amsterdam, 1969.

[25] Shuyu Takahashi. Kegonkyo Ni Miryousareta Suugakusya Suetsuna Joichi (Mathematician Suetsuna Joichi fascinated by Kegonkyo). *Gendai Mikkyo*, 22:135–152, 2011.

[26] Gaisi Takeuti. On a generalized logic calculus. *Japanese Journal of Mathematics*, 23:39–96, 1953.

[27] Gaisi Takeuti. Construction of the set theory from the theory of ordinal numbers. *Journal of the Mathematical Society of Japan*, 6(2):196–220, 1954.

[28] Gaisi Takeuti. On the fundamental conjecture of GLC IV. *Journal of the Mathematical Society of Japan*, 8(2):145–155, 1956.

[29] Gaisi Takeuti. On the fundamental conjecture of GLC V. *Journal of the Mathematical Society of Japan*, 10(2):121–134, 1958.

[30] Gaisi Takeuti. On the recursive functions of ordinal numbers. *Journal of the Mathematical Society of Japan*, 12(2):119–128, 1960.

[31] Gaisi Takeuti. Consistency proofs of subsystems of classical analysis. *The Annals of Mathematics*, 86(2):299–348, 1967.

[32] Gaisi Takeuti. Suugaku Ni Tsuite (About mathematics). *Kagakukisoron Kenkyu (Annals of the Japan Association for Philosophy of Science)*, (4):170–174, 1972.

[33] Gaisi Takeuti. Proof theory and set theory. *Synthese*, 62(2):255–263, 1985.

[34] Mark van Atten. *On Brouwer*. Wadsworth Philosophers Series. Thomson Wadsworth, Belmont, Calif., 2003.

[35] Jean van Heijenoort, editor. *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*. Harvard University Press, Cambridge, Mass., 1967.