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Contributed Paper

The Estimation of the Value of the Curvature of Photographic Space

Toshio Watanabe*

Abstract

In the present study, a method to estimate the value of the curvature of photographic space was presented. In photographic space, a visual triangle was made and the configuration of points of the visual triangle was mapped to physical space and the values of the curvature K and σ (the degree of depth perception) were estimated by using Luneburg's mapping functions. As the result, it was found that photographic space was hyperbolic and the absolute value of the curvature K and the value of σ of photographic space were smaller than those of visual space. The value of K showed that photographic space was closer to Euclidean space than visual space and the value of σ showed that photographic space was narrower space than visual space. The discovery of the method to estimate the value of the curvature of photographic space made us to compare various spaces quantitatively.

Key words: photographic space, visual space, depth, hyperbolic, curvature.

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The studies of depth perception of a photograph began in 1950s. It is known that the perceived distance from a camera position to an object in a photograph is affected by the distance between an observer and the photograph. Smith (1958a) found that the perceived distance was significantly longer in the long observation distance (2.9 m) than in the short observation distance (0.69 m). In another study (Smith, 1958b), he found that the observation distance did not affect the perceived height of the object, but affected the perceived distance to the objects. Further, Smith & Gruber (1958) found that the distance perceived in the photograph was shorter than in the direct observation when participants observed the photograph from the distance less than 2 m, but it was perceived longer when participants observed the photograph from the distance more than 2 m. Kraft, Patterson, & Mitchell (1986) found that the perceived distance from the camera position to the object was affected by the focal distance of the camera, but the lateral distance was not. Later, Hecht, van Doorn, & Koenderink (1999) found that the angle was perceived as larger both in the long observation distance (10 m) and in the short observation distance (1.5 m), but the distance was perceived as longer in the short observation distance and shorter in the long observation distance. These studies showed that perceived distance from the camera position to the object in the photograph was affected by the observation distance and the perceived angle was larger than the physical angle irrelevant to the observation condition. The common features of these studies were that the participant's task was to judge the distance from the camera position to the object in the photograph and the effect of the observation distance from the observer to the photograph was studied. Watanabe (2004, 2006) showed how the distances between two objects in the photograph and the angles of the objects from the median line were perceived. It was found that the perceived distance and the physical distance were nonlinearly related and the nonlinear relation was affected by the angle conditions. In the similar way, it was found that the perceived angle and the physical angle were also nonlinearly related and the nonlinear relationship was affected by the distance conditions. The results suggested that the photographic space was anisotropic.

As the next step, it is interesting to know what geometry describes photographic space. In observing a photograph of landscape, the previous studies mentioned before showed that we easily perceived depth, distances, angles and so on. This implies the possibility that photographic space has a geometrical structure. An important thing is what geometry describes photographic space most appropriately. In considering visual space, there are several candidates to describe the geometry of visual space. In most cases, Euclidean geometry is assumed. However, there are evidences that Euclidean geometry may not be appropriate as the geometry of visual space. For example, An anisotropic property (Foley, 1966) can not be explained by Euclidean geometry because Euclidean geometry presupposes that space should be homogenious. Luneburg (1947) assumed that visual space should be Riemannian space, especially the hyperbolic space of constant curvature, based on the discrepancy between parallel and distance alleys.

In considering the geometry of photographic space, it will be appropriate to use the same method as in the case of visual space. By using the same method, it makes us to compare photographic space with visual space also. In the previous study (Watanabe, 2006), It was found that the geodesic line of photographic space tended to be hyperbolic. Based on this result, the present study focuses on non-Euclidean properties of photographic space and compares them with visual space. For this purpose, we need to estimate the value of the curvature of two spaces. For visual space, we know the method to estimate the value of the curvature in Luneburg's paradigm. He assumed the following mapping functions between physical space and Euclidean map.

$$\left.\begin{aligned} \rho &= 2e^{-\sigma\gamma} \\ \varphi &= \phi \\ \vartheta &= \theta \end{aligned}\right] \tag{1}$$

$$\left.\begin{aligned} \tan\gamma &= (4/f)(x^2+z^2)^{0.5}/((4/f^2)(x^2+y^2+z^2)-1) \\ \tan 2\phi &= (8y/f^2)(x^2+z^2)^{0.5}/((4/f^2)(x^2+z^2-y^2)+1) \\ \tan\theta &= z/x \\ \xi &= \rho\cos(\varphi)\cos(\vartheta) \\ \eta &= \rho\sin(\varphi) \\ \zeta &= \rho\cos(\varphi)\sin(\vartheta) \end{aligned}\right] \tag{2}$$

Physical space is the space where stimuli are presented and Euclidean map is the Poincaré's model, which represents hyperbolic space with the Euclidean space of the same dimension. The feature of Poincaré's model is that hyperbolic angles are preserved (conformal mapping). Figure 1 shows the relationship between physical space (the left figure) and Euclidean map (the right figure). In physical space, a point is represented by Q and the coordinate value of point Q is represented by the bipolar coordinate system or the Cartesian coordinate system. In the bipolar coordinate system, point Q is represented by $Q(\gamma, \phi, \theta)$. γ shows the convergence angle of point Q subtended by participant's two eyes (L and R) and ϕ the bipolar latitude and θ the elevation angle of point Q. Further, σ is a parameter defined by Luneburg and it shows the degree of depth perception.

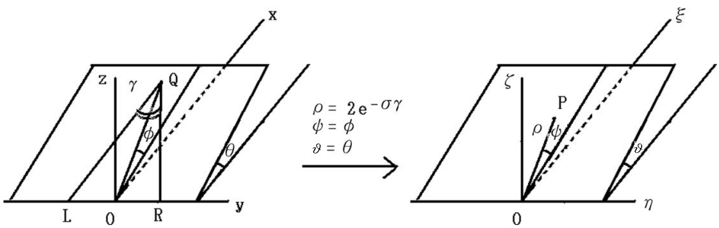

Figure 1. Luneburg's mapping functions between physical space and Euclidean map.

In the Cartesian coordinate system, x shows depth direction, y the lateral direction, z the vertical direction. Further, f shows the inter-pupil distance of the participant. In Euclidean map, a point is represented by point P and coordinate value of point P is represented by the polar coordinate system or the Cartesian coordinate system. In the polar coordinate system, point P is represented by $P(\rho, \phi, \vartheta)$. ρ shows the radial distance of point P, the distance from the origin O, and ϕ the polar latitude and ϑ the elevation angle. In the Cartesian coordinate system, ξ shows depth direction, η the lateral direction, ζ the vertical direction. In Euclidean map, a hyperbolic geodesic line is represented by the arc of circle orthogonal to the basic circle at two points. The basic circle represents infinity in hyperbolic space. And the hyperbolic distance of two points P_i and P_j (δ_{ij}) is represented by

$$\delta_{ij}=\sinh^{-1}(\rho_{ij}/[W_i W_j])/q \ ,$$

where

$$W_i=(q^{-2}-\rho_{0i}{}^2)^{0.5}\ , \quad q=(-K)^{0.5}/2\ , \quad -1\leqq K<0 \qquad (3)$$

In the case of visual space, by using Luneburg's mapping functions shown in (1), point Q is mapped into Euclidean map. Let's map three points Q_A, Q_B and Q_C into Euclidean map and define the corresponding three points in Euclidean map as P_A, P_B and P_C. Using these three points P_A, P_B and P_C, we can define the hyperbolic triangle ABC in Euclidean map as shown in Figure 2. Let's define points D, E and F as the hyperbolic midpoints of sides BC, AC and AB. Further, point G as the point which satisfies BG = EF and is on the side BC and point H as the point which satisfies CH = EF and is on the side BC. In Euclidean geometry, two points G and H coincide with each other. However, in hyperbolic geometry, two points G and H do not coincide with each other (Blank, 1961; Watanabe, 1996). Point G lies the left side of the midpoint D and point H the right side as shown in Figure 2. The locations of points D, E, F, G and H are the function of two parameters K and σ.

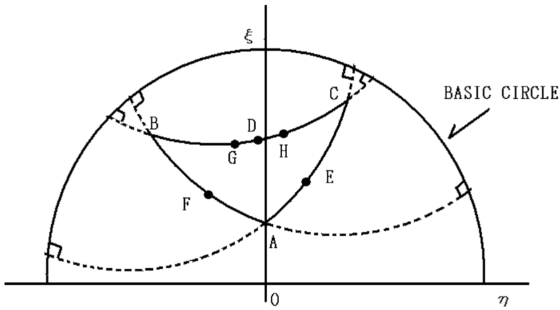

Figure 2. The hyperbolic triangle ABC in two dimensional Euclidean map.

In the hyperbolic triangle ABC, the equation of side BC passing through points B(ξ_2, η_2) and C(ξ_3, η_3) is represented by the arc of circle of center O$_{23}(\xi_{23}, \eta_{23})$ and radius r_{23}.

$$(\xi-\xi_{23})^2+(\eta-\eta_{23})^2=r_{23}{}^2 \tag{4}$$

where

$$\left.\begin{array}{l} r_{23}{}^2=\xi_{23}{}^2+\eta_{23}{}^2-q^{-2}, \quad q=(-K)^{0.5}, \quad -1\leqq K<0 \\ \begin{bmatrix}\xi_{23}\\ \eta_{23}\end{bmatrix}=0.5\begin{bmatrix}\xi_2 & \eta_2\\ \xi_3 & \eta_3\end{bmatrix}^{-1}\begin{bmatrix}\rho_{02}^2+q^{-2}\\ \rho_{03}^2+q^{-2}\end{bmatrix} \\ \rho_{ij}{}^2=(\xi_\mathrm{i}-\xi_\mathrm{j})^2+(\eta_i-\eta_j)^2 \end{array}\right]$$

When the midpoint of B and C is D(ξ_4, η_4), point D satisfies equation (5).

$$\sinh^{-1}(\rho_{24}/[W_2W_4])=\sinh^{-1}(\rho_{34}/[W_3W_4]) \tag{5}$$

where

$$W_i=(q^{-2}-\rho_{0i}{}^2)^{0.5}$$

From equation (5)

$$\begin{aligned}(\xi_4-(W_2{}^2\xi_3-W_3{}^2\xi_2)/(W_2{}^2-W_3{}^2))^2 \\ +(\eta_4-(W_2{}^2\eta_3-W_3{}^2\eta_2)/(W_2{}^2-W_3{}^2))^2 \\ =((W_2W_3\rho_{23})/(W_2{}^2-W_3{}^2))^2\end{aligned} \tag{6}$$

Therefore, point D is the intersection of equations (4) and (6) on the side BC.

Further, point G(ξ_7, η_7) which satisfies EF=BG on the side BC is represented by equations (4) and (7).

$$\sinh^{-1}(\rho_{27}/[W_2W_7])=\sinh^{-1}(\rho_{56}/[W_5W_6]) \tag{7}$$

where

$W_{\mathrm{i}}=(q^{-2}-\rho_{0i}{}^2)^{0.5}$

From equation (7)

$$(\xi_7-\xi_2W_5{}^2W_6{}^2/(W_5{}^2W_6{}^2+W_2{}^2\rho_{56}{}^2))^2 +(\eta_7-\eta_2W_5{}^2W_6{}^2/(W_5{}^2W_6{}^2+W_2{}^2\rho_{56}{}^2))^2 =(W_2{}^2\rho_{56}/(W_5{}^2W_6{}^2+W_2{}^2\rho_{56}{}^2))^2(\rho_{56}{}^2q^{-2}+W_5{}^2W_6{}^2) \tag{8}$$

Therefore, point G(ξ_7, η_7) is the intersection of equations (4) and (8) on the side BC. Similarly, point H (ξ_8, η_8) which satisfies EF＝CH on the side BC is represented by equations (4) and (9).

$$\sinh^{-1}(\rho_{38}/[W_3W_8])=\sinh^{-1}(\rho_{56}/[W_5W_6]) \tag{9}$$

where

$W_i=(q^{-2}-\rho_{0i}{}^2)^{0.5}$

From equation (9)

$$(\xi_8-\xi_3W_5{}^2W_6{}^2/(W_5{}^2W_6{}^2+W_3{}^2\rho_{56}{}^2))^2 +(\eta_8-\eta_3W_5{}^2W_6{}^2/(W_5{}^2W_6{}^2+W_3{}^2\rho_{56}{}^2))^2 =(W_3{}^2\rho_{56}/(W_5{}^2W_6{}^2+W_3{}^2\rho_{56}{}^2))^2(\rho_{56}{}^2q^{-2}+W_5{}^2W_6{}^2) \tag{10}$$

Therefore, point H(ξ_8, η_8) is the intersection of equations (4) and (10) on the side BC.

Next, let's consider to represent elliptic geometry in Poincaré's model as same as hyperbolic geometry. In elliptic geometry, the geodesic line passing through points P_i and P_j is represented by the arc of circle whose two intersections with the basic circle are symmetry at the center O of the basic circle. Elliptic angles are preserved in Poincaré's model. The elliptic distance of two points P_i and P_j (δ_{ij}) and the Euclidean distance of two points P_i and P_j (ρ_{ij}) is related as shown in equation (11).

$$\delta_{ij}=\sin^{-1}(\rho_{ij}/[W_iW_j])/q \tag{11}$$

where

$W_i=(q^{-2}+\rho_{0i}{}^2)^{0.5}$,　$q=K^{0.5}/2$,　$0<K\leqq 1$

In the elliptic triangle ABC, the equation of side BC passing through points B(ξ_2, η_2) and C(ξ_3, η_3) is represented as the arc of circle whose center is O_{23} (ξ_{23}, η_{23}) and whose radius is r_{23}.

$$(\xi-\xi_{23})^2+(\eta-\eta_{23})^2=r_{23}^2 \tag{12}$$

where

$$\left.\begin{array}{l} r_{23}^2=\xi_{23}^2+\eta_{23}^2+q^{-2}, \quad q=(K)^{0.5}, \quad 0\leqq K<1 \\ \begin{bmatrix}\xi_{23}\\ \eta_{23}\end{bmatrix}=0.5\begin{bmatrix}\xi_2 & \eta_2\\ \xi_3 & \eta_3\end{bmatrix}^{-1}\begin{bmatrix}\rho_{02}^2-q^{-2}\\ \rho_{03}^2-q^{-2}\end{bmatrix} \\ \rho_{ij}^2=(\xi_i-\xi_j)^2+(\eta_i-\eta_j)^2 \end{array}\right]$$

When the midpoint of B and C is $D(\xi_4, \eta_4)$, point D satisfies equation (13).

$$\sin^{-1}(\rho_{24}/[W_2W_4])=\sin^{-1}(\rho_{34}/[W_3W_4]) \tag{13}$$

where

$$W_i=(q^{-2}+\rho_{0i})^{0.5}$$

From equation (13)

$$\begin{aligned}&(\xi_4-(W_2^2\xi_3-W_3^2\xi_2)/(W_2^2-W_3^2))^2\\ &\qquad+(\eta_4-(W_2^2\eta_3-W_3^2\eta_2)/(W_2^2-W_3^2))^2\\ &\quad=((W_2W_3\rho_{23})/(W_2^2-W_3^2))^2\end{aligned} \tag{14}$$

Therefore, point D is the intersection of equations (12) and (14) on the side BC. Further, point $G(\xi_7, \eta_7)$ which satisfies EF$=$BG on the side BC is represented by equations (12) and (15).

$$\sin^{-1}\rho_{27}/[W_2W_7])=\sin^{-1}(\rho_{56}/[W_5W_6]) \tag{15}$$

where

$$W_i=(q^{-2}+\rho_{0i}^2)^{0.5}$$

From equation (15)

$$\begin{aligned}&(\xi_7-\xi_2W_5^2W_6^2/(W_5^2W_6^2-W_2^2\rho_{56}^2))^2\\ &\qquad+(\eta_7-\eta_2W_5^2W_6^2/(W_5^2W_6^2-W_2^2\rho_{56}^2))^2\\ &\quad=(W_2^2\rho_{56}/(W_5^2W_6^2-W_2^2\rho_{56}^2))^2(\rho_{56}^2q^{-2}+W_5^2W_6^2)\end{aligned} \tag{16}$$

Therefore, point $G(\xi_7, \eta_7)$ is the intersection of equations (12) and (16) on the side BC. Similarly, point $H(\xi_8, \eta_8)$ which satisfies EF$=$CH on the side BC is represented by equations (12) and (17).

$$\sin^{-1}(\rho_{38}/[W_3W_8])=\sin^{-1}(\rho_{56}/[W_5W_6]) \tag{17}$$

where

$$W_i=(q^{-2}+\rho_{0i}^2)^{0.5}$$

From equation (17)

$$(\xi_8-\xi_3 W_5{}^2 W_6{}^2/(W_5{}^2 W_6{}^2-W_3{}^2\rho_{56}{}^2))^2 + (\eta_8-\eta_3 W_5{}^2 W_6{}^2/(W_5{}^2 W_6{}^2-W_3{}^2\rho_{56}{}^2))^2 = (W_3{}^2\rho_{56}/(W_5{}^2 W_6{}^2-W_3{}^2\rho_{56}{}^2))^2(\rho_{56}{}^2 q^{-2}+W_5{}^2 W_6{}^2) \tag{18}$$

Therefore, point H(ξ_8, η_8) is the intersection of equations (12) and (18) on the side BC. By mapping these five points D, E, F, G and H into physical space, we can estimate the values of two parameters K and σ which minimize the difference of the data configuration and the theoretical configuration.

However, in the case of photographic space, there is no method to estimate the value of the curvature at present because Luneburg's mapping functions map physical space to Euclidean map, not to photographic space. In order to use Luneburg's mapping functions, we need to map photographic space to either physical space or Euclidean map or visual space. Further, no mapping functions are given between photographic space and any of three spaces at present. In the present study, we propose one method to estimate the value of the curvature of photographic space. It is the method to map the points in photographic space to physical space. Once we get the physical relationship between photographic space and physical space, we can map the points in photographic space to physical space and we can use the Luneburg's mapping functions to estimate the value of the curvature.

The physical relationship between photographic space and physical space was obtained in the following way: In the open space of 40 m in depth and 21 m in width, three objects (21 cm in diameter and 20 cm in height) were presented. Objects A and B were fixed points and were presented at A(10 m, 0 m) and B(14 m, 0 m). The first coordinate value shows the value along the depth direction and the second shows the value along the lateral direction (positive value for the right direction). Object X was presented in 30 various locations. These locations were given by AX = 4, 10, 16, 22, 28 m and angles BAX = 0, 4, 8, 12, 16, 20°. These 30 points were photographed from the origin with the camera (Nikon D100) with $f = 42$ mm (for 35 mm lens). The height of the camera lens was 93 cm from the ground. The photograph was presented by the personal computer (Toshiba Dynabook SS LX/190DR) and the physical distance AX and physical angle BAX in photographic space were measured. And they were compared with the physical distance AX and the physical angle BAX in physical space. They are shown in Figures 3 and 4.

Figure 3 shows the relationship between the physical distance AX in physical space and the physical distance AX in photographic space under the six angle conditions (*a*, *b*, *c*, *d*, *e*, *f*). As shown in Figure 3, the distance functions changed according to the angle conditions. This implies that anisotropy appears even in the physical dimension. Figure 4 shows the relationship between the physical angle ABX in physical space and the physical angle ABX in photographic space under 5 distance conditions. As shown in Figure 4, the angle functions are not affected by the distance conditions (*a*, *b*,

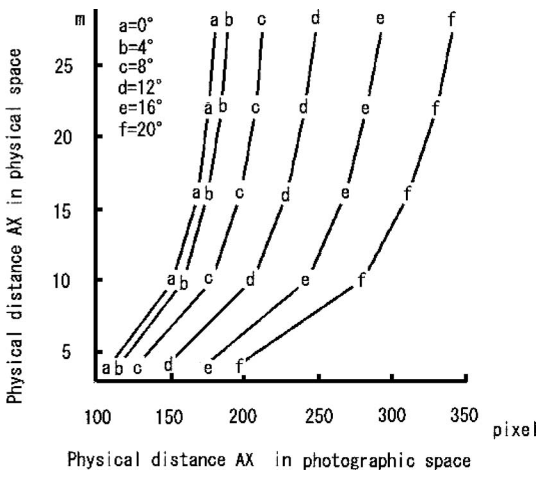

Figure 3. The relationship between the physical distance AX in photographic space and the physical distance AX in physical space.

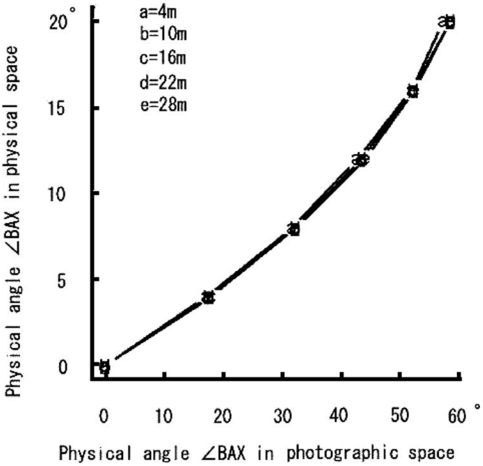

Figure 4. The relationship between the physical angle BAX in photographic space and the physical angle BAX in physical space.

c, *d*, *e*). This implies the physical geodesic line is a straight line in the physical dimension. Using Figures 3 and 4, the physical distance AX and the physical angle BAX in photographic space were mapped to physical space and the coordinate values of points were obtained by using distance AX and angle BAX.

Experiment

The purpose of the present experiment was to estimate the value of the curvature of photographic space (the condition 1) and visual space (the condition 2).

Methods

Participants. Eight undergraduate students (6 males and 2 females) participated in the conditions 1 and 7 undergraduate students (4 males and 3 females) in the condition 2.

Procedure. Under the photographic observation (the condition 1), a triangle experiment was conducted. In the open space used to get the relationship between photographic space and physical space, three objects (21 cm in diameter and 20 cm in height) were presented at A(4 m, 0 m), B(20.67 m, −7.52 m) and C(26.31 m, 9.57 m) and were photographed by the camera(Nikon D100 with $f = 42$ mm for 35 mm lens). The photograph in the personal computer (Toshiba Dynabook SS LX/10DR) was presented by the plasma display (Sanyo PDP-42HD6). As the aspect ratio of the plasma display was different from that of the computer, the photographic image in the plasma display was converted to get the same appearance as in the computer. The participant observed the photograph at the distance of 113 cm binocularly with a chin rest. Participant's first task was to point the perceptual midpoints of sides BC, AC and AB in a random order by the laser pointer. Suppose points D, E and F are the adjusted midpoints of sides BC, AC and AB in turn. The participant's second task was to point the position G to satisfy BG = EF and G was on the side BC perceptually and to point the position H to satisfy CH = EF and H was on the side BC perceptually. In pointing G and H, five points A, B, C, E and F were presented. Three objects were used for points A, B and C and small black points were used for points E and F. Similarly, participants pointed the locations I and J to satisfy DF = AI, DF = CJ on the side AC perceptually and the locations K and L to satisfy DE = AK, DE = BL on the side AB perceptually.

In the condition 2, the same experiment as photographic space was conducted with the real open space. The purpose of this experiment was to estimate the values of the curvature K and σ of visual space and compare them with those of photographic space. The participant observed the objects A, B and C binocularly at the camera position O. And they adjusted points D, E and F at the perceptual midpoints of sides BC, AC and AB in turn. For the points' adjustment, a radio controlled small car was used. The participant moved the car to points D, E and F. After adjusting midpoints, points G and H were adjusted. Points I, J, K and L were not adjusted in the condition 2 because there were no significant differences between I and J, and between K and L in the condition 1. In both conditions the experiment was conducted individually and participants had normal vision or normally corrected vision. The participants adjusted the points twice in the condition 1 and once in the condition 2.

Results

In the condition 1, each location was averaged over the participants and was plotted as shown in Figure 5. As the result, G and H were different significantly ($t[7]=2.601$, $p=0.018$) and showed hyperbolic property. However, there were no significant difference between I and J and between K and L ($p>0.05$). Points D, E, F, G and H were mapped to physical space with Figures 3 and 4. The configuration of points mapped to physical space is shown in Figure 6. Capital letters D, E, F, G and H show the average configuration of points in photographic space. And parameters K and σ were estimated to minimize RMS (Root Mean Squares). Obtained estimated values of K and σ were $K=-0.2$, $\sigma=150$ and RMS$=0.47$.

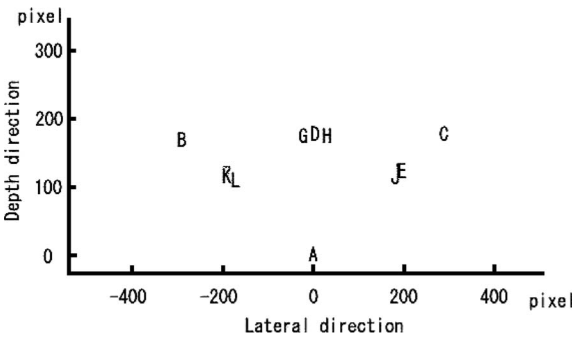

Figure 5. Average configuration of points of the triangle ABC in photographic space.

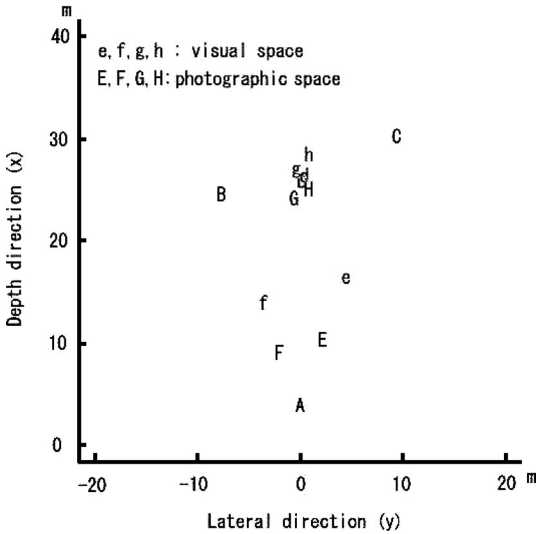

Figure 6. Average configuration of points of triangle ABC of photographic space and of visual space in physical space.

The lower case letters *d*, *e*, *f*, *g* and *h* of Figure 6 show the average configuration of points in visual space. Estimated values of K and σ were $K=-0.85$, $\sigma=375$ and RMS$=0.75$ in visual space.

Discussion

The purpose of the present study was to estimate the value of the curvature K of photographic space. It was completed by mapping photographic space to physical space and by using Luneburg's mapping functions. The value of the curvature of photographic space was $K=-0.2$ and the value of σ was $\sigma=150$. For the comparison the value of the curvature of visual space was $K=-0.85$ and $\sigma=375$. The result suggested that photographic space was hyperbolic as same as visual space (Blank, 1961; Indow, 1979, 2004; Luneburg, 1947). However, the absolute value of the curvature of photographic space was smaller than that of visual space. Further, the value of

σ of photographic space was smaller than that of visual space. Previous studies showed that as the value of σ was larger, depth perception became better (Indow and Watanabe, 1984). The value of σ of photographic space was much smaller than that of visual space. The value of σ suggested that photographic space was narrower than visual space. That is, depth perception in photographic space was worse than in visual space.

The present study gave the additional evidence that photographic space was hyperbolic. But, the value of the curvature was close to 0. In the case of visual space, the participants observed real 3 dimensional physical space. However, in the case of photographic space, the participants observed two dimensional physical space where three dimensional physical space was projected. This might be one reason that the value of the curvature of photographic space became smaller. The present study gave the method to estimate the value of the curvature of photographic space. This attempt gives a large possibility to compare the various spaces. For example, it might be possible to compare visual space and stereoscopic space by estimating of the value of the curvature. If stereoscopic space really produces three dimensional visual space, the values of the curvature K and σ are close to those of visual space. Finch (1977) showed that hyperbolic perspective produced better depth than the two-point perspective. This evidence is also related to the geometry of photographic space.

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