The Notion of Logical Rejection and a Proof of Ł-decidability of the Classical Propositional Logic

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I. A historical remark on the notion of 'rejection'

Because the notion of logical 'rejection' is not very familiar, I should like to begin this short article with a small historical remark on it.

As is well known, the notion of 'assertion' was introduced into logic by G. Frege for the first time in his 'Begriffsschrift' 1879. There he introduced the symbol '$\vdash$' for assertion, but it owed a philosophical meaning as a kind of 'last predicate' which could not be negated[1], so that such an expression as following was not considerable for him:

$$\neg \vdash \phi,$$

where ϕ is a proposition and $\neg$ is the symbol for meta-negation.

Though in the course of time, Frege has abandoned the interpretation of the assertion symbol as predicate, he inclined more to understand it as a symbol corresponding to an 'act of judgement[2]', it was impossible for him to put a negation-symbol before the assertion symbol throughout his life.

The assertion symbol was taken over by Whitehead and Russell, and used in 'Principia Mathematica' but since they used the assertion symbol almost in the same sense as Frege did[3], to negate the assertion of a proposition was not considerable, either.

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In their times when logic took the first step into its new stage, so good a philosophical meaning was imposed on the assertion symbol that it was impossible to negate it, and the metalogical treatment of deductive systems lacked, without which the notion of logical 'rejection' was impossible.

The notion of 'rejection' is, roughly speaking, the negation of assertion, while it will take its own form when considered with respect to its logical status in deductive systems, esp. in connection with the set of theorems in them.

In order to avoid a careless mistake, I should like to make a remark that there is a difference between the notion of 'rejection' and the notion of 'assertion of negation'. To reject a proposition ϕ in a deductive system means the same as not to accept ϕ as a thesis in the system, i.e. to negate the assertion of ϕ in the system. To assert the negation of ϕ is to accept $N\phi$, N being the symbol for negation on the object level[4], in the system. For example, a single letter 'p' is rejected, because '$\vdash p$' is not the case, and for the same reason, Np is also rejected[5].

The notion of 'rejection' was for the first time introduced into the stock of logical terminologies by J. Łukasiewicz in his 'Two-Valued Logic' in 1920, saying that he owed this notion to F. Brentano[6]. Though he there introduced this notion, he used it only to express whether a certain formula was asserted in the 'Two-Valued Logic' or rejected in it, and did not yet treat the relation between these two notions. To say more exactly, he used there the notion of 'rejection' only to express the relations between the Truth (1) and the False (0), for example:

$$U\!:1, \qquad N\!:0, \qquad N\!:1<0$$ [7],

where 'U' is for 'assertion', and 'N' for 'rejection', and read the first 'I assert truth', while the second 'I reject falsehood', the third 'I reject that truth implies falsehood'. From the axiomatic point of view, the notion of 'rejection' introduced here had little meaning.

It was in 1939 that he introduced this notion as an indispensable one in order to systematize logical systems. In that year, he held a lecture on Aristotle's syllogistic[8], stating the reason why it was necessary for him to introduce the notion of 'rejection' into his system of 'Aristotelian Syllogistic'.

Stating the logical status of 'rejection' in aristotelian syllogistic as following:

> Aristotle in his systematic investigation of syllogistic forms not only proves the true but also rejects the false ones.... It consists in reducing some false syllogistic forms to other ones, the errouneousness of which is already shown[9].

he proposes an axiomatic way of treatment of 'rejected propositions' as follows:

> No syllogism rejected by Aristotle follows from the above axioms and rules of inference but they do not entail, too, erroneousness of rejected syllogisms. In order to solve this problem the system should be extended in a way. The author has chosen here the following way which has not been used in deductive systems so far. *Besides asserted propositions in the system he names propositions which are rejected in the system and rejects axiomatically some of them and reduces the others to propositions which are already rejected by means of special rules of rejection.* (italicized by the present author)[12].

After stating this, he enumerates two special rejected axioms for aristotelian syllogism, and puts two rules of rejection.

1. the rule of rejection by substitution
   any expression can be rejected if its substitution implies a rejected expression.
2. the rule of rejection by detachment:
   if the implication $C\alpha\beta$ is asserted and the consequent β

is rejected, then the antecedent α can be rejected[11].

Rewriting in symbolic form, we obtain:

$$r_1^+:\ \vdash C\alpha\beta,\quad \dashv\beta \longrightarrow \dashv\alpha$$

$$r_2^-:\ \vdash\phi(p/\phi_1) \longrightarrow \dashv\phi(p).$$

In 1951, he published a book under the title 'Aristotle's Syllogistic from the Standpoint of Modern Formal Logic', where he repeated almost the same as was quoted above[12].

From what was explained above, the importance of the role of 'rejection' in the framework of Łukasiewicz may be clear. He has introduced the notion in order to make aristotelian syllogistic 'decidable' in his sense. He used this terminology in his 'O Syllogistyce Arystotelesa' 1939, in II 4 in the sense that amounted to:

> a system is decidable iff all the propositions which are not theses[13] are rejectable on the ground of the axiomatically rejected propositions by means of (positive) rules and rejective rules.

Let us call this kind of 'decidability' Ł-decidability according to Prof. Jerzy Słupecki[14].

But as Łukasiewicz states in the same place, the system of Aristotelian syllogistic is not Ł-decidable, as far as we do not add a special rule for rejection found by Prof. Słupecki[15].

But if we add this rule of rejection special for Aristotelian Syllogistic, then this system becomes Ł-decidable, and as he states:

> after added this rule, every well-formed formula of the system is either proved or demolished, i. e. either asserted or rejected.

He regards this rule as the most important discovery which was ever done from the time of Aristoteles in the field of syllogistic. Even he maintains at the end of the article:

> Herewith we can hold the investingation of the Aristotelian Syllogistic as completed in some sense[17].

Now it may be already clear why Łukasiewicz introduced such a notion as 'rejection'. Simply to say, logic is not only concerned with proof of theses, but also with such a kind of proof as proves that a formula is not a thesis, and the second kind of proof should be performed also as the first kind of proof is performed, i. e. axiomatically and completely in the sense that the given axiomatic method produces all the formulae which are not thesis of the deductive system[18].

II. General remarks on the deductive systems with rejected axioms and rejective rules

Every meaningful system has its rejected part, for otherwise it would be contradictory.

Let us take a deductive system $\mathcal{L}$ with L as the set of its well-formed formulae, Ax^+ as the set of positive axioms and R^+ as the set of its positive rules of inference.

Let $\mathcal{L}r$ be the deductive system obtained from $\mathcal{L}$ by supplying it with Ax^-, a set of rejected axioms, and R^-, a set of rules of rejection.

Let $Cn(R^*, X)$ be the set of all well-formed formulae which are obtainable from X, a set of wffs, by means of the rules in R^*, a set of rules.

Let L^+ be $Cn(R^+, Ax^+)$, and let L^- be $Cn(R, Ax)-Cn(R^+, Ax^+)$, where $R=R^+ \cup R^-$, $Ax=Ax^+ \cup Ax^-$. L^+ is the set of theses in the system $\mathcal{L}r$, while L^- is the set of rejected theses.

Now I shall give some definitions.

Def. 1. $\vdash\phi$ iff $\phi \in L^+$

Def. 2. $\dashv\phi$ iff $\phi \in L^-$

Def. 3. $\mathcal{L}r$ is Ł-decidable iff $\mathrm{L}=\mathrm{L}^+ \cup \mathrm{L}^-$.

Def. 4. $\mathcal{L}r$ is Ł-consistent iff $\mathrm{L}^- \cap \mathrm{L}^+=0$.

Def. 5. $\mathcal{L}r$ is Ł-complete iff $\mathcal{L}r$ is Ł-decidable and Ł-consistent.

If R^+ is given, then for each member of it, one can find a counter-rule of rejection. For, if X is a set of rejected wffs of $\mathcal{L}r$, and the following is the case,

$$\phi_1 \in Cn(\{r\}, \{\phi_2\}), \qquad \text{for some } \phi_1 \in X,$$

then this means that ϕ_2 is rejected on X by means of r. Defining as follows, we obtain the dual rejection rule 'r^-' for r:

Def. 6. r^- is r-dual iff the following is the case:
$\alpha \in Cn(\{r^-\}, X)$ iff $\exists \beta \in X(\beta \in Cn(\{r\}, \{\alpha\})$.

Def. 7. R^- is R-dual iff if every r^- in R$^-$ is r-dual for some r in R, and for every r in R, there is some r^- for which is r-dual.

III. A proof of Ł-completeness of the Classical Propositional Logic supplied with a rejected axiom and rules of rejection

Let PC the normal classical propositional logic, with 'C' and 'N' for primitive functors. As axioms we take the followings:

$$\vdash CpCNpq$$
$$\vdash CCNppp$$
$$\vdash CCpqCCqrCpr$$ [19].

The set of them will be denoted by Ax^+. As rules of inference, we take the followings:

$$r^+: \ \vdash C\alpha\beta, \ \vdash \alpha \longrightarrow \vdash \beta$$

$$r^+: \ \vdash \phi(p) \longrightarrow \vdash \phi(p/\alpha).$$

The set of them will be denoted by R^+.

Now let us consider the way in which PC will be extended to a Ł-complete PCr. On p. 109 of his 'Aristotelian Syllogistic', he takes as rejected axiom the variable 'p' and as rules of rejection the duals of r_1^+ and r_2^+, i.e. the following:

$$r_1^-: \ \vdash C\alpha\beta, \quad \dashv\beta \longrightarrow \dashv\alpha$$

$$r_2^-: \ \dashv\phi(p/\alpha) \longrightarrow \dashv\phi(p).$$

There he used them for the explanation of the notion of 'deductive equivalence', but did not give any proof of Ł-decidability of the classical propositional calculus. I shall now give a proof of Ł-decidability of it.

Let Ax^- be the set containing only '$\dashv p$', and let R^- be the set containing r_1^- and r_2^-.

Now let *PCr* be the system with L as its set of wffs and Ax^+, Ax^-, R^+, R^-.

In order to make the notion of rejective proof of ϕ easy to formulate, at first we must extend the notion of wffs of *PCr*. I shall give the definition in a morphological manner.

$\mathcal{F}$ is a wff of *PCr* iff $\mathcal{F}$ is of one of the following forms:

1. $\vdash\phi$,
2. $\dashv\phi$,
3. $\phi\vdash\phi_1$,

where ϕ, ϕ_1 are wffs of *PC*.

If ϕ_1 is a substitution instance of ϕ, then I shall call it a 'substitutional axiom'.

Now the notion of rejective proof of a wff 'ϕ' of *PC*, will be given as follows:

a sequence $\{c_i\}_{i=1,\cdots,m}$ is a rejective proof of ϕ iff each c_i is a wff of *PCr* and

1. $c_m = \dashv\phi$,

and for each c_i,

2. c_i is in Ax^+,
3. c_i is in Ax^-,
4. c_i is obtained by r_1^+ from c_j, c_k, where $j, k<i$, or

5. c_i is obtained by r_2^+ from c_j, where $j<i$, or
6. c_i is obtained by r_1^- from c_j, c_k, where $j, k<i$, or
7. c_i is obtained by r_2^{-*} from c_j, c_k, where $j, k<i$, and r_2^{-*} is the rule which has the following form:

$$r_2^- \quad \phi \vdash \phi_1, \ \dashv \phi_1 \longrightarrow \dashv \phi,$$

where $\phi \vdash \phi_1$ is a substitutional axiom.

Lemma 1. *PCr* is Ł-decidable.

Let $\phi(p_1, \cdots, p_n)$ be a wff containing only $p_1, \cdots, p_n$ as propositional variables.

Proof of the lemma 1.

A:	1.	$\neg\vdash \phi(p_1, \cdots, p_n)$	(Sup.)
	2.	$Tp = Cpp$	(Def.)
	3.	$Fp = NCpp$	(Def.)
	4.	$\vdash Cpp$	(th. in PC)
	5.	$\vdash Tp$	(2, 4)
	6.	$\vdash CpCNpq$	(th. in PC)
	7.	$\vdash CCppCNCppp$	(1.1, r_2^+, p/Cpp, p/q)
	8.	$\vdash CNCppp$	(4, 1.2, r_1^+)
	9.	$\dashv p$	(Ax^-)
	10.	$\dashv NCpp$	(8, 9, r_1^-)
	11.	$\dashv Fp$	(3, 10)
B:	12.	there exists at least one Tp-Fp combination for $p_1, \cdots, p_n$ which reduces $\phi(p_1, \cdots, p_n)$ to a contradictory one. I give to this combination the name θ, and write the result by the substitution with θ for $(p_1, \cdots, p_n)$ $\phi(\theta)$.	
	13.	$\phi(\theta)$ consists out of only 'p' and 'C, N', so rewrite $\phi(\theta)$ as $\phi^*(p)$.	
C:	14.	for all wffs α, $\phi^*(\alpha)$ is a contradictory one.	
	15.	$\vdash E\phi^*(p)Fp$	(14)
	16.	$\vdash NFp$	(th. in PC)
	17.	$\vdash EN\phi^*(p)NFp$	(15)

18. $\vdash N\phi^*(p)$ (16, 17)
19. $\vdash N\phi(\theta)$ (18, 13)

D: 20. $\vdash CNpCpq$ (th. in *PC*)
21. $\vdash CN\phi(\theta)C\phi(\theta)p$ (20, r_2^+, $p/\phi(\theta)$, q/p)
22. $\vdash N\phi(\theta)$ (19)
23. $\vdash C\phi(\theta)p$ (21, 22, r_1^+)
24. $\dashv\phi(\theta)$ (23, Ax^-, r_1^-)

E: 25. $\dashv\phi(p_1, \cdots, p_n)$ (24, r_2^-)

we now obtained:

$$\text{if} \quad \neg\vdash\phi(p_1, \cdots, p_n) \longrightarrow \dashv\phi(p_1 \ldots, p_n)$$

Lemma 2. *PCr* is Ł-decidable.

Proof:

1. $\dashv\phi$ (Sup.)
2. Rej$(n)=\{\phi : \phi$ is a formula of which the proof of rejection contains n-times use of $R^-\}$. (Def.)
3. $\phi \in \text{Rej}(1) \longrightarrow \neg\vdash\phi$

proof:

(a) case 1: $\phi \in Cn(\{r^-\}, \{c_j, c_k\})$, for some $j, k<i$.

1. let $\{c_i\}_m$ be the rejective proof of ϕ.
2. $j, k<m$

$$c_j = \vdash C\phi p, \quad c_k = \dashv p.$$

3. $c_m = \dashv\phi$
4. $\neg\vdash p$ (obvious)
5. $\vdash C\phi p$ (2)

   1.1 $\vdash\phi$ (aux, asp)
   1.2 $\vdash p$ (5, 1.1, r^+)
   1.3 contr. (4, 1.2)

6. $\neg\vdash\phi$. (1.1, 1.3)

(b) case 2: $\phi \in Cn(\{r_2^{-*}\}, \{c_j\})$, for some $j<m$

1. let $\{c_i\}_m$ be the rejective proof of ϕ.

2. $c_m = \dashv\phi$
3. $c_j = \phi\vdash\phi_1$, for some ϕ_1.
   But ϕ_1 is a rejected wff, so that it must be 'p', hence ϕ is also a single letter 'q'.
4. $\neg\vdash q$. (obvious)
5. $\neg\vdash\phi$. (4)

4. $\phi \in \mathrm{Rej}(k) \longrightarrow \neg\vdash\phi$, for all $k \leq n$. (Ind. Hyp.)
5. $\phi \in \mathrm{Rej}(n+1) \longrightarrow \neg\vdash\phi$.

proof:

(a) case 1: the last step to reach ϕ is carried out by r_1^-.

1. let $\{c_i\}_m$ be the rejective proof for ϕ.
2. $c_m = \dashv\phi$.
3. for some $i, j < m$ and ϕ_1,

$$c_i = \vdash C\phi\phi_1, \quad c_j = \dashv\phi_1.$$

4. $\vdash C\phi\phi_1$. (3)
5. $\phi_1 \in \mathrm{Rej}(n)$.
6. $\neg\vdash\phi_1$. (5, Ind. Hyp.)
7. $\neg\vdash\phi$. (4, 6)

(b) case 2. the last step to reach ϕ is carried out by r_2^-*.

1. let $\{c_i\}_m$ be the rejective proof of ϕ.
2. $c_m = \dashv\phi$.
3. for some $i, j < m$ and ϕ_1,

$$c_i = \phi\vdash\phi_1. \text{ and } \dashv\phi_1.$$

4. $\neg\vdash\phi_1$. (Ind. Hyp., 3).
5. $\neg\vdash\phi$. (3, 4)

6. $\dashv\phi \longrightarrow \neg\vdash\phi$. (6, 7)

Theorem. *PCr* is Ł-complete.

Proof: Lemma 1, 2, and by definition.

REMARK

1. cf. Frege [1], § 2.
2. This tendency is to find from the beginning of his philosophical activity. cf Frege, [2] p. 2, [5] p. 201*, p. 62, [7] p. 146. (the page numbers with * are from Nachlass, without * are from the originals.) On this, confer Waragai [1].
3. Whitehead and Russell [1], p. 8.
4. I shall use for '*C*', '*N*' for implication and negation on the object-level.
5. cf. Łukasiewicz [3], p. 95.
6. cf. also Łukasiewicz [3], p. 94. Also Frege [7].
7. There he uses '$<$' for implication.
8. Łukasiewicz [1]. This is also to find in Słupecki [1].
9. Translated in Słupecki-Bryll-Skardowska [1].
   cf. also Łukasiewicz [3], p. 67.
10. Translated in Słupecki-Bryll-Skardowska [1].
11. Translated in Słupecki-Bryll-Skardowska [1].
12. cf. § 20.
13. ϕ is a thesis iff ϕ is a wff and an axiom or a theorem.
14. cf. p. 76, Słupecki-Bryll-Skardowska [1].
15. Łukasiewicz [2], II. 4. (f). cf. also Słupecki [1], p. 226. cf also Łukasiewicz [3], § 30, p. 104.
16. Łukasiewicz [2], II. 4. p. 226. Translated by the present author.
17. Łukasiewicz [2], Słupecki [1], p. 227. Translated by the present author.
18. cf. Łukasiewicz [4], [5], [6].
19. This system is complete, and was found by Łukasiewicz in 1929.
20. This rule is essentially the same as r_2^-. But without this notion, the rejective proof will be very clumsy as in Łukasiewicz.

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