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| Abstract | The Fisher-Freidman definition classifies commodities as substitutable of complementary depending on the global shape of indifference maps. It thus requires the specification of the global mathematical properties of any model used to illustrate this definition. We utilized the simple model of a direct utility function of quadratic homogeneous form. As shown in Sections II-V, the substitution effect--theAllen elasticity of substitution-is always non-negative with this specification.Therefore, by the Hicks-Allen-Shultz definition, this model applies only to competing commodities and not to complementary commodities, a conclusion that is independent of the number of commodities. |
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# A Reinterpretation of the Fisher-Friedman Definition of Complementarity※


by

Kotaro Tsujimura and Sakiko Tsuzuki



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# A Reinterpretation of the Fisher-Friedman Definition of Complementarity※

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Contents

## I. Introduction--It's not the Satiation but the Proportion

Milton Friedman's definition of complementarity is best known through a quotation in Henry Schultz's seminal work, *Theory and Measurement of Demand* (1938).[1] Unfortunately, Schultz mistakenly classified Friedman's definition as a variation of that of Johnson-Allen, and Friedman's insightful approach has largely been lost to subsequent generations of scholars. Referring back to Friedman's original paper "The Fitting of Indifference Curves as a Method of Deriving Statistical Demand Functions" (Dec.,1933),[2] it becomes apparent that Shultz's summary deletes the following three extremely important statements from Friedman's original definition.

**Friedman I:** " Fisher took, I think, the first step in the direction of a precise definition of completing and competing commodities when he clarified the meaning of absolutely completing and absolutely competing commodities. …These definitions are entirely independent of the assumpton that utility is measurable.
There still remains the problem of defining the intermediate cases. I suggest that …"[3]

In other words, rather than the Johnson-Allen approach, Friedman attempted to complete Irving Fisher's definition of complementarity by bridging the gap between the two poles of perfect complements and perfect subsitutes, creating a definition continuous across commodities.

Irving Fisher himself had his own conception of the intermediate cases. He wrote, in his *Mathematical Investigations in the Theory of Value and Prices* (1892) that

"If the articles are completing (fig 22), a change in price will not cause the tangent line to very greatly alter the proportion of the consumption of the articles…; if substitutes (fig 20), a slight relative change in price will cause an enormous change in the proportion used."[4]

Fisher used a family of confocal ellipses to portray the indifference maps, if we may speak loosely, in the figures referred to above. The ellipses in the figure for competing commodities have a common major axis inclined downward to the right, while the ellipses for complementary commodities have a common major axis inclined upward to the right. However, it is extremely difficult to assign any economic meaning to the slope of the major axis, and it was to overcome this problem that Friedman most likely introduced the concept of "satiation curves" in his 1933 paper. These "satiation curves", which crystallized Fisher's basic concepts, made economic sense and appeared to be a step forward. Unfortunately, Friedman also inherited from Fisher the concept of absolute satiation, with the common center of the elliptical indifference curves denoting the "point of total satiaton".

This unattractive concept of total satiation in fact has nothing to do with the essence of the Fisher-Friedman definition of complementarity. Total satiation is a concept extrinsic not intrinsic to their approach. In our view, it is not the concept of "satiation" but that of "the limit of proportion" is what is really relevant to the FF definition. Choosing a hyperbolic shape for the indifference curves rather than an elliptical shape eliminates the concept of satiation in one stroke but keeps the essence of the FF definition unchanged. With hyperbolically shaped indifference curves, the concept of satiation is replaced by that of the limit of proportion, allowing us to reconstruct the Fisher-Friedman definition. Assuming both prices and quantities are non-negative, Fisher's definition can be restated as follows. In the case of complementary commodities, while no limit of proportion exists between prices, there exist both lower and upper limits of proportion between quantities of commodities, i.e.

(I.1.1) $0 \le p_i/p_0 \le \infty,\ 0 < (q_j/q_i)_{\min} \le q_j/q_i \le (q_j/q_i)_{\max} < \infty;\ i,j = 1,\cdots,n,\ i \ne j.$

On the other hand, in the case of competing commodities, there exist both lower and upper limits of proportion between prices but no limit of proportion between quantities of commodities.

(I.1.2) $0 < (p_i/p_j)_{\min} \le p_i/p_j \le (p_i/p_j)_{\max} < \infty,\ 0 \le q_j/q_i \le \infty.$

Further, following Friedman, we may introduce the relationship between the price ratio $p_i/p_j$ and the quantity ratio $q_j/q_i$ as follows: For complementary commodities, the lower and upper limits of proportion between quantities $(q_j/q_i)_{\min}$ and $(q_j/q_i)_{\max}$ correspond to the extreme situation of relative prices $(p_i/p_j) \to 0$ and $(p_i/p_j) \to \infty$ respectively. In short,

(I.2.1) $\lim\limits_{(p_i/p_j)\to 0} \dfrac{q_j}{q_i} = \left(\dfrac{q_j}{q_i}\right)_{\min}$ ; $\lim\limits_{(p_i/p_j)\to \infty} \dfrac{q_j}{q_i} = \left(\dfrac{q_j}{q_i}\right)_{\max}$

For competing commodities, the lower and upper limits of proportion between prices $(p_i/p_j)_{\min}$ and $(p_i/p_j)_{\max}$ correspond to the extreme situation of quantity ratios $(q_j/q_i) \to 0$ and $(q_j/q_i) \to \infty$ respectively. That is,

$$\text{(I.2.2)} \quad \lim_{(q_j/q_i)\to 0} \frac{p_i}{p_j} = \left(\frac{p_i}{p_j}\right)_{\min} \quad ; \quad \lim_{(q_j/q_i)\to \infty} \frac{p_i}{p_j} = \left(\frac{p_i}{p_j}\right)_{\max}$$

Hence, there is a remarkable and appealing symmetry in the reversal of the roles of prices and quantities between the definitions of complementarity and substitutability.

Friedman himself was extremely clear about what the above definition implied about the cross price effects on demand between commodities. He concluded that

**Friedman II:** "If the other goods are closely related and the relationship is one of competition, a rise in the price of the other goods will result in the demand schedule for the given commodity being raised; but if the closely related goods are complementary a rise in the price of one of them will result in the demand schedule being lowered."[5]

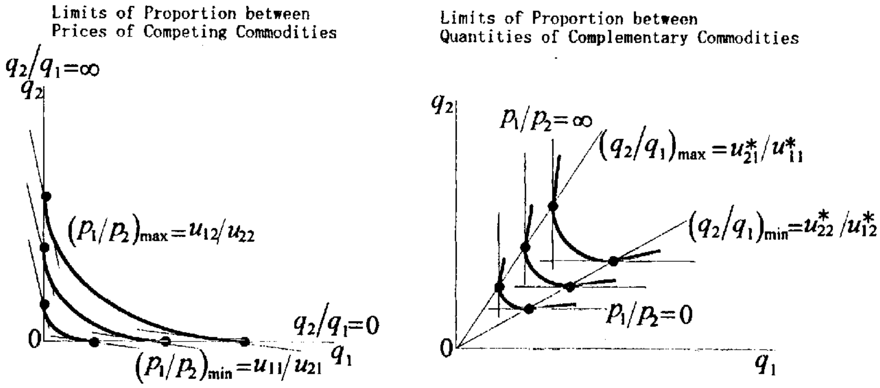
Thus, the Fisher-Friedman definition has implied since its inception positive cross price effects between competing commodities and negative cross price effects between complementary commodities.

Friedman also had a very clear idea about separability, apparent from the third quote below.

**Friedman III:** "If the two commodities are truly independent of the other commodities, the same set of indifference curves will be obtained no matter at what level the quantities of the other commodities are fixed."[6]

Hence, Friedman indicates that the concept of either complementary commodities or competing commodities should be applied within separable commodity groups, with separability defined as the independence of marginal rates of substitution between commodities in a group and quantities of commodities outside the group.

Fig.1 Indefference Maps: Visual Image of
the Fisher-Friedman Definition of complementarity Reinterpreted




## II. Duality

Equations (I.2.1) of the previous section for complementary commodities assume a function of the type $(q_j/q_i)=f(p_i/p_j)$ where $(p_i/p_j)$ plays the role of independent variables and $(q_j/q_i)$ the role of dependent variables. Conversely, equations (I.2.2) for competing commodities imply a function $(p_i/p_j)=g(q_j/q_i)$ where $(q_j/q_i)$ are the independent variables and $(p_i/p_j)$are the dependent variables. This reversal between prices and quantities as both dependent and independent variables immediatly suggests that the Fisher-Friedman definition of complementary and cometing commodities is very closely associated with the dual expression of the **direct** utility function $u(q)$ and the **indirect** utility function $v(p)$.[7]

Assume for a separable group of n commodities a **direct** utility function of quadratic homogeneous form.

(II.1) $u=\sum_i\sum_j u_{ij}q_iq_j$; $i,j=1,\cdots,n$, $u_{ij}=u_{ji}$.

Then, the **direct** marginal utility is

(II.2) $\dfrac{\partial u}{\partial q_i}=2\sum_j u_{ij}q_j$ .

Here, we define the **direct** prime form of the relative marginal utility (PFORMU)[8] as

$$\text{(II. 3)}\quad u_i = \frac{1}{2}\frac{\partial u}{\partial q_i} = \sum_j u_{ij} q_j$$

and using this notation we have, by Euler's theorem for homogeneous functions,

$$\text{(II. 1)'}\quad u = \sum_i u_i q_i \ .$$

The **direct** marginal rate of substitution is

$$\text{(II. 4)}\quad \frac{u_i}{u_j} = \frac{\sum_k u_{ik} q_k}{\sum_k u_{jk} q_k}$$

and we have the **direct** equilibrium equation

$$\text{(II.5)}\quad \frac{p_i}{p_j} = \frac{u_i}{u_j} = \frac{\sum_k u_{ik} q_k}{\sum_k u_{jk} q_k} \ .$$

From $u_i = p_i \lambda$ and the budget equation $\sum_i p_i q_i = m$ we deduce

$$\text{(II.6)}\quad \begin{bmatrix} u_{11} & \cdots & u_{1n} & -p_1 \\ \vdots & \cdots & \vdots & \vdots \\ u_{n1} & \cdots & u_{nn} & -p_n \\ p_1 & \cdots & p_n & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ m \end{bmatrix}$$

and define the **direct** determinant of consumer demand D as,

$$\text{(II.7)}\quad D = \begin{vmatrix} u_{11} & \cdots & u_{1n} & -p_1 \\ \vdots & \cdots & \vdots & \vdots \\ u_{n1} & \cdots & u_{nn} & -p_n \\ p_1 & \cdots & p_n & 0 \end{vmatrix}, \quad U = \begin{vmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & \cdots & \vdots \\ u_{n1} & \cdots & u_{nn} \end{vmatrix} .$$

Writing the cofactor of the $(n+1, j)$ element of $D$ as $D_{n+1,j}$, and the cofactor of $u_{ij}$ in $U$

as $u_{ij}^*$ , we have

(II.8) $D_{n+1,j} = \sum_k u_{jk}^* p_k$, $D_{n+1,n+1} = U$, and $D = \sum_j p_j D_{n+1,j}$ .

The **direct** demand equations are

(II.9) $q_j = \frac{D_{n+1,j}}{D} m$ , $\lambda = \frac{U}{D} m$ .

It must be noted, here, that $D$ is a quadratic homogeneous form of $p_i$

(II.10) $D = \sum_j p_j D_{n+1,j} = \sum_j \sum_k u_{jk}^* p_j p_k$ .

Inserting equations (II.8) and (II.9) into the **direct** utility function (II.1), we derive the **indirect** utility function

(II.1*) $u(q) = v(p) = Um^2/D.$

If we assume the income $m$ to be a constant, the **indirect** utility function $v(p)$ is homogeneous of degree (-2) with respect to prices $p_i$, in contrast to the second degree homogeneity of the **direct** utility function $u(q)$ with respect to $q_i$. From (II.1*) the **indirect** marginal utility can be expressed as

(II.2*) $\frac{\partial v}{\partial p_i} = \left(\frac{-Um^2}{D^2}\right)\frac{\partial D}{\partial p_i}$ , $\frac{\partial v}{\partial p_j} = \left(\frac{-Um^2}{D^2}\right)\frac{\partial D}{\partial p_j}$ .

Canceling the common factor $\left(-Um^2/D^2\right)$, we obtain the **indirect** PFORMU

(II.3*) $v_j = \frac{1}{2}\frac{\partial D}{\partial p_j} = D_{n+1,j} = \sum_k u_{jk}^* p_k$ .

The **indirect** marginal rate of substitution is

(II.4*) $\frac{\partial v}{\partial p_i} \bigg/ \frac{\partial v}{\partial p_j} = v_i / v_j = \sum_k u_{ik}^* p_k \bigg/ \sum_k u_{jk}^* p_k$ .

The **indirect** utility function (II.1*) also implies that

(II.11) $\frac{\partial v}{\partial m} = 2\frac{U}{D}m = 2\lambda$ .

We also have as Roy's identity

(II.12) $q_i = -\frac{\partial v}{\partial p_i}\bigg/\frac{\partial v}{\partial m}$ .

Inserting (II.2*), (II.3*) and (II.11) into equation (II.12) we obtain

(II.13) $q_i = \frac{D_{n+1,i}}{D}m$

which is coincident with the **direct** demand function (II.9). Hence, we have the **indirect** equilibrium equation

(II.5*) $\frac{q_j}{q_i} = \frac{D_{n+1,j}}{D_{n+1,i}} = \frac{\sum_k u^*_{jk} p_k}{\sum_k u^*_{ik} p_k} = \frac{v_j}{v_i}$

which is the counterpart of equation (II.5) of the **direct** model. From $v_i/q_i = \lambda^*$ , $v_i - q_i\lambda^* = o$ and $\sum_i p_i q_i = m$ the **indirect** model of consumer demand is written as

(II.6*) $$\begin{bmatrix} u^*_{11} & \cdots & u^*_{1n} & -q_1 \\ \vdots & \cdots & \vdots & \vdots \\ u^*_{n1} & \cdots & u^*_{nn} & -q_n \\ q_1 & \cdots & q_n & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ p_n \\ \lambda^* \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ m \end{bmatrix},$$

and the **indirect** determinant of consumer demand becomes

(II.7*) $$D^* = \begin{vmatrix} u^*_{11} & \cdots & u^*_{1n} & -q_1 \\ \vdots & \cdots & \vdots & \vdots \\ u^*_{n1} & \cdots & u^*_{nn} & -q_n \\ q_1 & \cdots & q_n & 0 \end{vmatrix}, \quad U^* = \begin{vmatrix} u^*_{11} & \cdots & u^*_{1n} \\ \vdots & \cdots & \vdots \\ u^*_{n1} & \cdots & u^*_{nn} \end{vmatrix}.$$

Writing the cofactor of the $(n+1,j)$ element of $D^*$ as $D^*_{n+1,j}$ , and the cofactor of $u^*_{ij}$ in $U^*$ as $u^{**}_{ij}$ , we have

$$(\text{II.8}^*)\quad D^*_{n+1,j}=\sum_k u^{**}_{jk}q_k\,,\ D^*_{n+1,n+1}=U^* \text{ and } D^*=\sum_j q_j D^*_{n+1,j}=\sum_j\sum_k u^{**}_{jk}q_j q_k\,.$$

Hence, the **indirect** demand equations are

$$(\text{II.9}^*)\quad p_j=\frac{D^*_{n+1,j}}{D^*}m \text{ and } \lambda^*=\frac{U^*}{D^*}m\ .$$

Equation (II.8*) and (II.9*) include cofactors $u^{**}_{ij}$ of elements $u^*_{ij}$ of the adjugate determinant $U^*$ of $U$. Now, making use of Cauchy's theorem[9] on the value of the adjugate determinant that

$$(\text{II.14})\quad \text{adj }U=U^*=U^{n-1}=UU^{n-2}$$

and the definition that

$$(\text{II.15})\quad U=\sum_{j=1}^{m}u_{ij}u^*_{ij} \text{ and } U^*=\sum_{j=1}^{n}u^*_{ij}u^{**}_{ij}\ ,$$

we obtain $U^*=\sum_{j=1}^{n}u^*_{ij}u^{**}_{ij}=UU^{n-2}=U^{n-2}\sum_{j=1}^{n}u_{ij}u^*_{ij}$ which can be rewritten as

$$(\text{II.16})\quad \sum_{j=1}^{n}u^*_{ij}\left(u^{**}_{ij}-U^{n-2}u^*_{ij}\right)=0\ .$$

In order for equation (II.16) to hold at any arbitrary value of $u^*_{ij}$ but zero we must have

$$(\text{II.17})\quad u^{**}_{ij}=u_{ij}U^{n-2}\ ,$$

as a corollary to the Cauchy's theorem above. Inserting (II.17) into equation (II.8*) we obtain

$$(\text{II.8}^*.1)\quad D^*_{n+1,j}=\sum_k u^{**}_{jk}q_k=U^{n-2}\sum_k u_{jk}q_k \text{ and}$$

(II. 8*.2) $D^* = \sum_j \sum_k u_{jk}^{**} q_j q_k = U^{n-2} \sum_j \sum_k u_{jk} q_j q_k$ .

Inserting (II.8*.1) and (II. 8*.2) into equation (II.9*) gives

(II.9*.1) $p_j = \frac{D^*_{n+1,j}}{D^*} m = \frac{\sum_k u_{jk} q_k}{\sum_j \sum_k u_{jk} q_j q_k} m$ and

(II. 9*.2) $\lambda^* = \frac{U^*}{D^*} m = \frac{\sum_k u_{jk} u^*_{jk}}{\sum_j \sum_k u_{jk} q_j q_k} m$

where, by equation (II.1), the denominator coincides with the **direct** utility function $u(q)$. By equations (II.3) and (II.7) the numerator of (II.9*.1) coincides with the **direct** PFORMU $u_j$ , and the numerator of (II.9*.2) coincides with U of equation (II.7). Thus we can rewrite the **indirect** demand functions (II. 9*.1) and (II.9*.2) as

(II.9*)' $p_j = \frac{u_j}{u} m$ , $\lambda^* = \frac{U}{u} m$ .

Comparing the **direct** demand functions (II.9) with the **indirect** demand functions (II. 9*) we observe that the parameters $u^*_{jk}$ and variables $p_j$ of the former correspond to the parameters $u_{jk}$ and variables $q_j$ of the latter. Thus, they constitute strictly symmetrical factor-reversal forms.

## III. Specified Definition of Complementarity

In the case of the two-commodity model the **direct** equilibrium equation (II.5) of the previous section becomes,

(III.1) $\frac{p_1}{p_2} = \frac{u_{11} q_1 + u_{12} q_2}{u_{21} q_1 + u_{22} q_2}$ .

If we apply the FF definition of competing commodities of the form of (I.2.2) of Section I to this equation, we get on the $q_1$-axis where $q_2 = 0,\ q_1 > 0$, (III.2.1) $\left(p_1/p_2\right)_{\min} = \lim\limits_{q_2 \to 0} \dfrac{u_{11}q_1 + u_{12}q_2}{u_{21}q_1 + u_{22}q_2} = u_{11}/u_{21}$ ,

and on the $q_2$-axis where $q_1 = 0$, $q_2 > 0$ we have

(III.2.2) $\left(p_1/p_2\right)_{\max} = \lim\limits_{q_1 \to 0} \dfrac{u_{11}q_1 + u_{12}q_2}{u_{21}q_1 + u_{22}q_2} = u_{12}/u_{22}$ .

Since the price ratio must always be non-negative, $sgn\ u_{ii} = sgn\ u_{ij}$ holds in general. Further, since the marginal utility must be non-negative, the sign of both $u_{ij}$ and $u_{ii}$ must be positive.

Conditions (III.2.1) and (III.2.2) imply

(III.3.1) $u_{11}/u_{21} \le u_{12}/u_{22}$ i.e. $u_{11}u_{22} \le u_{12}^2$ and

(III.3.2) $u_{11}/u_{21} \le p_1/p_2 \le u_{12}/u_{22}$

in other words $0 \le u_{21}p_1 - u_{11}p_2$ , $0 \le u_{12}p_2 - u_{22}p_1$ .

For three competing commodities we have three **direct** equilibrium equations corresponding to $p_1/p_2$, $p_1/p_3$ and $p_2/p_3$. Let us take the first one for example,

(III.4.1) $\dfrac{p_1}{p_2} = \dfrac{u_{11}q_1 + u_{12}q_2 + u_{13}q_3}{u_{21}q_1 + u_{22}q_2 + u_{23}q_3}$ .

On the $q_1$-axis where $q_1 > 0$, $q_2 = q_3 = 0$, we obtain from (III.4.1),

(III.5.1) $\left(p_1/p_2\right)_{\min} = u_{11}/u_{21}$ .

On the $q_2$-axis where $q_2 > 0$, $q_1 = q_3 = 0$, we obtain from (III.4.1),

(III.5.2) $\left(p_1/p_2\right)_{\max} = u_{12}/u_{22}$ .

On the $q_3$-axis where $q_3 > 0$, $q_1 = q_2 = 0$, we obtain

(III.5.3) $\left(p_1/p_2\right)_{med} = u_{13}/u_{23}$ .

Inserting these results into

$$\text{(III.6)}\quad (p_1/p_2)_{\min} \le (p_1/p_2)_{med} \le (p_1/p_2)_{\max},$$

we obtain the inequalities,

$$\text{(III.7)}\quad u_{11}/u_{21} \le u_{13}/u_{23} \le u_{12}/u_{22}.$$

These inequalities can be generalized to the case of $n(\ge 2)$ competing commodities, and we obtain the following definition of competing commodities:

$$\text{(III.8)}\quad u_{ii}u_{jj} \le u_{ii}u_{ij} \le u_{ij}^2\ ;\ i,j,=1,\cdots,n,\ \ i \ne j\ ,$$

in general. Perfectly competing commodities are defined as

$$\text{(III.9)}\quad (p_i/p_j)_{\min} = (p_i/p_j)_{\max}$$

which implies that

$$\text{(III.10)}\quad u_{ii}u_{jj} = u_{ii}u_{jk} = u_{ij}u_{jk} = u_{ij}^2;\ i,j,k=1,\cdots,n,\ i \ne j,\ i \ne k,\ j \ne k\ .$$

This suggests that the opposite extreme of the weakest case of competing commodities may be defined as

$$\text{(III.11)}\quad u_{ii} = u_{jj} \to 0,\ 0 < u_{ij}^2\ .$$

In the case of two complementary commodities the **indirect** equilibrium equation (II.5*) of the previous section is written as

$$\text{(III.1*)}\quad \frac{q_2}{q_1} = \frac{u_{21}^{*}p_1 + u_{22}^{*}p_2}{u_{11}^{*}p_1 + u_{12}^{*}p_2}.$$

Applying the definition of complementarity in Section I to the above equation, we obtain for the case of $p_1 = 0$ and $p_2 > 0$

(III.2.1*) $(q_2/q_1)_{\min} = \lim_{p_1 \to 0} \frac{u^*_{21}p_1 + u^*_{22}p_2}{u^*_{11}p_1 + u^*_{12}p_2} = u^*_{22}/u^*_{12}$ ,

and for the case of $p_1 > 0$ and $p_2 = 0$

(III.2.2*) $(q_2/q_1)_{\max} = \lim_{p_2 \to 0} \frac{u^*_{21}p_1 + u^*_{22}p_2}{u^*_{11}p_1 + u^*_{12}p_2} = u^*_{21}/u^*_{11}$ .

Thus we obtain the inequalities

(III.3.1*) $u^*_{22}/u^*_{12} \le u^*_{21}/u^*_{11}$ i.e. $u^*_{11}u^*_{22} \le u^{*2}_{12}$ ,

(III.3.2*) $u^*_{22}/u^*_{12} \le q_2/q_1 \le u^*_{21}/u^*_{11}$ .

For three complementary commodities we have three **indirect** equilibrium equations corresponding to $q_2/q_1$, $q_3/q_1$ and $q_3/q_2$. Taking for example the first equation

(III.4.1*) $\frac{q_2}{q_1} = \frac{u^*_{21}p_1 + u^*_{22}p_2 + u^*_{23}p_3}{u^*_{11}p_1 + u^*_{12}p_2 + u^*_{13}p_3}$ ,

we obtain for the case of $p_1 = 0,\ p_2 > 0,\ p_3 = 0$

(III.5.1*) $(q_2/q_1)_{\min} = u^*_{22}/u^*_{12}$.

For the case of $p_1 > 0,\ p_2 = 0,\ p_3 = 0$ we obtain

(III.5.2*) $(q_2/q_1)_{\max} = u^*_{21}/u^*_{11}$ ,

and for the case of $p_1 = p_2 = 0,\ p_3 > 0$

(III.5.3*) $(q_2/q_1)_{med} = u^*_{23}/u^*_{13}$ .

Inserting these results into

(III.6*) $(q_2/q_1)_{\min} \le (q_2/q_1)_{med} \le (q_2/q_1)_{\max}$ , we obtain the inequalities,

$$(\text{III.7}^*)\ \ u_{22}^*/u_{12}^* \le u_{23}^*/u_{13}^* \le u_{21}^*/u_{11}^* \ .$$

Quite analogous to the case of competing commodities, in this case of complementary commodities the inequalities (III.7$^*$) can be generalized for a group of $n(\ge 2)$ commodities, providing the following definition of complementarity.

$$(\text{III.8}^*)\ \ u_{ii}^* u_{jj}^* \le u_{jj}^* u_{ik}^* \le u_{ij}^* u_{jk}^* \le u_{ij}^{*2}\ ,\ i,j,k = 1,\cdots,n; i \ne j, i \ne k, j \ne k\ .$$

Perfectly complementary commodities are defined as

$$(\text{III.9}^*)\ \ \left(q_j/q_i\right)_{\min} = \left(q_j/q_i\right)_{\max}$$

which implies that in this case

$$(\text{III.10}^*)\ \ u_{ii}^* u_{jj}^* = u_{ij}^{*2}\ .$$

This suggests that the opposite extreme of the weakest case of complementary commodities to be defined as

$$(\text{III.11}^*)\ \ u_{ii}^* u_{jj}^* \to 0,\ 0 < u_{ij}^{*2}\ .$$

## IV. Intuitive Elasticity of Substitution

We have the **direct** equilibrium equation (II.5) in Section II

$$(\text{IV.1})\ \ \frac{p_i}{p_j} = \frac{u_i}{u_j} = \frac{\sum_h u_{ih} q_h}{\sum_h u_{jh} q_h},\ h = 1,\cdots,n.$$

Dividing both the numerator and the denominator of above equation by $q_i$, we obtain

(IV.2) $\dfrac{p_i}{p_j} = \dfrac{u_i}{u_j} = \dfrac{u_i/q_i}{u_j/q_i} = \dfrac{u_{ij}(q_j/q_i) + \sum_{k \neq j} u_{ik}(q_k/q_i)}{u_{jj}(q_j/q_i) + \sum_{k \neq j} u_{jk}(q_k/q_i)}$ .

Thus

(IV.3) $\dfrac{\partial(u_i/q_i)}{\partial(q_j/q_i)} = u_{ij} \, , \; \dfrac{\partial(u_j/q_i)}{\partial(q_j/q_i)} = u_{jj}$ .

From (IV.1) and (IV.2), we deduce that

(IV.4) $\dfrac{\partial(p_i/p_j)}{\partial(q_j/q_i)} = \dfrac{\partial\left(\dfrac{u_i}{q_i} \Big/ \dfrac{u_j}{q_i}\right)}{\partial(q_j/q_i)} = \dfrac{q_i}{u_j^2}\left\{u_j u_{ij} - u_i u_{jj}\right\}$ .

With a little manipulation the above equation for the **indirect** intuitive elasticity of substitution between competing commodities becomes

(IV.5) $\dfrac{\partial(p_i/p_j)}{\partial(q_j/q_i)} \dfrac{(q_j/q_i)}{(p_i/p_j)} = \dfrac{(u_{ij}u_j - u_{jj}u_i)q_j}{u_i u_j}$ .

The reciprocal of this **indirect** intuitive elasticity of substitution provides the ordinary form of the **direct** intuitive elasticity of substitution $\sigma$ between competing commodities

(IV.6) $\sigma_{ij} = \dfrac{\partial(q_j/q_i)}{\partial(p_i/p_j)} \dfrac{(p_i/p_j)}{(q_j/q_i)} = \dfrac{u_i u_j}{(u_{ij}u_j - u_{jj}u_i)q_j}$ .

Now, we rewrite PFORMU (II.3) as

(IV.7) $u_i = u_{ij}q_j + \sum_{k \neq j} u_{ik}q_k \, , \; u_j = u_{jj}q_j + \sum_{k \neq j} u_{jk}q_k$ ,

and inserting this into (IV.6) we obtain

$$\text{(IV.8)}\quad \sigma_{ij}=\frac{\left(u_{ij}\sum_{k\neq j}u_{ik}q_k+u_{jj}\sum_{k\neq j}u_{ik}q_k\right)q_j+\left(\sum_{k\neq j}u_{ik}q_k\right)\left(\sum_{k\neq j}u_{jk}q_k\right)+u_{ij}u_{jj}q_j^2}{\left(u_{ij}\sum_{k\neq j}u_{jk}q_k-u_{jj}\sum_{k\neq j}u_{ik}q_k\right)q_j}.$$

Comparing each first term only we find that the numerator is always larger than the denominator. Furthermore, as we saw in Section III, the Fisher-Friedman definition of cmpeting commodities requires that the inequalities (III.8) $u_{ii}u_{jj}\le u_{ii}u_{ij}\le u_{ij}^2$ must always hold. This implies that the denominator of (IV.8) is positive. Therefore, the intuitive elasticity of substitution between competing commodities $\sigma$ (equation (IV.8)) is always positive and larger than unity.

In the case of perfectly competing commodities, we saw in Section III that the condition of (III.10) $u_{ii}u_{jj}=u_{ii}u_{ij}=u_{ij}^2$ is imposed. Since the denominator of equation (IV.8) becomes zero, the intuitive elasticity of substitution $\sigma_{ij}$ must be infinite

$$\text{(IV.9)}\quad \sigma_{ij}=\infty$$

no matter how many are the number of commodities. Conversely, in the weakest case of competing commodities, the magnitude of the intuitive elasticity of substitution is not independent of the number of commodities $n$. Inserting the condition (III.11) $u_{ii}=u_{jj}\to 0$ of Section III into equation (IV.8), we obtain

$$\text{(IV.10)}\quad \sigma_{ij}=\left(u_{ij}q_j+\sum_{k\neq i,j}u_{ik}q_k\right)\Big/u_{ij}q_j\,.$$

Thus, when $n=2$ we have $\sigma=u_{12}^2q_1q_2/u_{12}^2q_1q_2=1$. When $n=3$, $\sigma=(u_{12}q_2+u_{13}q_3)/u_{12}q_2$ and in the simplest case of $u_{12}=u_{13}$, $q_2=q_3$, we have $\sigma=2$. When $n=4$, $\sigma=(u_{12}q_2+u_{13}q_3+u_{14}q_4)/u_{12}q_2$, and in the simplest case of $u_{12}=u_{13}=u_{14}$ and $q_2=q_3=q_4$, we have $\sigma=3$; etc.. These examples suggest that the intuitive elasticity of substitution $\sigma$ tends to have values positively correlated with the number of commodities $n$ in the weakest cases of competing goods and in simplest for $n$ becomes

$$\text{(IV.11)}\quad \sigma=n-1\,.$$

Hence for a given number of commodities $n$, the probable range of the magnitude of the intuitive elasticity of substitution is

$$(\text{IV.12})\quad n-1 \le \sigma \le \infty \; .$$

Thus, the lower bound for the value of $\sigma$ for competing commodities is not constant at $\sigma = 1$ but rises as the number of commodities increases.

More generally, assuming a certain single value of $\beta \equiv u_{ii}/u_{ij}$, $0 \le \beta \le 1$, throughout every possible combinations of competitive commodities in a separable group, we have

$$(\text{IV.13})\quad \sigma = \frac{n}{1-\beta} - 1 \; ,$$

at the initial state of $q_1 = \cdots = q_n$. This suggests that, generally speaking, the elasticity of substitution between competitive commodities $\sigma$ tends to increase with the number of commodities $n$ as well as with the degree of competitiveness $\beta$.

For complementary commodities, we have the **indirect** equilibrium equation in Section II,

$$(\text{II.5}^*)\quad \frac{q_j}{q_i} = \frac{v_j}{v_i} = \frac{\sum_h u^*_{jh} p_h}{\sum_h u^*_{ih} p_h}, \quad h = 1, \cdots, n \; .$$

Dividing both the numerator and the denominator of this equation by $p_j$, we obtain

$$(\text{IV.1}^*)\quad \frac{q_j}{q_i} = \frac{v_j}{v_i} = \frac{v_j/p_j}{v_i/p_j} = \frac{u^*_{ji}(p_i/p_j) + \sum_{k \ne i} u^*_{jk}(p_k/p_j)}{u^*_{ii}(p_i/p_j) + \sum_{k \ne i} u^*_{ik}(p_k/p_j)} \; .$$

Differentiating this gives

$$(\text{IV.2}^*)\quad \frac{\partial(v_j/p_j)}{\partial(p_i/p_j)} = u^*_{ji} \; , \quad \frac{\partial(v_i/p_j)}{\partial(p_i/p_j)} = u^*_{ii} \; .$$

From (IV.1*) and (IV.2*), we deduce that

$$(\text{IV.3}^*)\quad \frac{d(q_j/q_i)}{d(p_i/p_j)} = -\frac{d\left(\frac{v_j}{p_j}\Big/\frac{v_i}{p_j}\right)}{d(p_i/p_j)} = \frac{p_j}{v_i^2}\left\{v_i u_{ji}^* - v_j u_{ii}^*\right\}.$$

Next, we define the intuitive elasticity of substitution for complementary commodities as

$$(\text{IV.5}^*)\quad \sigma_{ij}^* = \frac{d(q_j/q_i)}{d(p_i/p_j)} \cdot \frac{(p_i/p_j)}{(q_j/q_i)} = \frac{d(q_j/q_i)}{d(p_i/p_j)} \cdot \frac{(p_i/p_j)}{(v_j/v_i)}.$$

Inserting $(\text{IV.3}^*)$ into $(\text{IV.5}^*)$, we obtain

$$(\text{IV.6}^*)\quad \sigma_{ij}^* = \frac{\left(v_i u_{ji}^* - v_j u_{ii}^*\right) p_i}{v_i v_j}.$$

Now, we rewrite the PFORMU $(\text{II.3}^*)$ as

$$(\text{IV.7}^*)\quad v_i = u_{ii}^* p_i + \sum_{k \neq i} u_{ik}^* p_k,\quad v_j = u_{ji}^* p_i + \sum_{k \neq i} u_{jk}^* p_k.$$

Inserting $(\text{IV.7}^*)$ into $(\text{IV.6}^*)$ we obtain

$$(\text{IV.8}^*)\quad \sigma_{ij}^* = \frac{\left(u_{ji}^* \sum_{k \neq i} u_{ik}^* p_k - u_{ii}^* \sum_{k \neq i} u_{jk}^* p_k\right) p_i}{\left(u_{ji}^* \sum_{k \neq i} u_{ik}^* p_k + u_{ii}^* \sum_{k \neq i} u_{jk}^* p_k\right) p_i + \left(\sum_{k \neq i} u_{jk}^* p_k\right)\left(\sum_{k \neq i} u_{ik}^* p_k\right) + u_{ji}^* u_{ii}^* p_i^2}.$$

Comparing each first term of the numerator and of the denominator we immediately note that the former is always smaller than the latter. At the same time, our restatement of the Fisher-Friedman definition of complementary commodities in Section III gives the inequalities

$$(\text{III.8}^*)\quad u_{ii}^* u_{jj}^* \le u_{jj}^* u_{ij}^* \le u_{ij}^{*2} \quad \text{for } i,j = 1,\cdots,n.\ \ i \neq j$$

which must always hold. Hence the numerator of equation $(\text{IV.8}^*)$ is non-negative. Thus, the intuitive elasticity of substitution for complementary commodities $\sigma_{ij}^*$ is always less than unity and non-negative.

If the commodities are perfectly complementary, the numerator of (IV.8$^*$) must be zero, because of the condition (III.10$^*$) $u^*_{ii}u^*_{jj} = u^*_{ii}u^*_{ij} = u^{*^2}_{ij}$, $i,j = 1,\cdots,n; i \neq j$. That is, we have

(IV.9$^*$) $\sigma^*_{ij} = 0$

for perfectly complementary commodities, irrespective of the number of commodities. However, in the weakest case of complementarity, the magnitude of $\sigma$ is correlated with the number of commodities $n$. Inserting the definition of weakest complementarity ((III.11$^*$) $u^*_{ii} \to 0$, $0 < u^{*^2}_{ij}$, $i,j = 1,\cdots,n; i \neq j$) into equation (IV.8$^*$) we obtain

(IV.10$^*$) $\sigma^*_{ij} = u^*_{ji} \Big/ \left( u^*_{ji}p_i + \sum_{k \neq i,j} u^*_{jk}p_k \right)$.

Therefore, we have, when $n = 2$, $\sigma^* = u^*_{21}p_1 / u^*_{21}p_1 = 1$. When $n = 3$, $\sigma^* = u^*_{21}p_1 / (u^*_{21}p_1 + u^*_{23}p_3)$, and in the simplest case of $u^*_{21}p_1 = u^*_{23}p_3$ we have $\sigma^* = 1/2$. When $n = 4$, $\sigma^* = u^*_{21}p_1 / (u^*_{21}p_1 + u^*_{23}p_3 + u^*_{24}p_4)$ and in the simplest case of $u^*_{21}p_1 = u^*_{23}p_3 = u^*_{24}p_4$ we have $\sigma^* = 1/3$; etc. . Hence the intuitive elasticity of substitution $\sigma^*$ for commodities at the weakest level of complementarity is inversely correlated with the number of commodities $n$. The simplest case would be represented as

(IV.11$^*$) $\sigma = \dfrac{1}{n-1}$.

For a given number of commodities $n$ the probable range of the magnitude of the intuitive elasticity of substitution between complementary commodities would be

(IV.12$^*$) $0 \leq \sigma^* \leq 1/(n-1)$.

In this manner, the upper bound of $\sigma^*$ at the weakest level of complementarity is not fixed at the level of $\sigma^* = 1$ but falls toward zero as the number of commodities increases.

In the same fashion as in the case of competitive commodities above, for $n$ complementary commodities, with $0 \le \beta^* \equiv u^*_{ii}/u^*_{ij} \le 1$, at the initial state of $p_1 = \cdots = p_n$, we obtain a more general form,

$$\text{(IV.13}^*\text{)} \quad \sigma^* = \frac{1-\beta^*}{n+\beta^*-1}.$$

This implies that $\sigma^*$ is a decreasing function not only of the number of complementary commodities $n$ but of the degree of complementarity $\sigma^*$.

## V. The Slutsky Equation

To derive the Slutsky equation for complementary commodities we make use of the **direct** demand equation (II.9) of Section II

$$\text{(II.9)} \quad q_j = \frac{D_{n+1,j}}{D} m .$$

Differentiating and manipulating the above, we obtain

$$\text{(V.1)} \quad \frac{\partial q_j}{\partial m} = \frac{D_{n+1,j}}{D} \quad \text{and} \quad \frac{q_j}{m} = \frac{D_{n+1,j}}{D}$$

which give the income elasticity of demand

$$\text{(V.2)} \quad \frac{\partial q_j}{\partial m} \bigg/ \frac{q_j}{m} = 1 ,$$

as a natural consequence of the homothetic preference field. On the other hand, we derive the (**direct**) Slutsky equation in the usual fashion, using (II.8), (II.9) etc. to obtain

$$\text{(V.3)} \quad \frac{\partial q_j}{\partial p_i} = \frac{m \cdot D_{n+1,i} D_{n+1,j}}{D^2} \left\{ \frac{u^*_{ji} D - D_{n+1,i} D_{n+1,j}}{D_{n+1,i} D_{n+1,j}} - 1 \right\} = \frac{q_i q_j}{m} \{ \sigma^*_A - \textit{income elasticity} \}$$

where $\sigma^*_A$ stands for the (**direct**) Allen-Uzawa partial elasticity of substitution[10] between complementary commodities. That is

$$\text{(V.4)}\quad \sigma_A^* = \frac{u_{ji}^* D - D_{n+1,i} D_{n+1,j}}{D_{n+1,i} D_{n+1,j}} ,$$

where the denominator is the same as that of the intuitive elasticity of substitution between complementary commodities $\sigma^*$ (IV.8*) obtained in the previous section. Inserting (II.8) into (V.4) we rewrite the numerator of $\sigma_A^*$ as

$$u_{ji}^* D - D_{n+1,i} D_{n+1,j} = \{I\} + \{II\} + \{III\} + \{IV\};$$

$$\{I\} = p_i \left\{ \left( u_{ij}^* \sum_{k \neq i} u_{ik}^* - u_{ii}^* \sum_{k \neq i} u_{jk}^* \right) p_k \right\} > 0,$$

$$\text{(V.5)}\qquad \{II\} = p_j \left\{ \sum_{k \neq i,j} \left( u_{ij}^* u_{jk}^* - u_{jj}^* u_{ik}^* \right) p_k \right\} > 0,$$

$$\{III\} = \left\{ \sum_{k \neq i,j} \left( u_{ij}^* u_{kk}^* - u_{ik}^* u_{jk}^* \right) p_k^2 \right\} < 0,$$

$$\{IV\} = \left\{ \sum_{g,h \neq i,j; g \neq h} \left( u_{ij}^* u_{gh}^* - u_{ig}^* u_{jh}^* \right) p_g p_h \right\} \quad 0.$$

The Fisher-Friedman definition of complementarity specified in Section III requires that the following inequalities hold

$$\text{(III.8)}\quad u_{ii} u_{jj} \leq u_{ii} u_{ij} \leq u_{ij}^2, \quad i, j = 1, \cdots, n, \; i \neq j.$$

Therefore, $\{I\}$ must be strongly positive compared to the other brackets. $\{II\}$ must be positive on the one hand and $\{III\}$ must be negative but the absolute value of these terms is of the same order of magnitude. $\{IV\}$ may be either positive or negative. However, even if the summation of $\{II\}$, $\{III\}$ and $\{IV\}$ is negative, it is extremly unlike to offset the positive value of $\{I\}$. we may safely conclude that the numerator of $\sigma_A^*$(V.4) is non-negative, in general. we know on the other hand that the denominator of $\sigma_A^*$(V.4) is positive. we can write it in a simillar fashion as

$$D_{n+1,i} \cdot D_{n+1,j} = \{I\}' + \{II\}' + \{III\}' + \{IV\}';$$

(V.6)
$$\{I\}' = p_i \left\{ \left( u^*_{ij} \sum_{k \neq i} u^*_{jk} + u^*_{ii} \sum_{k \neq i} u^*_{jk} \right) p_k \right\},$$
$$\{II\}' = p_j \left\{ \sum_{k \neq i,j} \left( u^*_{ij} u^*_{jk} + u^*_{ij} u^*_{ik} \right) p_k \right\},$$
$$\{III\}' = \left\{ \sum_{k \neq i,j} u^*_{ik} u^*_{jk} p_k^2 \right\},$$
$$\{IV\}' = \left\{ \sum_{g,h \neq i,j; g \neq h} \left( u^*_{ig} u^*_{jh} + u^*_{ih} u^*_{jg} \right) p_g p_h \right\}.$$

Comparing $\{I\}' \sim \{IV\}'$ with $\{I\} \sim \{IV\}$, we confirm that the denominator of $\sigma^*_A$ is always larger than the numerator. Therefore the Allen-Uzawa partial elasticity of substitution between complementary commodities is non-negative and not larger than unity, i.e.

(V.7) $0 \leq \sigma^*_A \leq 1$.

Comparing the Allen-Uzawa partial elasticty of substitution $\sigma^*_A$(V.4) with the intuitive elasticity of substitution $\sigma^*$(IV.8*), we find that the former is more sophisticated than the latter having the supplementary terms $\{II\}$, $\{III\}$ and $\{IV\}$ in the numerator when $n > 2$. In the case of two-commodity complementarity, $\sigma^*_A$ and $\sigma^*$ are equal.

(V.8)
$$\sigma^* = \frac{\left( u^{*2}_{12} - u^*_{11} u^*_{22} \right) p_1 p_2}{\left( u^*_{11} p_1 + u^*_{12} p_2 \right)\left( u^*_{21} p_1 + u^*_{22} p_2 \right)} = \sigma^*_A \ .$$

For three-commodity complementarity, however, the numerator of $\sigma^*_A$ is not the same as that of $\sigma^*$.

(V.9) the numerator of $\sigma^* = \left( u^{*2}_{12} - u^*_{11} u^*_{22} \right) p_1 p_2 + \left( u^*_{12} u^*_{13} - u^*_{23} u^*_{11} \right) p_1 p_3;$

the numerator of $\sigma^*_A = \left( u^{*2}_{12} - u^*_{11} u^*_{22} \right) p_1 p_2 + \left( u^*_{12} u^*_{13} - u^*_{23} u^*_{11} \right) p_1 p_3$
$+ \left( u^*_{12} u^*_{23} - u^*_{13} u^*_{22} \right) p_2 p_3 - \left( u^*_{13} u^*_{23} - u^*_{12} u^*_{33} \right) p_3^2 .$

In the case of perfect complementarity, the Fisher-Friedman definition assumes that

(III.10$^*$) $u_{ii}u_{jj} = u_{ij}u_{jj} = u_{ij}^2$ , $i,j = 1,\cdots,n$, $i \neq j$.

Hence, the numerator of the Allen-Uzawa partial elasticity of substitution $\sigma_A^*$ becomes zero regardless of the number of commodities $n$, as well as does the intuitive elasticity of substitution $\sigma^*$. Thus, we have

(V.10) $\sigma_A^* = \sigma^* = 0$,

for perfect complementarity. On the other hand, in the weakest instance of complementarity, if we assume the simplest case of

(III.11$^*$) $u_{ii}^* \to 0$ , $0 < u_{ij}^* = u_{ik}^*$; $i,j,k = 1,\cdots,n$, $i \neq j$, $i \neq k$, $j \neq k$, and $p_1 = \cdots = p_n$

we obtain from equations (V.8) and (V.9) $\sigma_A^* = u_{12}^{*\,2} p_1 p_2 / u_{12}^* u_{21}^* p_1 p_2 = 1$ for $n = 2$ and $\sigma_A^* = \dfrac{u_{12}^{*\,2} p_1 p_2 + u_{12}^* u_{13}^* p_1 p_3 + u_{12}^* u_{23}^* p_2 p_3 - u_{13}^* u_{23}^* p_3^2}{\left(u_{12}^* p_2 + u_{13}^* p_3\right)\left(u_{21}^* p_1 + u_{23}^* p_3\right)} = 2u_{12}^{*\,2} p_1^2 / 4u_{12}^{*\,2} p_1^2 = 1/2$ for $n = 3$.

In the same fashion, we obtain, for $n = 4$, $\sigma_A^* = 1/3$ and can confirm that, in the simplest case assumed above for the weakest instance of complementarity, we have the general form of

(V.11) $\sigma_A^* = \dfrac{1}{n-1}$.

This of course is equal to the value of the intuitive elasticity of substitution $\sigma^*$ between complementary commodities (IV.11$^*$). These equivalence of values of $\sigma_A^*$ and $\sigma^*$ at the two extreme cases of perfect and weakest complementarity strongly suggest that even though the value of the Allen-Uzawa partial elasticity of substitution $\sigma_A^*$ may differ to some extent from that of the intuitive elasticity of substitution $\sigma^*$ because of the supplementary terms of $\{II\}$, $\{III\}$ and $\{IV\}$ in the numerator of the former, the range of magnitude of $\sigma_A^*$ (V.7) is exactly the same as that of $\sigma^*$.

Assuming that the value of the Allen-Uzawa partial elasticity of substituton between complementary commodities $\sigma_A^*$ stays within the range of $0 \leq \sigma_A^* \leq 1$, we can conclude that the inequality

$$(V.12)\quad \frac{\partial q_j}{\partial p_i} = \frac{q_i q_j}{m}\left\{\sigma_A^* - 1\right\} \le 0$$

always holds for complementary commodities. Additionally, we can obtain the following equation for the cross-price elasticity of demand for complementary commodities,

$$(V.13)\quad \eta_{ij}^* = \frac{\partial q_j}{\partial p_i}\frac{p_i}{q_j} = \frac{p_i q_i}{m}\left\{\sigma_A^* - 1\right\} \le 0 .$$

Thus, the Fisher-Friedman definition suggests that the cross-price effect between complementary commodities must be negative in most cases, even though the elasticity of substitution $\sigma^*$ itself is positive.

In order to deduce the Slutsky equation for competing commodities we make use of the **indirect** demand equation derived in Section II,

$$(II.9^*)\quad p_i = \frac{D_{n+1,i}^*}{D^*} m = \frac{u_i}{u} m .$$

Differentiating the above **indirect** demand equation, and arranging terms, we obtain the **indirect** Slutsky equation

$$(V.14)\quad \frac{\partial p_i}{\partial q_j} = \frac{p_i p_j}{m}\left\{\frac{u_{ij}D^* - D_{n+1,i}^* D_{n+1,j}^*}{D_{n+1,i}^* D_{n+1,j}^*} - 1\right\} .$$

Here, the first term in brackets in the right-hand side of the **indirect** Slutsky equation is equivalent to the **indirect** Allen-Uzawa partial elasticity of substitution between competing commodities. Quite analogous to the relation between the **indirect** and **direct** intuitive elasticities of substitution for competing commodities observed in the previous section, the reciprocal of the **indirect** Allen-Uzawa partial elasticity of substitution is equal to the **direct** Allen-Uzawa partial elasticity of substitution between competing commodities. The validity of this analogy is confirmed by the fact that, in the case of the two-commodity model, the form of the Allen-Uzawa partial elasticity of substitution equals that of the intuitive elasticity of substitution. Thus, we have

$$\text{(V.15)}\quad \sigma_A = \frac{D^*_{n+1,i}D^*_{n+1,j}}{u_{ij}D^* - D^*_{n+1,i}D^*_{n+1,j}} = \frac{u_i u_j}{u_{ij}u - u_i u_j} = \sigma,$$

for the **(direct)** Allen-Uzawa partial elasticity of substitution between competing commodities. Inserting (II.8$^*$) of Section II we get for the numerator,

$$u_i u_j = \left(\sum_j u_{ij} q_j\right)\left(\sum_k u_{jk} q_k\right) = \{I\} + \{II\} + \{III\} + \{IV\}:$$

$$\text{(V.16)}\quad \begin{aligned} \{I\} &= q_i\left\{\left(u_{ij}\sum_{k\neq i} u_{jk} + u_{ii}\sum_{k\neq i} u_{jk}\right)q_k\right\}, \\ \{II\} &= q_j\left\{\sum_{k\neq i,j}\left(u_{ij}u_{jk} + u_{jj}u_{ik}\right)q_k\right\}, \\ \{III\} &= \left\{\sum_{k\neq i,j} u_{ik}u_{jk}q_k^2\right\}, \\ \{IV\} &= \left\{\sum_{g,h\neq i,j;g\neq h}\left(u_{ig}u_{jh} + u_{ih}u_{jg}\right)q_g q_h\right\}. \end{aligned}$$

Thus, it is obvious that the numerator of the Allen-Uzawa partial elasticity of substitution $\sigma_A$ for competing commodities is always positive. Similarly, we obtain for the denominator of $\sigma_A$ of (V.15)

$$uu_{ij} - u_i u_j = \{I\}' + \{II\}' + \{III\}' + \{IV\}'.$$

$$\text{(V.17)}\quad \begin{aligned} \{I\}' &= q_i\left\{\left(u_{ij}\sum_{k\neq i} u_{ik} - u_{ii}\sum_{k\neq i} u_{jk}\right)q_k\right\}, \\ \{II\}' &= q_j\left\{\sum_{k\neq i,j}\left(u_{ij}u_{jk} - u_{jj}u_{ik}\right)q_k\right\}, \\ \{III\}' &= \left\{\sum_{k\neq i,j}\left(u_{ij}u_{kk} - u_{ik}u_{jk}\right)q_k^2\right\}, \\ \{IV\}' &= \left\{\sum_{g,h\neq i,j;g\neq h}\left(u_{ij}u_{gh} - u_{ig}u_{jh}\right)q_g q_h\right\}. \end{aligned}$$

Applying the rule of the Fisher-Friedman definition of competitive commodities in Section III,

(III.8) $u_{ii}u_{jj} \le u_{ij}u_{jj} \le u_{ij}^2$ , $i,j = 1,\cdots,n;\ i \ne j$,

we find that $\{I\}' >> 0$, $\{II\}' > 0$, $\{III\}' < 0$, $\{IV\}' \doteqdot 0$ and $\left|\{II\}'\right| \doteqdot \left|\{III\}'\right|$. Thus, we conclude it is highly probable that the value of the denominator of $\sigma_A$ is non-negative. Noting that the numerator (V.16) is always larger than the denominator (V.17), it is clear that the Allen-Uzawa partial elasticity of substitution $\sigma_A$ for competing commodities always takes a positive value not less than unity, i.e.

(V.18) $1 \le \sigma_A$ .

In the case of perfectly competing goods where $u_{ii}u_{jj} = u_{ij}u_{jj} = u_{ij}^2$ holds, the denominator (V.17) becomes zero while the numerator is positive. Hence the Allen-Uzawa partial elasticity of substitution $\sigma_A$ is infinite, no matter how many are the number of commodities $n$. On the other hand, if we take the simplest case for the most weakly competing goods where $u_{ii} \to 0$, $0 < u_{ij} = u_{ik}$ , $i,j,k = 1,\cdots,n$, $i \ne j, i \ne k, j \ne k$ and $q_1 = \cdots = q_n$ hold, we have from (V.15), (V.16) and (V.17), $\sigma_A = 1$ for $n = 2$, $\sigma_A = 2$ for $n = 3$, and in general

(V.19) $\sigma_A = n - 1$.

These results are exactly the same as those for the intuitive elasticity of substitution for competing commodities obtained in the previous section, and support the validity of the definition of the Allen-Uzawa partial elasticity of substitution for competing cmmodities $\sigma_A$ given by (V.15).

If we replace the Allen-Uzawa partial elasticity of substitution for complementary commodities $\sigma_A^*$ with the one for competing commodities $\sigma_A$ , obtained from the **indirect** model above, equation (V.13) gives

(V.13*) $\eta_{ij} = \dfrac{\partial q_j}{\partial p_i}\dfrac{p_i}{q_j} = \dfrac{p_i q_i}{m}\{\sigma_A - 1\} \ge 0$ .

Because of the general range of (V.18), the sign of the bracketed term in (V.13*) must alway be non-negative, and positive in most cases. Generally speaking, the cross-price elasticity of demand $\eta_{ij}$ for competing commodities is always positive.

## VI. The Degree of Complementarity

As we noted in Section I, Friedman intended to bridge the gap between Fisher's two poles of perfectly complementary and perfectly substitutable commodities by defining the intermediate cases. To accomplish this goal, we require some measure to express rigorously and continuously the degree of complementarity and substitutablility between commodities. Let us begin by illustrating the problem through the simple example of the two commodity model. In Section III, we specified the definition of two competing commodities with the following inequalities involving the limits of proportion between prices.

(III.3) $u_{11}/u_{21} \le p_1/p_2 \le u_{12}/u_{22}$; $u_{11}u_{22} \le u_{12}^2$.

We further defined perfectly competing commodities as $u_{11}u_{22} = u_{12}^2$ and the weakest case of competing commodities as $u_{11} = u_{22} = 0$. Hence, the degree of substitutability between the two commodities is defined as

(VI.1) $u_{11}u_{22} = \beta^2 u_{12}^2$, $0 \le \beta \le 1$,

where $\beta = 1$ represents the case of perfectly competing goods and $\beta = 0$ represents the case of the most weakly competing goods. In the case of the two commodity model, the intuitive elasticity of substitution $\sigma$ and the Allen-Uzawa partial elasticity of substitution $\sigma_A$ are equivalent and can be written as

$$(\text{VI.2}) \quad \sigma_{ij} = \sigma_{Aij} = \frac{(u_{11}q_1 + u_{12}q_2)(u_{21}q_1 + u_{22}q_2)}{(u_{12}^2 - u_{11}u_{22})q_1 q_2}.$$

For simplicity, assume a normalized model where $u_{11} = u_{22} = u_{ii}$, $u_{12} = u_{21} = u_{ij}$, and $p_1 = p_2$ corresponds to $q_1 = q_2 = q_i$. This implies (from VI.1) that $u_{ii}^2 = \beta^2 u_{ij}^2$ which, when inserted in (VI.2), gives

$$(\text{VI.3}) \quad \sigma_{ij} = \sigma_{Aij} = \frac{u_{ij}^2 q_i^2 \{1 + \beta^2 + 2\beta\}}{u_{ij}^2 q_i^2 (1 - \beta^2)} = \frac{1+\beta}{1-\beta}.$$

We use (VI.3) to generate Table 1 showing the correspondence between the degree of substitutability $\beta$ and the magnitude of the elasticity of substitution between two competing commodities. This table implies that the cross-price elasticity of demand between competing commodities $\eta_{ij}$ (V.13*) is non-negative.

Table 1. Degree of Substitutability $\beta$

| | weakest | | | medium | | | perfect |
|---|---|---|---|---|---|---|---|
| $\beta$ | 0 | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 | 1 |
| $\sigma_{ij} = \sigma_{Aij}$ | 1 | 1.2 | 1.9 | 3 | 5.7 | 19 | $\infty$ |

In Section III we specified the case of complementarity between two commodities as follows, based upon the limits of proportion between quantities.

(III.3') $u^*_{22}/u^*_{12} \leq q_2/q_1 \leq u^*_{21}/u^*_{11}$, $u^*_{11}u^*_{22} \leq u^{*2}_{12}$

where $u^*_{11}u^*_{22} = u^{*2}_{12}$ corresponds to perfect complementarity and $u^*_{11} = u^*_{22} = 0$ corresponds to the case of weakest complementarity. Defining the degree of complementarity $\beta^*$ between the two commodities as

(VI.4) $u^*_{11}u^*_{22} = \beta^{*2} u^{*2}_{12}$, $0 \leq \beta^* \leq 1$ ,

$\beta^* = 1$ represents perfect complementarity while $\beta^* = 0$ represents the weakest case of complementarity. We have also previously obtained the following equation for both the intuitive elasticity of substitution $\sigma^*$ and the Allen-Uzawa partial elasticity of substitution $\sigma^*_A$.

$$\text{(VI.5)} \quad \sigma^*_{ij} = \sigma^*_{Aij} = \frac{\left(u^{*2}_{12} - u^*_{11}u^*_{22}\right)p_1 p_2}{\left(u^*_{11}p_1 + u^*_{12}p_2\right)\left(u^*_{21}p_1 + u^*_{22}p_2\right)} .$$

Again assume a normalized model where $u^*_{11} = u^*_{22} = u^*_{ii}$ ,$u^*_{12} = u^*_{21} = u^*_{ij}$ and $q_1 = q_2$ corresponds to $p_1 = p_2 = p_i$. Thus, $u^*_{ii} = \beta u^*_{ij}$ and

$$\text{(VI.6)} \quad \sigma^*_{ij} = \sigma^*_{Aij} = \frac{u^{*2}_{ij} p_i^2\left(1 - \beta^{*2}\right)}{u^{*2}_{ij} p_i^*\left\{1 + \beta^{*2} + 2\beta^*\right\}} = \frac{1-\beta^*}{1+\beta^*} .$$

From the above we obtain a table showing the correspondence between the degree of complementarity $\beta^*$ and the value of the elasticity of substitution ($\sigma^* = \sigma^*_A$) for two complementary commodities.

Table 2. Degree of Complementarity $\beta^*$

| | weakest | | | medium | | | perfect |
|---|---|---|---|---|---|---|---|
| $\beta^*$ | 0 | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 | 1 |
| $\sigma^*_{ij} = \sigma^*_{Aij}$ | 1 | 0.82 | 0.54 | 0.33 | 0.18 | 0.05 | 0 |

Table 2 implies that the cross price elasticity of demand $\eta^*_{ij}$ must be non-positive (also by equation (V.13)). Since the own elasticity of substitution has a value of the same magnitude but with a sign opposite to that of the cross elasticity of substitution shown in Tables 1 and 2, the Fisher-Friedman definition implies that the own price elasticity of demand for competing commodities has a large negative value while that for complementary commodities has some negative value absolutely less than unity.

The constant elasticity of substitution (CES) function[11] provides values for the elasticity of substitution ranging from $0 \leq \sigma \leq \infty$ and includes $\sigma = 1$ as the special case of the Cobb-Douglas form. Hence, it is not the values of the elasticity of substitution that mark the FF definition as unique. Rather, the distinct feature of our extension of the FF definition is in the variation of the elasticity of substitution with relative prices. This is illustrated in Tables 3 and 4 for particular degrees of substitutability $\left(\beta = u_{ii}/u_{ij} = 0.5\right)$ and complementarity $\left(\beta^* = u^*_{ii}/u^*_{ij} = 0.5\right)$.

Inserting the **direct** demand function (II.9) into equation (VI.2) we obtain the elasticity of substitution between two competing commodities.

$$\text{(VI.7)} \quad \sigma_{12} = \frac{\left(u_{12}^2 - u_{11}u_{22}\right)p_1p_2}{\left(u_{22}p_1 - u_{12}p_2\right)\left(u_{11}p_2 - u_{12}p_1\right)}.$$

Making use of $\beta = u_{ii}/u_{12}$ and $u_{11} = u_{22} = u_{ii}$, this can be rewritten as

$$\text{(VI.8)} \quad \sigma_{12} = \frac{\left(1-\beta^2\right)\frac{p_1}{p_2}}{\left(\beta\frac{p_1}{p_2} - 1\right)\left(\beta - \frac{p_1}{p_2}\right)}$$

where by definition the restriction $u_{ii}/u_{12} \leq p_1/p_2 \leq u_{12}/u_{ii}$ (i.e. $\beta \leq p_1/p_2 \leq 1/\beta$ ) is imposed. Assuming that $\beta = 0.5$, we obtain the results summarized in Table 3.

Table 3. Relative Prices & Competing $\sigma$ at $\beta = 0.5$

| $p_1/p_2$ | 0.5 | 0.51 | 0.55 | 0.6 | 0.8 | 1.0 | 1.25 | 1.67 | 1.82 | 1.96 | 2.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\sigma_{12}$ | $\infty$ | 51.32 | 11.38 | 6.43 | 3.33 | 3.00 | 3.33 | 6.43 | 11.38 | 51.32 | $\infty$ |

Here the elasticity of substitution $\sigma_{12}$ between competing commodities is lowest at the center of the normalized indifference map and rises symmetrically toward the lower and upper limits of proportion between prices where it approaches infinity.

The elasticity of substitution between two complementary cmmodities is given by equation (VI.5). Inserting $u_{ii}^{*}/u_{12}^{*} = \beta^{*}$ into this we obtain

$$(\text{VI.9})\quad \sigma_{12}^{*} = \frac{\left(1-\beta^{*2}\right)\dfrac{p_1}{p_2}}{\left(\beta^{*}\dfrac{p_1}{p_2}+1\right)\left(\beta^{*}+\dfrac{p_1}{p_2}\right)}$$

where by definition the restriction $\beta^{*} \le q_2/q_1 \le 1/\beta^{*}$ (corresponding to $0 \le p_1/p_2 \le \infty$) is imposed. If we assume that $\beta^{*} = 0.5$, the variation in the value of the elasticity of substitution $\sigma_{12}^{*}$ with changes in relative prices $p_1/p_2$ is:

Table 4. Relative Prices & Complementary $\sigma^{*}$ at $\beta^{*} = 0.5$

| $p_1/p_2$ | 0 | 0.1 | 0.2 | 0.5 | 1.0 | 2.0 | 5.0 | 10.0 | $\infty$ |
|---|---|---|---|---|---|---|---|---|---|
| $\sigma_{12}^{*}$ | 0 | 0.12 | 0.20 | 0.30 | 0.33 | 0.30 | 0.20 | 0.12 | 0 |

Here the value of $\sigma_{12}^{*}$ is highest at the center of the normalized preference field and decreases toward zero as the quantities approach both their lower and upper limits of proportion .

Under our revised Fisher-Friedman definition, it is noteworthy that the elasticity of substitution $\sigma_{ij}$ of competing commodities approaches infinity at the limits of proportion between prices while the elasticity of substitution $\sigma_{ij}^{*}$ of complementary commodities approaches zero at the limits of proporton between quantities. This holds for any particular degree of substitutability $\beta$ or complementarity $\beta^{*}$ excepting the cases of $\beta = 0$ or $\beta^{*} = 0$.

Finally, making use of equations (IV.13) or (IV.13)* obtained in Section IV, we can illustrate the two dimensional determination of the value of the elasticity of substitution by the number of commodities $n$ and the degree of competitiveness $\beta$ (or of complementarity $\beta^*$). For a group of competitive commodities, we have, e.g.

Table 5. Two Dimensional Determination of $\sigma$

| $\beta$ \ $n$ | 2 | 4 | 6 | 8 | 10 | $\infty$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 3 | 5 | 7 | 9 | $\infty$ |
| 1/2 | 3 | 7 | 11 | 15 | 19 | • |
| 2/3 | 5 | 11 | 17 | 23 | 29 | • |
| 3/4 | 7 | 15 | 23 | 31 | 39 | • |
| 4/5 | 9 | 19 | 29 | 39 | 49 | • |
| 1 | $\infty$ | • | • | • | • | $\infty$ |

It may be of some interest to observe the correspondence between effects of $n$ and those of $\beta$ on $\sigma$, in this table, especially from view point of the theory of the industrial organization.

## VII. Independent Commodities

In Tables 1 and 2 of the previous section we observed that for a two commodity model the value of the elasticity of substitution $\sigma = \sigma_A$ ranged from $\infty$ to 1 in the case of competing goods and that the value of the elasticity of substitution $\sigma^* = \sigma^*_A$ ranged between 0 and 1 in the case of complementary goods. Therefore, the two series $\sigma$ and $\sigma^*$ appear to constitute a unified continuous series from $\sigma = 0$ for perfect complementarity $(\beta^* = 1)$ to $\sigma^* = \infty$ for the case of perfectly competing goods $(\beta = 1)$. The point of overlap between the two series, $\sigma = 1 = \sigma^*$ $(\beta = 0 = \beta^*)$ suggests that we can reinterpret both the weakest case of competing goods $(\beta = 0)$ and the weakest case of complementary goods $(\beta^* = 0)$ as the state of independent commodities. Since $u_{11} = u_{22} = 0$ and $u^*_{11} = u^*_{22} = 0$ hold at this point, the **indirect** PFORMU defined by (II.3*) becomes

(VII.1) $v_1 = u^*_{12} p_2, \quad v_2 = u^*_{21} p_1,$

and the **direct** PFORMU can be written as

(VII.2) $u_1 = u_{12} q_2$, $u_2 = u_{21} q_1$ .

These specifications initially appear counter-intuitive, given that the relative marginal utility of both commodity 1 and commodity 2 depends not on its own price or quantity but on the price or quantity of the other commodity. This dilemma can be overcome as follows. Using the **direct** and **indirect** marginal rates of substitution given by (VII.2) and (VII.3) we obtain

(VII.3) $\frac{u_1}{u_2} = \frac{q_2}{q_1}$, $\frac{v_1}{v_2} = \frac{p_2}{p_1}$ .

We can now redefine the **direct** and **indirect** relative marginal utilities as

(VII.4) $u_1 = \frac{1}{q_1}$, $u_2 = \frac{1}{q_2}$; $v_1 = \frac{1}{p_1}$, $v_2 = \frac{1}{p_2}$ .

The marginal utility of each commodity is now a function of its own quantity or price, and this specification can be made slightly more sophisticated by defining the **direct** form of the relative marginal utility for each independent commodity as

(VII.5) $u_i = \frac{a_i}{q_i}$ .

This type of PFORMU corresponds to a Cobb-Douglas utility function of the form

(VII.6) $u = \sum_k a_k \ln q_k$ ; or, $u = \prod_k q_k^{a_k}$ , $k = 1, \cdots, n$ .

In other words, we are now assuming a utility function of a different form from that assumed throughout this paper. While this might initially seem unacceptable, in fact it is not. Even though we have assumed a quadratic homogeneous utility function for the **direct** model of complementary commodities (and its dual for competing commodities), only in the case of the two commodity model is there any overlap (at $\beta = 0 = \beta^*, \sigma = 1 = \sigma^*$). As noted previously (from (IV.11), (IV.11$^*$), (V.11), and (V.19)), in cases where $n > 2$, the value of the elasticity of substitution ($\sigma$ or $\sigma_A$) between the most weakly competing goods increases with the number of commodities $n$ in that separable group. Conversely, the value of the elasticity of substitution between the

most weakly complementary commodities decreases with the number of commodities $n$ in that separable group. Therefore, when there are more than two commodities, the value of $\sigma$ or $\sigma_A$ in the most weakly competing case must be larger than unity and the value of $\sigma^*$ or $\sigma_A^*$ must be less than unity in the weakest case of complementarity.
In other words, when $n > 2$, the two extremes of ($\beta = 0, \sigma > 1$) and ($\beta^* = 0, \sigma^* < 1$), for competing and complementary commodities do not share any common values. There exists a gap between the state of the weakest complementarity and that of the most weakly competing goods. Thus the state of independence between commodities is distinct from that of the most weakly complementary or competing commodities. Because the state of independent commodities is not in general on the midpoint of a continuous series between perfect complementarity and perfect substitutability, independent commodities can and should be defined separately from our extended Fisher-Friedman definition. Thus, the Cobb-Douglas specification[12] of (VII.5) and (VII.6) is compatible with our definition of complementary and competing commodities outlined in Section II and III.

## VIII. Off-budget Complementarity and Engel's law

A tropical climate is complementary to air conditioners but competes with room heaters. Roads are highly complementary to automobiles. These examples illustrate some of the many cases where the natural condition of a country or its social capital is complementary to or competes with commodities purchased by consumers. Since these factors are not included in the budget of the consumer, we will follow standard practice by denoting these relationships as off-budget complementarity. Such off-budget complementarity obviously has a singnificant impact on consumer behavior in many cases.

Off-budget complementarity is not limited to a nation's attributes but also extends to the physical condition of the consumer himself. For example, healthy teeth are needed to bite into a fresh apple. Indeed, one of the oldest examples of this type of off-budget cmplementarity is found in Paul's first letter to the Corinthians where he quotes "Food is meant for the stomach and the stomach for food" (Corinthians I: 6)

In certain instances, it is necessary to introduce these factors of off-budget complementarity into the utility functions of consumers, including for example the climate in utility functions used to describe demand for air conditioners. The most common example of this necessity in the analysis of household consumption is the case of Engel's Law.

When comparing expenditure on food from household to household, we must consider not only the household budget but also the number of people (stomachs) in each household.

For simplicity, assume a two commodity model which consists of total food consumption $q_1$ and the total consumption of other commodities $q_2$. To denote the off-budget complementarity between the consumption of food $q_1$ and family size $N$, we introduce the latter into the PFORMU of the former specified as an independent commodity and obtain

$$\text{(VIII.1)} \quad u_1 = \frac{a_1}{q_1 - b_1 N}; \quad a_1 > 0,\ b_1 > 0,\ (q_1 - b_1 N) > 0.$$

Partially differentiating this gives

$$\text{(VIII.2)} \quad \frac{\partial u_1}{\partial N} = \frac{a_1}{(q_1 - b_1 N)^2 b_1} > 0.$$

The positive sign of $\partial u_1 / \partial N$ implies complementarity between $q_1$ and $N$. This Klein-Rubin type[13] PFORMU is of course consistent with a demand function which expresses Engel's law.

Introducing off-budget complementarity into our extended FF definition for separable groups of complementary and competing commodities would complicate the analysis considerably. Fortunately, the need to do so should seldom arise. The complementarity between pancakes, maple syrup and butter is not affected by household size and the substitutability between pancakes and cereal is also separable from the number of persons in a household. Moreover, the complementarity between autos and gasoline is nearly independent of the highway system, and the substitutability between types of heaters (kerosene versus electric) is separable from the surrounding climate. Thus, off-budget complementarity is largely irrelevant to the analysis of the complementarity or substitutability of individual commodities. It is important in the case of composite commodity groups where independent groups can be formed through aggregating individual commodities, but analysis would follow the above example for total food consumption and family size.

In short, our extended Fisher-Friedman definition of complementarity is consistent with the empirical fact of Engel's law even though it assumes a homothetic preference field for each separable group of complementary or competing commodities. However, we leave for future work the development of a precise theory of the preference tree based on our extended Fisher-Friedman definition.

## IX. Summary

The Fisher-Friedman definition classifies commodities as substitutable or complementary depending on the global shape of indifference maps. It thus requires the specification of the global mathematical properties of any model used to illustrate this definition. We utilized the simple model of a **direct** utility function of quadratic homogeneous form. As shown in Sections II-V, the substitution effect--the Allen elasticity of substitution--is always non-negative with this specification. Therefore, by the Hicks-Allen-Shultz definition[14], this model applies only to competing commodities and not to complementary commodities, a conclusion that is independent of the number of commodities.

Our extended FF definition is more flexible and more realistic. Fisher initially focused on the difference in magnitude in the elasticities of substitution between competing and complementary commodities while Friedman noted that the sign of the cross price elasticities must be positive for competing commodities and negative for complementary commodities. We specified Fisher's criterion as $\sigma_{ij}$ 1, and Friedman's as $\eta_{ij} = \frac{p_i p_j}{m}\left\{\sigma_{ij} - 1\right\}$ 0, specifications that are consistent with one another. By so doing we derived an extended Fisher-Friedman definition capable of classifying commodities within each separable group as substitutable or complementary no matter how large the number of commodities in each group. Unlike the HAS definition, the Fisher-Friedman defiition specified here can classify tea and lemon to be complementary and at the same time tea and coffee to be substitutable.

The late Professor Stigler once noted that "(the Hicks-Allen definition) cannot be applied introspectively to classify commodities ⋯, so they offer no avenue to the utilization of introspection ⋯. As a result, such criteria can be applied concretely only if one has full knowledge of the demand functions. ⋯The chief reason for presenting criteria in terms of utility, I suspect, is that, when familiar names are given to unknown possibilities, an illusion of definiteness of results is frequently conferred"[15]. Our reconstructed Fisher-Friedman definition of complementarity can provide the useful a priori information in advance of any empirical research. So, it may meet Stigler's requirement, we hope.

Footnotes

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1. Starting our research we took Samuelson's comprehensive survey article on the theory of complementarity to be standard-setting. See Samuelson [15]-(II).

2. We owe thanks to Professor Milton Friedman who very kindly gave us a xerox copy of his unpublished paper of December 1933, at our request in July 1982.

3. See p.13 of Friedman [6].

4. See p.73 of Fisher [5].

5. See p.44 of Friedman [6].

6. See p.46 of Friedman [6]. Concerning the concept of separability, see Leontief [13], Goldman and Uzawa [8], and Geary and Morishima [7]. Friedman may be the earliest in setting and using the concept although not the specific terminology.

7. We have extensively borrowed Houthakker's terminology in presenting our dual model. See Houthakker [11]-(I), (II).

8. The concept of PFORMU comes from Samuelson [15]-(II), pp.95-99.

9. For more on Cauchy's theorem, see Aitken [1], p.81 and p.92.

10. See Uzawa [20], Mcfadden [14], Nadiri [15], and pp.340-343 and pp.503-505 of Allen [2].

11. See Arrow, Chenery, Minhas, and Solow [3].

12. This remindes us the transcendental logarithmic function which includes the Cobb-Douglas form as a special case. However, we have not investigated in detail yet. See Christensen, Jorgenson and Lau [4]-(I), (II).

13. See Klein and Rubin [12], Stone [18], Tsujimura [19], and Tsujimura and Sato [21].

14. See Hickes [9], [10], Allen [2]-(I), and Schultz [16].

15. See pp.134-35 of Stigler [17]; and Samuelson [16]-(II).

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