Title	Inference on incomplete information games with multi-dimensional actions
Sub Title	
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Publisher	Keio Economic Observatory Sangyo Kenkyujo
Publication year	2021
Jtitle	KEO discussion paper No.161 (2021. 4)
JaLC DOI	
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Notes	
Genre	Technical Report
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=AA10715861-00000161-0001

Inference on incomplete information games with multi-dimensional actions

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April 27, 2021

KEO Discussion Paper No. 161

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INFERENCE ON INCOMPLETE INFORMATION GAMES WITH MULTI-DIMENSIONAL ACTIONS

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Abstract. This note extends partial identification analyses by de Paula and Tang (2012) and Aradillas-López and Gandhi (2016) to games with multi-dimensional actions. We discuss two models of players' payoff functions in which strategic parameters can be partially identified without assuming equilibrium selection mechanisms or distribution forms of unobservables.

1. Introduction

Multiplicity of equilibria often causes problems when econometricians estimate game theoretic models. de Paula and Tang (2012) derived partial identification results for incomplete information games where players take binary actions. Specifically, correlations between players' actions generated by multiple equilibria provide an insight into signs of interaction effects. Aradillas-López and Gandhi (2016) extended their results to games where players have ordered choices. Their identification strategy is based on covariance restrictions between actions and strategic parts of players' payoff functions. Although their model and identification analysis are quite insightful, they focus on the case where each player's choice set is one-dimensional. This note extends Aradillas-López and Gandhi's (2016) analysis and derives covariance restrictions in games where players have multi-dimensional actions. Examples of games with multi-dimensional actions include Cournot, Bertrand, and entry games with multi-product firms. In addition, the model can allow firms to determine both prices and levels of advertisements.

2. Main results

We closely follow the notation in Aradillas-López and Gandhi (2016). Let P be the number of players. Each player $p \in \{1, ..., P\}$ has K^p-dimensional action space $\mathcal{A}^p = \prod_{k=1}^{K^p} \mathcal{A}_k^p$, where $\mathcal{A}_k^p$ denotes player p's action set for the k-th dimension. Assume that the set $\mathcal{A}_k^p$ has an ordinal structure for each p and k and can be finite, countably infinite, or uncountable. Let $Y^p = (Y_1^p, ..., Y_{K^p}^p) \in \mathcal{A}^p$ be player p's action variable. Let $\mathcal{A}^{-p} = \prod_{q \neq p} \mathcal{A}^q$ be the action space of all players other than p, and $Y^{-p} = (Y^q)_{q \neq p}$ be a profile of action variables for p's opponent. Lowercases y^p and y^{-p} represent potential actions for p, and for all players other than p, respectively. Player p's payoff function is given by $\nu^p(y^p, y^{-p}; \xi^p)$, where ξ^p is a vector of p's payoff shifters. ξ^p can be decomposed into observable exogenous variables X and p's private payoff shock ε^p, that is, $\xi^p = (X, \varepsilon^p)$. We note that X and ε^p can be correlated in an arbitrary way, and the dimension of ε^p is unrestricted. Thus, our specification encompasses not only one-dimensional shock but also K^p-dimensional shock, which equals to the dimension of the player's action.

We impose the following assumption on players' information structure.

Assumption 1. *X is public information and ε^p is observed only by player p. ε^p is independent of ε^{-p} conditional on X. The distribution of (X, ε^p) and payoff structures are common knowledge among the players.*

The conditional independence assumption on the private shock is prevalent in the literature on estimation of games using covariance restrictions (see, de Paula and Tang, 2012; Aradillas-López and Ghandi, 2016). It is worthwhile to note that the elements of ε^p can be arbitrarily correlated. Based on the above notation, player p's expected payoff is written as

$$\bar{\nu}^p_\sigma(y^p; \xi^p) = \sum_{y^{-p} \in \mathcal{A}^{-p}} \sigma^{-p}(y^{-p}) \cdot \nu^p(y^p, y^{-p}; \xi^p),$$

where $\sigma^{-p} : \mathcal{A}^{-p} \to [0, 1]$ is p's belief over the other players' action. A typical solution concept of static incomplete information games is Bayesian Nash equilibrium (BNE), in which each player chooses an action that maximizes its expected utility given the equilibrium belief. Given that players' shocks are independent (Assumption 1), BNE can be characterized as a collection of choice probabilities $\sigma_*(X) := \{\sigma^p_*(\cdot|X) : \mathcal{A}^p \longrightarrow [0, 1]\}_{p=1}^P$ conditional on X, where

$$\sigma^p_*(y_p|X) = \mathbb{E}_{\xi^p|X}\left[\mathbb{I}\left\{y_p = \arg\max_{y \in \mathcal{A}^p} \bar{\nu}^p_{\sigma_*}(y; \xi^p)\right\}\right],$$

for each $y_p \in \mathcal{A}^p$, $\mathbb{E}_{\xi^p|X}[\cdot]$ is expectation under the conditional distribution of ξ^p given X, and $\mathbb{I}\{\cdot\}$ is the indicator function. Hereafter, we assume that $\arg\max_{y \in \mathcal{A}^p} \bar{\nu}^p_{\sigma_*}(y; \xi^p)$ is singleton with probability one. This assumption is widely employed in the literature.

The next assumption requires that the observed data are generated according to some BNE.

Assumption 2. *Each observation is generated according to a BNE, i.e.,*

$$Y^p = \arg\max_{y \in \mathcal{A}^p} \bar{\nu}^p_{\sigma_*}(y; \xi^p) \quad \textit{for some BNE } \sigma_*(X).$$

We allow that observations are generated from multiple equilibria after conditioning on X. It is important to note that this assumption does not impose any equilibrium selection mechanisms.

Aradillas-López and Gandhi (2016) considered the case of a univariate action variable (i.e., $K^p = 1$) with an ordinal structure on the action set, and derived a covariance restriction between p's action and some strategic component of p's payoff function, which can used for inference on the strategic component. Their key idea for partial identification is to explore certain separability and monotonicity conditions for the payoff function. This paper extends their analysis to the case of multi-dimensional action variables, where it is not trivial how to extend shape constraints on the payoff functions, such as separability and monotonicity.

2.1. **First model: Multi-dimensional separability.** Take any player p of interest. We first consider an extension of the separability assumption in Aradillas-López and Gandhi (2016, Assumption 1) to the multi-dimensional case. In particular, we impose the following assumption on p's payoff.

Assumption 3. *The payoff function ν^p can be expressed as*

$$\nu^p(y^p, y^{-p}; \xi^p) = \nu^{p,a}(y^p; \xi^p) - \sum_{k=1}^{K^p} \nu_k^{p,b}(y_k^p; \xi^p) \cdot \eta_k^p(y^{-p}; X),$$

for some $\nu^{p,a}$, $\{\nu_k^{p,b}\}_{k \in K^p}$, and $\{\eta_k^p\}_{k \in K^p}$.

In words, the payoff function can be decomposed so that the strategic part (i.e., the second term) is additively separable with respect to each dimension of the action. For the univariate case (i.e., $K^p = 1$), this assumption reduces to Aradillas-López and Gandhi (2016, Assumption 1).

For each belief σ^{-p}, the expected payoff for p from choosing y^p can be expressed as

$$\bar{\nu}_\sigma^p(y^p; \xi^p) = \sum_{y^{-p} \in \mathcal{A}^{-p}} \sigma^{-p}(y^{-p}) \cdot \nu^p(y^p, y^{-p}; \xi^p) = \nu^{p,a}(y^p; \xi^p) - \sum_{k=1}^{K^p} \nu_k^{p,b}(y_k^p; \xi^p) \cdot \bar{\eta}_{\sigma,k}^p(X),$$

where $\bar{\eta}_{\sigma,k}^p(X) = \sum_{y^{-p} \in \mathcal{A}^{-p}} \sigma^{-p}(y^{-p}) \cdot \eta_k^p(y^{-p}; X)$. Then for each $y_{-k}^p \in \mathcal{A}_{-k}^p$, pair of actions $v > u$ in $\mathcal{A}_k^p$, and pair of beliefs σ and σ', we obtain the following characterization for the changes in the expected payoff between (v, y_{-k}^p) and (u, y_{-k}^p):

$$\begin{aligned} & [\bar{\nu}_\sigma^p(v, y_{-k}^p; \xi^p) - \bar{\nu}_\sigma^p(u, y_{-k}^p; \xi^p)] - [\bar{\nu}_{\sigma'}^p(v, y_{-k}^p; \xi^p) - \bar{\nu}_{\sigma'}^p(u, y_{-k}^p; \xi^p)] \\ = \; & [\bar{\eta}_{\sigma',k}^p(X) - \bar{\eta}_{\sigma,k}^p(X)] \cdot [\nu_k^{p,b}(v; \xi^p) - \nu_k^{p,b}(u; \xi^p)]. \end{aligned} \tag{1}$$

We note that due to separability in Assumption 3, the right hand side of this expression is independent of y_{-k}^p.

Hereafter we fix the dimension k of interest and focus on inference for parameters contained in the component η_k^p. To derive moment inequalities from this characterization, we impose monotonicity of $v_k^{p,b}(y_k^p; \xi^p)$ with respect to y_k^p.

Assumption 4. *For each $v > u$ in $\mathcal{A}_k^p$, it holds $\nu_k^{p,b}(v; \xi^p) \geq \nu_k^{p,b}(u; \xi^p)$ with probability one.*

Under this assumption and (1), the event $\bar{\eta}_{\sigma,k}^p(X) \geq \bar{\eta}_{\sigma',k}^p(X)$ implies

$$\bar{\nu}_\sigma^p(v, y_{-k}^p; \xi^p) - \bar{\nu}_\sigma^p(u, y_{-k}^p; \xi^p) \leq \bar{\nu}_{\sigma'}^p(v, y_{-k}^p; \xi^p) - \bar{\nu}_{\sigma'}^p(u, y_{-k}^p; \xi^p),$$

for each k and $v > u$. Based on this, we obtain the following lemma for optimal choices under given beliefs. Let $y_\sigma^p(\xi^p) = (y_{\sigma,1}^p(\xi^p), ..., y_{\sigma,K^p}^p(\xi^p)) = \arg\max_{y \in \mathcal{A}^p} \bar{\nu}_\sigma^p(y; \xi^p)$.

Lemma 1. *Under Assumptions 1-4, the following logical relation holds:*

$$\bar{\eta}_{\sigma,k}^p(X) \geq \bar{\eta}_{\sigma',k}^p(X) \text{ and } y_{\sigma,-k}^p(\xi^p) = y_{\sigma',-k}^p(\xi^p) \implies \mathbb{I}\{y_{\sigma,k}^p(\xi^p) \leq y_k^p\} \geq \mathbb{I}\{y_{\sigma',k}^p(\xi^p) \leq y_k^p\},$$

with probability one, for each $y_k^p \in \mathcal{A}_k^p$, ξ^p, σ, and σ'.

By taking conditional expectation given X and Y^p_{-k}, we can derive the covariance restrictions (or moment inequalities) for observables.

Theorem 1. *Suppose Assumptions 1-4 hold. Then, for each $y^p_k \in \mathcal{A}^p_k$, it holds*

$$E[\mathbb{I}\{Y^p_k \leq y^p_k\} \cdot \eta^p_k(Y^{-p}; X)|X, Y^p_{-k}] \geq E[\mathbb{I}\{Y^p_k \leq y^p_k\}|X, Y^p_{-k}] \cdot E[\eta^p_k(Y^{-p}; X)|X, Y^p_{-k}],$$

with probability one.

Based on these moment inequalities, we can conduct inference on parameters that specify $\eta^p_k(Y^{-p}; X) = \eta^p_k(Y^{-p}; X|\theta^p_k)$. To implement inference on θ^p_k, we can employ several existing econometric methods for (conditional) moment inequalities, such as Andrews and Shi (2013) and Chernozhukov, Lee and Rosen (2011). Note that conditioning on the other action variables Y^p_{-k} is crucial to derive valid moment inequalities.

We can also show that if the BNE is unique, then the above moment inequalities become equalities. Thus, we can also conduct a statistical test for uniqueness of the BNE by testing the zero covariance restrictions. Such a test is considered as a multi-dimensional version of de Paula and Tang's (2012) test for uniqueness of the BNE.

Example 1. [Entry game with multi-product firms] Consider an entry game by P firms. Each firm p has K^p formats (e.g., high-priced and low-priced brands). Firm p determines the number of outlets with respect to each format, $y^p = (y^p_1, \ldots, y^p_{K^p})$. Let X be variables which affect profitability (e.g., population and income). Then consider firm p's profit from one outlet of the k-th format given by

$$\pi^p_k(y^p, y^{-p}, \xi) = \sum_{q\in\{1,\ldots,P\}} \sum_{l\in\{1,\ldots,K^q\}} (X'\theta_{p_k,q_l}) \cdot y^q_l + \varepsilon^p_k,$$

where $X'\theta_{p_k,q_l}$ represents the business stealing effect of firm q's l-th format on firm p's k-th format. Assuming that each firm's profit is the sum of profits from all of their outlets, firm p's payoff function is written as

$$\begin{aligned}
&\nu^p(y^p, y^{-p}; \xi) := \sum_{k\in\{1,\ldots,K^p\}} y^p_k \cdot \pi^p_k(y^p, y^{-p}, \xi) \\
= &\sum_{k\in\{1,\ldots,K^p\}} \sum_{l\in\{1,\ldots,K^p\}} y^p_k \cdot (X'\theta_{p_k,p_l}) \cdot y^p_l + \sum_{k\in\{1,\ldots,K^p\}} y^p_k \cdot \varepsilon^p_k + \sum_{k\in\{1,\ldots,K^p\}} y^p_k \sum_{q\neq p} \sum_{l\in\{1,\ldots,K^q\}} (X'\theta_{p_k,q_l}) \cdot y^q_l.
\end{aligned}$$

This setup fits into our Assumption 3 by setting

$$\begin{aligned}
\nu^{p,a}(y^p; \xi^p) &= \sum_{k\in\{1,\ldots,K^p\}} \sum_{l\in\{1,\ldots,K^p\}} y^p_k \cdot (X'\theta_{p_k,p_l}) \cdot y^p_l + \sum_{k\in\{1,\ldots,K^p\}} y^p_k \cdot \varepsilon^p_k, \\
\nu^{p,b}_k(y^p_k; \xi^p) &= y^p_k, \\
\eta^p_k(y^{-p}; X) &= \sum_{q\neq p} \sum_{l\in\{1,\ldots,K^q\}} (X'\theta_{p_k,q_l}) \cdot y^q_l.
\end{aligned}$$

Thus, Theorem 1 can be applied to conduct inference on the strategic parameters θ_{p_k,q_l} for $k \in \{1, ..., K^p\}$ and $l \in \{1, ..., K^q\}$ with $q \neq p$, which allows us to see whether firms have incentives to locate outlets similar to those of their competitors. This is an important empirical question since on the one hand, similar types of outlets cannibalize each other's demand, but on the other hand, they act as complementary goods. Although here we present an entry model, our multi-dimensional model encompasses Cournot and Bertrand games with multi-product firms as well.

2.2. **Second model: Strategic interaction through one channel.** As another example, this subsection considers the situation where only one channel directly affects the strategic interaction term. Again take any player p of interest. We now impose the following assumption.

Assumption 5. *ν^p can be expressed as*

$$\nu^p(y^p, y^{-p}; \xi^p) = \nu^{p,a}(y^p; \xi^p) - \nu^{p,b}(y_1^p; \xi^p) \cdot \eta^p(y_{-1}^p, y^{-p}; X),$$

for some $\nu^{p,a}$, $\nu^{p,b}$, and η^p.

In words, there exists only one channel y_1^p which directly affects the strategic interaction term $\nu^{p,b}$. In this case, the expected payoff for p of choosing y^p under belief σ can be written as

$$\bar{\nu}_\sigma^p(y^p; \xi^p) = \sum_{y^{-p} \in \mathcal{A}^{-p}} \sigma^{-p}(y^{-p}) \cdot \nu^p(y^p, y^{-p}; \xi^p) = \nu^{p,a}(y^p; \xi^p) - \nu^{p,b}(y_1^p; \xi^p) \cdot \bar{\eta}_\sigma^p(y_{-1}^p, X),$$

where $\bar{\eta}_\sigma^p(y_{-1}^p, X) = \sum_{y^{-p} \in \mathcal{A}^{-p}} \sigma^{-p}(y^{-p}) \cdot \eta^p(y_{-1}^p, y^{-p}; X)$. Then for each $y_{-1}^p \in \mathcal{A}_{-1}^p$, pair of actions $v > u$ in $\mathcal{A}_1^p$, and pair of beliefs σ and σ', we obtain the following characterization for the changes in the expected payoff between (v, y_{-1}^p) and (u, y_{-1}^p):

$$\begin{aligned} & [\bar{\nu}_\sigma^p(v, y_{-1}^p; \xi^p) - \bar{\nu}_\sigma^p(u, y_{-1}^p; \xi^p)] - [\bar{\nu}_{\sigma'}^p(v, y_{-1}^p; \xi^p) - \bar{\nu}_{\sigma'}^p(u, y_{-1}^p; \xi^p)] \\ (2) \qquad = \; & [\bar{\eta}_{\sigma'}^p(y_{-1}^p, X) - \bar{\eta}_\sigma^p(y_{-1}^p, X)] \cdot [\nu^{p,b}(v; \xi^p) - \nu^{p,b}(u; \xi^p)]. \end{aligned}$$

In addition, we maintain the assumption on monotonicity of $v^{p,b}(y_1^p; \xi^p)$ with respect to y_1^p.

Assumption 6. *For each $v > u$ in $\mathcal{A}_1^p$, it holds $\nu^{p,b}(v; \xi^p) \geq \nu^{p,b}(u; \xi^p)$ with probability one.*

Under this assumption (2), the event $\bar{\eta}_\sigma^p(y_{-1}^p, X) \geq \bar{\eta}_{\sigma'}^p(y_{-1}^p, X)$ implies

$$\bar{\nu}_\sigma^p(v, y_{-1}^p; \xi^p) - \bar{\nu}_\sigma^p(u, y_{-1}^p; \xi^p) \leq \bar{\nu}_{\sigma'}^p(v, y_{-1}^p; \xi^p) - \bar{\nu}_{\sigma'}^p(u, y_{-1}^p; \xi^p),$$

for each $v > u$. Based on this, we obtain the following lemma for optimal choices under given beliefs. Let $y_\sigma^p(\xi^p) = (y_{\sigma,1}^p(\xi^p), y_{\sigma,-1}^p(\xi^p)) = \arg\max_{y \in \mathcal{A}^p} \bar{\nu}_\sigma^p(y; \xi^p)$.

Lemma 2. *Under Assumptions 1-2 and 5-6, the following logical relation holds:*

$$\bar{\eta}_\sigma^p(y_{-1}^p, X) \geq \bar{\eta}_{\sigma'}^p(y_{-1}^p, X) \text{ and } y_{\sigma,-1}^p(\xi^p) = y_{\sigma',-1}^p(\xi^p) \implies \mathbb{I}\{y_{\sigma,1}^p(\xi^p) \leq y_1^p\} \geq \mathbb{I}\{y_{\sigma',1}^p(\xi^p) \leq y_1^p\},$$

with probability one, for each $y_1^p \in \mathcal{A}_1^p$, ξ^p, σ, and σ'.

By taking conditional expectation given X and Y_{-1}^p, we can derive covariance restrictions (or moment inequalities) for observables.

Theorem 2. *Suppose the Assumptions 1-2 and 5-6 hold. Then, for each $y_1^p \in \mathcal{A}_1^p$, it holds*

$$E[\mathbb{I}\{Y_1^p \leq y_1^p\} \cdot \eta^p(Y_{-1}^p, Y^{-p}; X)|X, Y_{-1}^p] \geq E[\mathbb{I}\{Y_1^p \leq y_1^p\}|X, Y_{-1}^p] \cdot E[\eta^p(Y_{-1}^p, Y^{-p}; X)|X, Y_{-1}^p].$$

Similar comments to Theorem 1 apply. Inference on parameters to specify $\eta^p(Y_{-1}^p, Y^{-p}; X)$ can be conducted by the existing econometric methods.

Example 2. [Bertrand game with advertisements] We consider a Bertrand game with advertisements. Each firm decides price and the level of advertisements simultaneously. Let $y^p = (y_1^p, y_2^p)$ be player p's two-dimensional action, where y_1^p denotes the price of its product and y_2^p denotes the level of its advertisements. Let X be demand shifters. We assume the following log-linear demand function for firm p's product,

$$\log Q^p = \sum_{q=1}^{P} a^{p,q}(X) \cdot \log y_1^q + \sum_{q=1}^{P} b^{p,q}(X) \cdot \log y_2^q + \varepsilon_3^p.$$

Firm p's cost function is given by

$$C^p(Q^p, y_2^p) = (c_1 + \varepsilon_1^p) \cdot Q^p + (c_2 + \varepsilon_2^p) \cdot y_2^p.$$

As discussed above, we allow that $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ are arbitrarily correlated. Then, we define firm p's payoff function as

$$\begin{aligned} & \nu^p(y^p, y^{-p}; \xi) := y_1^p Q^p - C^p(Q^p, y_2^p) \\ = \; & -(c_2 + \varepsilon_2) \cdot y_2^p + \{(y_1^p - c_1 - \varepsilon_1) \cdot \exp(a^{p,p}(X) \cdot \log y_1^p + \varepsilon_3^p)\} \\ & \times \left[\exp(b^{p,p}(X) \cdot \log y_2^p) \cdot \prod_{q \neq p} \exp(a^{p,q}(X) \cdot \log y_1^q) \cdot \prod_{q \neq p} \exp(b^{p,q}(X) \cdot \log y_2^q) \right] \end{aligned}$$

This setup fits into our Assumption 5 by setting

$$\begin{aligned} \nu^{p,a}(y^p; \xi^p) &= -(c_2 + \varepsilon_2) \cdot y_2^p, \\ \nu^{p,b}(y_1^p; \xi^p) &= (y_1^p - c_1 - \varepsilon_1) \cdot \exp(a^{p,p}(X) \cdot \log y_1^p + \varepsilon_3^p), \\ \eta^p(y_{-1}^p, y^{-p}; X) &= \exp(b^{p,p}(X) \cdot \log y_2^p) \cdot \prod_{q \neq p} \exp(a^{p,q}(X) \cdot \log y_1^q) \cdot \prod_{q \neq p} \exp(b^{p,q}(X) \cdot \log y_2^q). \end{aligned}$$

Thus, Theorem 2 can be applied to conduct inference on the parameters of η^p. In particular, the sign of $b^{p,q}(X)$ is of great concern since it is theoretically ambiguous. On the one hand, advertisements take away competitors' demand (i.e., business-stealing effects), but on the other hand, they increase consumers' awareness of all products in the market (i.e., complementary effects).

Appendix A. Mathematical appendix

Since the proofs of Lemma 2 and Theorem 2 are similar to those of Lemma 1 and Theorem 1, respectively, here we only present the proofs for Lemma 1 and Theorem 1.

A.1. **Proof of Lemma 1.** Take any ξ, $y_k^p \in \mathcal{A}_k^p$, $y_{-k}^p \in \mathcal{A}_{-k}^p$, σ, and σ'. Then define

$$\mathbb{I}_\sigma^p(y_k^p, y_{-k}^p; \xi^p) = \max_{u \leq y_k^p} \min_{v > y_k^p} \mathbb{I}\{\bar{\nu}_{\sigma,k}^p(v, y_{-k}^p; \xi^p) - \bar{\nu}_{\sigma,k}^p(u, y_{-k}^p; \xi^p) \leq 0\}.$$

By Assumption 4, if $\bar{\eta}^p_{\sigma,k}(X) \geq \bar{\eta}^p_{\sigma',k}(X)$, then

$$\mathbb{I}^p_\sigma(y^p_k, y^p_{-k}; \xi^p) \geq \mathbb{I}^p_{\sigma'}(y^p_k, y^p_{-k}; \xi^p). \tag{3}$$

Also, if $y^p_{\sigma,-k}(\xi^p) = y^p_{\sigma',-k}(\xi^p) = y^p_{-k}$, then the definition of $\mathbb{I}^p_\sigma(\cdot)$ yields

$$\mathbb{I}\{y^p_{\sigma,k}(\xi^p) \leq y^p_k\} = \mathbb{I}^p_\sigma(y^p_k, y^p_{-k}; \xi^p), \quad \mathbb{I}\{y^p_{\sigma',k}(\xi^p) \leq y^p_k\} = \mathbb{I}^p_{\sigma'}(y^p_k, y^p_{-k}; \xi^p). \tag{4}$$

Therefore, by combining (3) and (4), the conclusion follows.

A.2. **Proof of Theorem 1.** The proof is analogous to that of Aradillas-López and Gandhi (2016, Theorem 1). Given X, let $\{\sigma_{*j}(X)\}_{j=1}^J$ be the set of BNE and $P^S_j(X)$ be the probability that the equilibrium $\sigma_{*j}(X)$ is selected. The probability that equilibrium $\sigma_{*j}(X)$ is selected conditional on Y^p_{-k} is given by

$$P^S_j(X, Y^p_{-k}) = \frac{P^S_j(X) \cdot \sigma^p_{*j}(Y^p_{-k}|X)}{\sum_{j'=1}^J P^S_{j'}(X) \cdot \sigma^p_{*j'}(Y^p_{-k}|X)},$$

where with slight abuse of notation, $\sigma^p_{*j}(Y^p_{-k}|X)$ represents the probability function of Y^p_{-k} under the equilibrium $\sigma_{*j}(X)$. Observe that

$$\begin{aligned}
& E[\mathbb{I}\{Y^p_k \leq y^p_k\} \cdot \eta^p_k(Y^{-p}; X)|X, Y^p_{-k}] \\
= & \sum_{j=1}^J P^S_j(X, Y^p_{-k}) \cdot E_{\xi|X,Y^p_{-k}}[\mathbb{I}\{y^p_{\sigma_{*j},k}(\xi^p) \leq y^p_k\} \cdot \eta^p_k(y^{-p}_{\sigma_{*j}}(\xi^{-p}); X)\big|X, Y^p_{-k}] \\
= & \sum_{j=1}^J P^S_j(X, Y^p_{-k}) \cdot E_{\xi^p|X,Y^p_{-k}}[\mathbb{I}\{y^p_{\sigma_{*j},k}(\xi^p) \leq y^p_k\}\big|X, Y^p_{-k}] \cdot E_{\xi^{-p}|X}[\eta^p_k(y^{-p}_{\sigma_{*j}}(\xi^{-p}); X)\big|X] \\
= & E_{\xi^p|X,Y^p_{-k}}\left[\sum_{j=1}^J P^S_j(X, Y^p_{-k}) \cdot \mathbb{I}\{y^p_{\sigma_{*j}}(\xi^p) \leq y^p_k\} \cdot \bar{\eta}^p_{\sigma_{*j},k}(X)\middle| X, Y^p_{-k}\right],
\end{aligned}$$

where the second equality follows from $\xi^p \perp \xi^{-p}|X$ and $Y^p_{-k} \perp \xi^{-p}|X$ (by Assumption 1). We also have

$$\begin{aligned}
& E[\mathbb{I}\{Y^p_k \leq y^p_k\}|X, Y^p_{-k}] \cdot E[\eta^p_k(Y^{-p}; X)|X, Y^p_{-k}] \\
= & \sum_{j=1}^J P^S_j(X, Y^p_{-k}) \cdot E_{\xi^p|X,Y^p_{-k}}[\mathbb{I}\{y^p_{\sigma_{*j},k}(\xi^p) \leq y^p_k\}|X, Y^p_{-k}] \times \sum_{j=1}^J P^S_j(X, Y^p_{-k}) \cdot E_{\xi^{-p}|X}[\eta^p_k(y^{-p}_{\sigma_{*j}}(\xi^{-p}); X)|X] \\
= & \sum_{j=1}^J P^S_j(X, Y^p_{-k}) \cdot E_{\xi^p|X,Y^p_{-k}}[\mathbb{I}\{y^p_{\sigma_{*j},k}(\xi^p) \leq y^p_k\}|X, Y^p_{-k}] \times \sum_{j=1}^J P^S_j(X, Y^p_{-k}) \cdot \bar{\eta}^p_{\sigma_{*j},k}(X) \\
= & E_{\xi^p|X,Y^p_{-k}}\left[\left(\sum_{j=1}^J P^S_j(X, Y^p_{-k}) \cdot \mathbb{I}\{y^p_{\sigma_{*j},k}(\xi^p) \leq y^p_k\}\right) \times \left(\sum_{j=1}^J P^S_j(X, Y^p_{-k}) \cdot \bar{\eta}^p_{\sigma_{*j},k}(X)\right)\middle| X, Y^p_{-k}\right].
\end{aligned}$$

Combining these equations, we obtain

$$\begin{aligned}
& E[\mathbb{I}\{Y_k^p \le y_k^p\} \cdot \eta_k^p(Y^{-p};X)|X,Y_{-k}^p] - E[\mathbb{I}\{Y_k^p \le y_k^p\}|X,Y_{-k}^p] \cdot E[\eta_k^p(Y^{-p};X)|X,Y_{-k}^p] \\
= \; & E_{\xi^p|X,Y_{-k}^p}\left[\sum_{j=1}^J P_j^S(X,Y_{-k}^p)\cdot \mathbb{I}\{y_{\sigma*j,k}^P(\xi^p) \le y_k^p\} \cdot \bar{\eta}_{\sigma*j,k}^p(X)\right. \\
& \left. - \left(\sum_{j=1}^J P_j^S(X,Y_{-k}^p)\cdot \mathbb{I}\{y_{\sigma*j,k}^p(\xi^p) \le y_k^p\}\right) \times \left(\sum_{j=1}^J P_j^S(X,Y_{-k}^p)\cdot \bar{\eta}_{\sigma*j,k}^p(X)\right)\middle| X,Y_{-k}^p\right].
\end{aligned}$$

Note that conditioning on Y_{-k}^p implies conditioning on the event $\{y_{\sigma,-k}^p(\xi^p) = y_{\sigma',-k}^p(\xi^p)\}$. Now the object inside the above conditional expectation is nonnegative since it can be expressed as

$$\begin{aligned}
& \sum_{j=1}^J P_j^S(X,Y_{-k}^p)\cdot \mathbb{I}\{y_{\sigma*j,k}^p(\xi^p) \le y_k^p\} \cdot \bar{\eta}_{\sigma*jk}^p(X) \\
& - \left(\sum_{j=1}^J P_j^S(X,Y_{-k}^p)\cdot \mathbb{I}\{y_{\sigma*j,k}^p(\xi^p) \le y_k^p\}\right) \times \left(\sum_{j=1}^J P_j^S(X,Y_{-k}^p)\cdot \bar{\eta}_{\sigma*j,k}^p(X)\right) \\
= \; & \sum_{\ell=1}^J\sum_{j=1}^J P_\ell^S(X,Y_{-k}^p)P_j^S(X,Y_{-k}^p)\cdot \mathbb{I}\{y_{\sigma*j,k}^p(\xi^p) \le y_k^p\}\cdot(1-\mathbb{I}\{y_{\sigma*\ell,k}^p(\xi^p) \le y_k^p\})\cdot\{\bar{\eta}_{\sigma*j,k}^p(X) - \bar{\eta}_{\sigma*\ell,k}^p(X)\} \\
\ge \; & 0,
\end{aligned}$$

where the inequality follows from Lemma 1.

References

[1] Andrews, D. W. K. and X. Shi (2013) Inference based on conditional moment inequalities, *Econometrica*, 81, 609-666.

[2] Aradillas-Lopez, A. and A. Gandhi (2016) Estimation of games with ordered actions: an application to chain-store entry, *Quantitative Economics*, 7, 727-780.

[3] Chernozhukov, V., Lee, S. and A. M. Rosen (2013) Intersection bounds: estimation and inference, *Econometrica*, 81, 667-737.

[4] de Paula, A. and X. Tang (2012) Inference of signs of interaction effects in simultaneous games with incomplete information, *Econometrica*, 80, 143-172.

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