A Theory of 
"Economies of Diversification"

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Recently, a firm's diversification is frequently observed in many kinds of business fields. So far, an economic theory to explain the merit of business diversification is the concept of economies of scope, which means a better cost performance of joint production than separate production. More precisely we define economies of scope as the case where the cost of producing multiple output is less than the summation of cost of producing a single output by a single firm. We would like to start this analysis by considering the source of economies of scope.

In Baumol, Panzar, and Willig[1] the following explanation can be found concerning this point.

According to their explanation there are two main factors which cause economies of scope. One is a public input, and another is a sharable input. As an example of a public input, information, technology, and knowhow can be picked up. The electronic technology for producing a handy calculator can be easily diverted to the production of a digital watch. Once a firm gets these inputs, they become public goods because of the non-rivalness which makes it possible to divert them to a different purpose with little cost. On the other hand, as a sharable input we can choose usual production factors, labor and capital. A talented secretary can take care of two bosses in different sections at the same time, and a head office building can be shared by different activities. In a word the merit of a sharable input is to make use of many kinds of abilities of an employee, a machine, and a building.

The examples mentioned above, however, are observed not only in the case of business diversification. Because a public input is available for multiple activities in common, it usefully contributes to production in the existing activity. For the efficient utilization of a public input, it is enough to use it for the expansion of the existing activity. The same argument can be applied to a sharable input. In order to share the multiple talents of labor and capital it is necessary that they not be utilized to the fullest extent. The presence of economies of scope
caused by a sharable input, therefore, means a cost saving by the utilization of
an idle input. Just to raise the utilization ratio, diversification is not necessary,
but expanding the existing activity is sufficient. To summarize, neither a public
input nor a sharable input are distinctive sources of economies of scope.

What is a distinctive source of economies of scope? That is the benefit which
can be received by not doing a single activity but multiple activities. We can
give some examples of this benefit; the merger of Toyota Automobile Manu-
factoring Company with Toyota Automobile Sales Company, JR(Japan Railway
Company)'s entry into a business of travel agency and development of station
buildings, Seven-Eleven's network system, shops and restaurants in gas stations,
exhibitions and concerts in a department store, and so forth. The common theme
in all the examples is the utilization of factors which are acquired through the
existing activity and remain unused.

In the case of Toyota, JR's travel agency, and Seven-Eleven, economies of
scope arise by exchanging the information acquired in each activity. In Toyota
Manufacturing the technological information regarding automobiles is accumu-
lated, while in Toyota Sales market information including customer's preference
is collected. A merger of them makes it possible to exchange their information
and has a good effect on their business performance. The manufacturing sec-
tion can produce a car suitable for customer's needs, and the sales section can
advertise a car's quality to customers more precisely. In JR's case, also, the in-
formation exchange contributes to both activities. JR produces the information
of a timetable through the usual transport activity. This information had never
been utilized at all before JR started a travel agency business. After the diver-
sification JR's travel agency can freely get the information of timetables which
is necessary to produce a travel tour. On the other hand, the information from
JR's travel agency is useful for JR transport activity to make a timetable that
customers really need. Let me consider Seven-Eleven. Seven-Eleven is not only
a retail trade but a information production company with a huge network. The
basic activity is, needless to say, retail sales. More than 4000 branch shops of
Seven-Eleven, when they sell goods to customers, input the data of a date, time,
a good's name, and a customer's characteristics into the register and send this
information to Seven-Eleven's head office. The head office adds to the value of
the collected information by aggregating and processing it, and sells the valuable
information to manufacturing companies. Without the network system, the in-
formation acquired in each branch shop would be wasted. The characteristic of
Seven-Eleven is to collect the data of sales and to use it for information service
activity.

In the examples of JR's development of station buildings, stores and restau-
rants in gas stations, and exhibitions in department stores, we can find a source of
economies of scope peculiar to a service industry. In a service industry such as a
railway, a gas station, and a department store, production and consumption take
place at the same time. JR can produce transport service only when people ride on a train. Both in a gas station and a department store production means sales to customers. In these businesses, it is quite important knowing how to attract people together and keep them for a while before they leave. JR can make use of a large number of people who pass through stations every day by constructing a high station building with a hotel and a shopping center. A store in a gas station has an effect of making them stay and buy goods after providing gasoline. The purpose of an exhibition in a department store is to bring customers together. Needless to say, an exhibition itself does not yield a profit to a department store, although it is possible to make a profit from sales to people who come intending to see an exhibition.

As mentioned above, the aim of recently observed business diversification is not passive such as diversion of existing knowhow and utilization of idle inputs but a positive one to make use of factors which have been unused in an existing activity. By taking notice of this point, it becomes easier to predict a new direction of the diversification. There may still remain various kinds of unused factors in a firm, and it is a key of a firm's development how to utilize them in a new activity.

II

In the previous section we picked up economies of scope as a factor of business diversification. Is economies of scope, however, the only factor that we have to mention? Are there any factors we failed to notice?

The concept of economies of scope is originally introduced in industrial organization theory to derive the condition of natural monopoly in the case of multiple output. The necessary and sufficient condition of natural monopoly is equal to the subadditivity of the cost function. While in the case of single output this condition attributes to economies of scale, in the multiproduct case we have to pay attention to the effect of the change of output's composition on cost. Economies of scope treats the extreme case of this change, that is, the comparison between complete specialization and diversification. Economies of scope, therefore, fundamentally involves a different issue from business diversification. Industrial organization needs the concept of economies of scope to investigate whether one industry is in the situation of natural monopoly in the multiproduct case. On the other hand when we focus on the merit of business diversification, the main issue is what kind of benefit will arise by starting a new activity in addition to an existing one.

The difference of their analytical points of view stands out in relief when we undertake an empirical research concerning economies of scope. The existence

\(^1\) For the details of the characteristics of service industry, see Ihara[3].
of economies of scope is empirically confirmed through the estimation of a multiproduct cost function with the data of a firm which has already diversified its business. This kind of research, however, gives us no fruitful information about the merit of diversification, because a firm has already enjoyed the benefit of joint production. We are really interested in what kind of benefit will arise by adding a new activity that a firm has not been permitted to undertake alongside an existing activity. The empirical analysis of economies of scope, therefore, only gives ex-post interpretation for the benefit of already observed business diversification.

On the basis of the problems concerning economies of scope mentioned above, we would like to take up another source of diversification economy, which is an external effect. Classical examples of an external effect are the relationship between an orchard and an apiculturist and between an upstream factory and a downstream fisherman. In addition to these examples, recently, information including technology and knowhow is noticed as an important factor. One of the distinctive features of information as an economic good is externality, because the value of information that an economic unit owns highly depends on whether other units have already known it or not. As is well known, externality is treated as an obstacle that hinders the realization of Pareto optimality. One of the ways to solve the problem is a merger of interested firms. A merger internalizes the external effect and makes the profit after a merger greater than the total profit of the two firms before. In this sense the existence of externality can be a powerful incentive to business diversification through a merger.

III

On the basis of the argument in section I, we will construct a model of economies of scope where its occurrence mechanism is reflected.

As mentioned before, since the presence of economies of scope means better cost performance of joint production, a theoretical model is usually built with a cost function. A multiproduct cost function which is consistent with a production function of n kinds of input and m kinds of output can be written as

$$C = C(p_1, p_2, \ldots, p_n, y_1, y_2, \ldots, y_m),$$

where $p_i$ is i-th input price and $y_j$ is j-th output. For simplification we continue the argument below removing factor prices from (1). Economies of scope can be defined using (1) as follows.

$$C(y_1, y_2, \ldots, y_m) < \sum_{j=1}^{m} C(y_j, 0, \ldots, 0)$$

In the case of two outputs, (2) can be rewritten as

$$C(y_1, y_2) < C(y_1, 0) + C(0, y_2),$$

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and degree of economies of scope is defined as

$$SCOPE = \frac{C(y_1, 0) + C(0, y_2) - C(y_1, y_2)}{C(y_1, y_2)}. \quad (4)$$

Using (3), we can test whether economies of scope exists or not between the two output. In order for the inequality (3) to be available, however, we need the values of the two terms in the right side of (3) as well as that of the left side. It is quite difficult to calculate these values in the neighborhood of all the observations, and nearly impossible to build the null hypothesis of parameters for statistical test. The test of economies of scope, therefore, is transformed into a test of its sufficient condition, which is called a condition of cost complementarity between multiple products. Cost complementarity is defined as

$$\frac{\partial^2 C}{\partial y_1 \partial y_2} < 0. \quad (5)$$

Here we have to pay attention to the difference of meaning between cost complementarity in (5) and degree of economies of scope in (4). The value of (4) means to what degree a firm now enjoys the benefit of economies of scope. On the other hand, the magnitude of cost complementarity reveals the marginal benefit of an increase of one product measured by another product's marginal cost. Using (5) instead of (4), it becomes much easier to test the existence of economies of scope, and an usual statistical methodology is available with a specified cost function.

The methodology described above is certainly suitable to check whether economies of scope exist or not. However, as shown in section I while the source of economies of scope is discussed on the basis of production function, an empirical analysis is done with a flexible cost function. This approach makes the relationship between the source and the empirical consequence quite ambiguous. It is, therefore, desirable to specify the production function explicitly including the source of economies of scope, to derive the cost function consistent with it, and to test its existence. On this point of view, we will build a model where the information exchange is treated as the source of economies of scope in the production function.

The information exchange can be expressed in the following four production functions.

$\begin{align*}
y_1 &= f_1(x_1, I_2) \\
I_1 &= g_1(y_1) \\
y_2 &= f_2(x_2, I_1) \\
I_2 &= g_2(y_2),
\end{align*}$

$^{2}$Since this is a test of sufficient condition, we cannot deny the existence of economies of scope when the null hypothesis is rejected.
where \( x_1 \) and \( x_2 \) are input vectors used in activity 1 and activity 2 respectively, and \( y_1 \) and \( y_2 \) are their output. A firm producing \( y_1 \) gets information \( I_1 \) in activity 1, which can be used as an input in activity 2, and vice versa. The amount of information depends on the scale of each activity, \( y_1 \) and \( y_2 \).

If we assume that a firm minimizes its cost at a given output level, and also assume the homotheticity of the functions \( f_1 \) and \( f_2 \), the sufficient condition of economies of scope is derived as follows:

\[
\frac{\partial^2 C}{\partial y_1 \partial y_2} = -\left( \sum \frac{\partial^2 f_1}{\partial x_1 \partial x_1} r_{1i} r_{1i} \right) \frac{\partial f_1}{\partial y_1} \frac{\partial g_2}{\partial y_2} + \left( \sum \frac{\partial^2 f_1}{\partial x_1 \partial x_1} \right) \frac{\partial g_2}{\partial y_2} \left( \sum \frac{\partial f_1}{\partial x_1} r_{1i} \right)
\]

\[
- \left( \sum \frac{\partial^2 f_2}{\partial x_2 \partial x_2} r_{2i} r_{2i} \right) \frac{\partial f_2}{\partial y_1} \frac{\partial g_1}{\partial y_1} + \left( \sum \frac{\partial^2 f_2}{\partial x_2 \partial x_2} \right) \frac{\partial g_1}{\partial y_1} \left( \sum \frac{\partial f_2}{\partial x_2} r_{2i} \right)
\]

where \( r_{1i} \) and \( r_{2i} \) are positive multipliers, and \( p_{1i} \) and \( p_{2i} \) are prices of \( x_{1i} \) and \( x_{2i} \) respectively. According to (10), it can be shown that the cost complementarity is realized if the following conditions are satisfied.

- The production functions \( f_1 \) and \( f_2 \) are concave,
- the marginal productivity of \( I \) in the production functions is positive,
- the function \( g \) is an increasing function with respect to \( y \), and
- \( I \) and \( x \) are complementary in the sense that the increase of \( I \) raises the marginal productivity of \( x \).

On the basis of this model we can test economies of scope by specifying the production functions \( f \) and information generating functions \( g \). To check the sufficient condition, the four conditions above are available. Instead we can measure the degree of economies of scope directly in production functions (6) to (9) by comparing the cost of joint production with that of separate production.

\[\text{For the details about the derivation, see Appendix A.}\]

\[\text{In the case of separate production, zero value is substituted for } I \text{ in (6) and (8).}\]
In this section we build a model of economies of diversification concerned with the existence of externality mentioned in section II. Suppose there are two activities which affect external effect on each other written as the following production functions.

\[ y_1 = f_1(x_{11}, x_{12}, \ldots, x_{1m})g_1(x_{21}, x_{22}, \ldots, x_{2n}), \]
\[ y_2 = f_2(x_{21}, x_{22}, \ldots, x_{2n})g_2(x_{11}, x_{12}, \ldots, x_{1m}). \]

where \( y_1 \) and \( y_2 \) are output in activity 1 and 2 respectively, and \( x_{1i} \) is \( i \)-th input used in activity 1, and \( x_{2j} \) is \( j \)-th input used in activity 2. \( g \) function stands for the externality affected by the other activity.

Suppose there exist two firms named as A and B which independently do activity 1 and 2 respectively. Their profit can be defined as

\[ \pi_A = p_1f_1(x_{11}, \ldots, x_{1m})g_1(x_{21}, \ldots, x_{2n}) - w_{11}x_{11}, \]
\[ \pi_B = p_2f_2(x_{21}, \ldots, x_{2n})g_2(x_{11}, \ldots, x_{1m}) - w_{2j}x_{2j}, \]

where \( p_1 \) and \( p_2 \) are prices of \( y_1 \) and \( y_2 \), and \( w_{1i} \) and \( w_{2j} \) are prices of \( x_{1i} \) and \( x_{2j} \) respectively. Since firm A and B independently maximize their profit, the necessary condition of maximization can be written as

\[ \frac{\partial \pi_A}{\partial x_{1i}} = p_1 \frac{\partial f_1}{\partial x_{1i}} g_1(x_{21}, \ldots, x_{2n}) - w_{1i} = 0, \quad i = 1, \ldots, m, \]
\[ \frac{\partial \pi_B}{\partial x_{2j}} = p_2 \frac{\partial f_2}{\partial x_{2j}} g_2(x_{11}, \ldots, x_{1m}) - w_{2j} = 0, \quad j = 1, \ldots, n. \]

Multiplying \( f_1(x_{11}, \ldots, x_{1m}) \) on both side of (15) and \( f_2(x_{21}, \ldots, x_{2n}) \) on both side of (16), we get

\[ \frac{\partial f_1}{\partial x_{1i}} - w_{1i}f_1(x_{11}, \ldots, x_{1m}) = 0, \]
\[ \frac{\partial f_2}{\partial x_{2j}} - w_{2j}f_2(x_{21}, \ldots, x_{2n}) = 0. \]

Solving (17) and (18) with respect to \( x_{1i} \) and \( x_{2j} \) respectively, we derive the input demand function as follows.

\[ x_{1i} = h_{1i}(p_1, w_{1i}, \ldots, w_{1m}, y_1) \]
\[ x_{2j} = h_{2j}(p_2, w_{2j}, \ldots, w_{2n}, y_2) \]

*Bacharach* derived the characteristics of the root of non-cooperative game by using a cost function when externality between \( n \) kinds of activities exists.
Substituting (19) and (20) in (11) and (12), we get

\[ y_1 = f_1(h_{11}(y_1), \ldots, h_{1m}(y_1), y_2, \ldots, h_{2n}(y_2, \ldots)), \]
\[ y_2 = f_2(h_{21}(y_2, \ldots), h_{2m}(y_2, \ldots), y_1, \ldots, h_{1m}(y_1, \ldots)). \]

(21) and (22)

If we apply the Cournot-Nash equilibrium to this model, (21) and (22) stand for the reaction curves of a firm A and B respectively and the equilibrium output levels of \( y_1 \) and \( y_2 \) are determined as the intersection point of the two curves.

To reveal the shape of the curves we calculate their slope. Totally differentiating (21) with respect to \( y_1 \) and \( y_2 \), we get

\[
d_{y_1} = \left( \sum \frac{\partial f_1}{\partial x_{1i}} \frac{\partial h_{1i}}{\partial y_1} \right) g_1(h_{21}(y_2), \ldots, h_{2n}(y_2)) dy_1 + \left( \sum \frac{\partial f_2}{\partial x_{2j}} \frac{\partial h_{2j}}{\partial y_2} \right) g_2(h_{11}(y_1), \ldots, h_{1m}(y_1)) dy_2. \tag{23}
\]

To rewrite (23),

\[
\frac{dy_2}{dy_1} = \frac{1 - \left( \sum \frac{\partial f_1}{\partial x_{1i}} \frac{\partial h_{1i}}{\partial y_1} \right) g_1(h_{21}(y_2), \ldots, h_{2n}(y_2))}{\left( \sum \frac{\partial g_1}{\partial x_{2j}} \frac{\partial h_{2j}}{\partial y_2} \right) f_1(h_{11}(y_1), \ldots, h_{1m}(y_1))}
= \frac{f_1(h_{11}(y_1), \ldots, h_{1m}(y_1)) - y_1 \left( \sum \frac{\partial f_1}{\partial x_{1i}} \frac{\partial h_{1i}}{\partial y_1} \right)}{\left( \sum \frac{\partial g_1}{\partial x_{2j}} \frac{\partial h_{2j}}{\partial y_2} \right) [f_1(h_{11}(y_1), \ldots, h_{1m}(y_1))]^2} \tag{24}
\]

is derived. If we assume the concavity of \( f_1 \), it can be shown that the numerator of (24) is negative. The sign of (24), therefore, depends on that of \( \sum \frac{\partial g_1}{\partial x_{2j}} \frac{\partial h_{2j}}{\partial y_2} \) in the denominator. Since \( \frac{\partial h_{2j}}{\partial y_2} \) is usually positive, the sign is determined by \( \frac{\partial g_1}{\partial x_{2j}} \), which stands for the marginal external effect of \( j \)-th input used for activity 2 on activity 1. If \( \sum \frac{\partial g_1}{\partial x_{2j}} \frac{\partial h_{2j}}{\partial y_2} \), which means weighted summation of externality from activity 2 to 1, is positive, the slope of (21) is positive. To sum up, the reaction curve of firm A is right ascending when the external effect from activity 2 to 1 is positive and right descending when the effect is negative.

\[ ^4 \text{It is shown that the first order partial derivative of the numerator with respect to } y_1 \text{ is non-negative. Because the numerator has the value of zero when } y_1 \text{ is equal to zero, it is obvious that the numerator is non-negative.} \]
The slope of firm B's reaction curve also can be reduced as follows.

\[
\frac{dy_2}{dy_1} = \frac{\left( \sum \frac{\partial f_2}{\partial x_{y_1}} \frac{\partial y_{y_1}}{\partial x_{y_1}} \right) \left[ f_2(h_{21}(y_2), \ldots, h_{2n}(y_2)) \right]^2}{f_2(h_{21}(y_2), \ldots, h_{2n}(y_2)) - y_2 \left( \sum \frac{\partial f_2}{\partial x_{y_2}} \frac{\partial y_{y_2}}{\partial y_{y_2}} \right)}
\]

(25)

If the concavity of \( f_2 \) is assumed, the sign of \( \frac{dy_2}{dy_1} \) depends on whether the external effect of activity 1 on activity 2 is positive or negative. If positive, firm B's reaction curve is right ascending, and if negative, the curve is right descending.

Here we classify the types of the Cournot-Nash equilibrium into four cases in Table 1.

| [Table 1] Four Types of Externality between A and B |
|---------------------------------|-----------------|-----------------|-----------------|-----------------|
| Type 1  | economy | economy | right ascending | right descending |
| Type 2  | economy | diseconomy | right descending | right ascending |
| Type 3  | diseconomy | economy | right ascending | right descending |
| Type 4  | diseconomy | diseconomy | right descending | right descending |

The four types can be revealed in the following figures.
In all the figures above the convergence to the equilibrium is realized. It can be shown that the condition for the convergence is greater absolute value of the slope of A's reaction curve than that of B's. The condition, therefore, can be written as follows:\(^7\).

\(^7\)See Appendix B for the details.
If the inequality (26) is satisfied, the Cournot-Nash equilibrium is realized. The equilibrium, however, is optimal neither for A nor B. There exists an area where profits of both A and B are improved. The figures below show the area by adding the iso-profit curves of A and B in [Figure 1].

\[
\left| \frac{f_1(h_{11}(y_1), \ldots, h_{1m}(y_1)) - y_1 \left( \sum \frac{\partial f_1}{\partial x_{1i}} \frac{\partial h_{1i}}{\partial y_1} \right)}{\left( \sum \frac{\partial y_1}{\partial x_{2i}} \frac{\partial h_{2i}}{\partial y_2} \right) [f_1(h_{11}(y_1), \ldots, h_{1m}(y_1))]^2} \right| > \\
\left| \frac{\left( \sum \frac{\partial y_2}{\partial x_{2i}} \frac{\partial h_{2i}}{\partial y_1} \right) [f_2(h_{21}(y_2), \ldots, h_{2n}(y_2))]^2}{f_2(h_{21}(y_2), \ldots, h_{2n}(y_2)) - y_2 \left( \sum \frac{\partial f_2}{\partial x_{2i}} \frac{\partial h_{2i}}{\partial y_2} \right)} \right| .
\]
In the shadowed areas both A's profit and B's profit are greater than those at the Cournot-Nash equilibrium point. A merger is one of the ways for A and B to move into these areas. The merged company can choose the combination of $y_1$ and $y_2$ which gives the maximum profit.

As mentioned above, when there exist two activities which have external effects on each other, the separate production by two firms does not realize the optimal situation. However, if one of them starts the other activity by means of a merger, the external effect is internalized and total profit increases.
In this paper I picked up two factors of "Economies of Diversification" which are economies of scope and externality. Economies of scope means the better cost performance of joint production than separate production. Externality, on the other hand, gives incentive to a firm to diversify its activity in the sense that the total profit after diversification is greater than before because of the internalization of the external effect.

The source of economies of scope is also a main issue in this paper. It was shown that public inputs and sharable inputs, which are pointed out as the sources of economies of scope in Baumol, Panzar, and Willig[1], are not distinctive sources, because these inputs are available even when a firm expands only its existing activity. As the distinctive source, here, I picked up the existence of some inputs which are acquired through the existing activity but unused there. If these inputs are useful in another activity, a firm doing its existing activity leaves them wasted, but the entry into a new activity gives benefit to the firm. As shown in section III, the exchange of a useful input between two activities saves the cost.

Externality has been usually treated as an obstacle to realize Pareto optimality in a textbook of economics. This point of view is certainly useful, for example, when we search the way to solve environmental problems. Another important point that we cannot fail to notice, however, is the role of 'information' as the source of externality. Information, including technology and knowhow, has recently become a valuable input for production according to the development of a communications network. Information has two aspects of externality: external economy and external diseconomy. Technology, knowhow, and information concerning stock prices can be picked up as the sources of external economy. A firm has a chance to make a profit by getting this information earlier than its competitors. On the other hand, there exists information which becomes effective when many people know it. The information, for example, concerning date, time, and method of transaction in a market is valuable for a firm when many other firms get it, because a market transaction does not work at all without many participants.

As mentioned above, information should be focused on as a source of economies of diversification. There frequently occur mergers and diversification between activities which are seemingly unrelated to each other. We have to notice the invisible existence of information behind them more carefully in the future.
Appendix A

First we calculate the marginal change of cost with respect to the marginal change of \( y_1 \) leaving \( y_2 \) constant. Totally differentiating (6) and (8), we get

\[
dy_1 = \sum \frac{\partial f_1}{\partial x_{1j}} dx_{1j} \tag{27}
\]

\[
dy_2 = \sum \frac{\partial f_2}{\partial x_{2j}} dx_{2j} + \frac{\partial f_2}{\partial l_1} dl_1
\]

\[
= \sum \frac{\partial f_2}{\partial x_{2j}} dx_{2j} + \frac{\partial f_2}{\partial l_1} \frac{\partial g_1}{\partial y_1} dy_1
\]

\[
= \sum \frac{\partial f_2}{\partial x_{2j}} dx_{2j} + \frac{\partial f_2}{\partial l_1} \frac{\partial g_1}{\partial y_1} \sum \frac{\partial f_1}{\partial x_{1j}} dx_{1j}
\]

\[
= 0. \tag{28}
\]

If we assume that \( f \) is a homothetic function, the expansion of output under a firm’s cost minimization increases the amount of each input proportionally. Defining \( r_{ij} \) and \( r_{2j} \) as some positive constant numbers, we get the following equations.

\[
dx_{1j} = dx_1 r_{1j} \tag{29}
\]

\[
dx_{2j} = dx_2 r_{2j} \tag{30}
\]

Using (29) and (30), we can rewrite (27) and (28) as

\[
dy_1 = dx_1 \sum \frac{\partial f_1}{\partial x_{1j}} r_{1j}, \tag{31}
\]

\[
dx_2 = -\frac{\sum \frac{\partial f_2}{\partial x_{2j}} r_{2j}}{\partial l_1 \frac{\partial g_1}{\partial y_1} \sum \frac{\partial f_1}{\partial x_{1j}}} dx_1. \tag{32}
\]

Totally differentiating the equation defining cost, \( C = p_{1j} x_{1j} + p_{2j} x_2 \), we get

\[
dC = \sum p_{1j} x_{1j} + \sum p_{2j} x_{2j} \]

\[
= dx_1 \sum p_{1j} r_{1j} + dx_2 \sum p_{2j} r_{2j}
\]

\[
= dx_1 \sum p_{1j} r_{1j} - \frac{\sum \frac{\partial f_1}{\partial x_{1j}}}{\sum \frac{\partial f_2}{\partial x_{2j}}} dx_1. \tag{33}
\]
Appendix B

Suppose that the reaction curves of A and B can be written as follows respectively.

\[ y_1 = F(y_{2,-1}) \]  \hspace{1cm} (41)
\[ y_2 = G(y_1), \]  \hspace{1cm} (42)

where \( y_{2,-1} \) is firm B's output level in the previous term. The two production functions above imply that firm A determines its current output level according to firm B's output level in the previous term, and that firm B determines its current output level according to A's current output level. Removing \( y_1 \) from (41) and (42) we get

\[ y_2 = G[F(y_{2,-1})]. \]  \hspace{1cm} (43)

Totally differentiating (43) we get

\[ dy_2 = \frac{\partial G}{\partial y_1} \frac{\partial F}{\partial y_{2,-1}} dy_{2,-1}. \]  \hspace{1cm} (44)

For the convergence of \( y_2 \), the current marginal change of \( y_2 \), \( dy_2 \), has to converge to zero. The equation (44) shows that the condition for the convergence is

\[ \left| \frac{\partial G}{\partial y_1} \frac{\partial F}{\partial y_{2,-1}} \right| < 1. \]  \hspace{1cm} (45)

Defining the inverse function of \( F \) as \( F^{-1} \), (45) can be rewritten as

\[ \left| \frac{\partial G}{\partial y_1} \right| \left| \frac{\partial F^{-1}}{\partial y_{2,-1}} \right| < 1. \]  \hspace{1cm} (46)

Finally we get

\[ \left| \frac{\partial F^{-1}}{\partial y_{1}} \right| > \left| \frac{\partial G}{\partial y_1} \right|. \]  \hspace{1cm} (47)
References

