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Measuring Marginal Utility:
The Problem of Irving Fisher Revisited

by

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Abstract

The method of measuring marginal utility devised by Irving Fisher is discussed. Deterministic nature of the method is illuminated and an attempt for statistical extension is made. The discussion is a step for filling the gap between the classical methods of measuring marginal utility and modern econometric methods of estimating utility functions.

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1. Introduction

1.1 The rise of the utility theory also gave rise to a discussion on necessity and possibility of empirical measurement of the notion of utility, Jevons [4]. I. Fisher [1] and R. Frisch [2,3] developed Jevons' idea by elaborating practical methods for utility measurement. Since then, the notions of the indifference curve and the marginal rate of substitution have replaced the utility function and the marginal utility in the theory of consumers' behavior. It appears now that the work of Fisher and Frisch is only of a historical interest in the field of Econometrics.

In applications of Fisher and Frisch method, we are liable to get confused with inconsistent measurements provided by their method: Employing Fisher method in its original form and Engel curves at two time points (or places), we can obtain a measurement of a marginal utility curve. If one more independent Engel curve is available, we end up with three measurements of the curve. These measured curves have to be identical in principle or close with each other at least approximately. Actual measurements, however, do not show this identity or close approximation. This instability of measurements by Fisher and Frisch method may have been one of the reasons why one casts doubt upon the validity of their method.

1.2 The problem of utility measurement has gradually earned a modern outlook through the work of Wald [9], Stone [8], Parks [6] and others. Principles of modern methods of utility measurement or statistical estimation of utility functions are no different from those of the classical methods, for utility measurement is impossible without the equilibrium equation of the theory of consumers' behavior. Statistical principles and the equilibrium equation together constitute modern versions of econometric method for estimating utility functions. It may be thought that Fisher and Frisch method is sensitive to statistical error and small error cause large variation of their measurement. And it is felt that a certain statistical principle should be added to Fisher and Frisch method.

1.3 In this article we examine Fisher's principle of utility measurement and try to find out what makes his method yield unstable results. A suggestion for a statistical extension is also made to resolve the problem of instability.

2. I. Fisher's method

2.1 We consider a two good model of the consumers' behavior. Let (q_F, p_F) and (q_H, p_H) be quantity consumed and price of goods F and H respectively.

Total expenditure E satisfies the budget equation,

$$(2.1) \qquad p_F\, q_F + p_H\, q_H = E\ .$$

We set the functions

$$(2.2) \qquad \begin{aligned} u_F &= u_F(q_F)\ , \\ u_H &= u_H(q_H)\ , \end{aligned}$$

to represent marginal utilities of the goods F and H, where the functions u_F and u_H are assumed to be dependent only on q_F and q_H respectively.

The marginal utility of money λ is a function of p's and E

$$(2.3) \qquad \lambda = \lambda\ (p_F\ ,\ p_H\ ,\ E).$$

The first order condition for the utility maximization is

$$(2.4) \qquad u_F\ /\ p_F = u_H\ /\ p_H\ .$$

We rewrite this equation into the form

$$(2.5) \qquad q_H = f\ (q_F\ ,\ E),$$

which corresponds to the expansion path given p_F and p_H fixed. By C we denote the expansion path under the relative price $\phi = p_F\ /\ p_H$.

2.2 Fisher devised a procedure for measuring the marginal utility (of money) under the assumptions:

(a) A set of budget data, which represents two expansion paths C_1 and C_2 under two different relative price situations ϕ_1 and ϕ_2 , is available.

(b) The utility function underlying the budget data is uniform.

(c) The utility function is additive so that the marginal utility functions take the form of (2.2).

The assumptions (a), (b), (c) provide a sufficient condition for the possibility of utility measurement. Frisch presented different sufficient conditions.

2.3 An actual problem we encounter in measurement work is not whether the assumptions (a), (b), (c) are really sufficient condition, but whether the hypotheses (b), (c) are empirically valid to explain variations of the data (a).

J. N. Morgan [5] takes up Fisher method, regarding Boston and Food as C_1 and F respectively. Substituting several cities and several consumption items for C_2 and H, he gets a number of combination of data and therefore different measurements of the marginal utility of money in Boston. The result which does not show much unifomity among the measurements may contribute to doubts on the validity of Fisher method or even the possibility of utility measurement.

However, if we want to determine empirically an additive utility function which is intended to explain budget data of more than two goods and of more than two places, Fisher's method should be extended to analyze more complicated models.

2.4 We set here another assumption:

(d) The expansion path C is approximated by a linear equation,

$$q_H = \alpha + \beta q_F . \tag{2.6}$$

This assumption is not necessary for Fisher's principle but this kind of operationality is required in actual analysis.

The assumption (d) in addition to the additivity assumption implies that the utility function is one of the member of Pollak family of utility functions, Pollak 6.

2.5 Let C_1 and C_2 in Figure 1 be the expansion paths of times 1 and 2. From the fixed initial point a we determine points b, c, ... , so that the following equations are satisfied,

$$\begin{array}{ll} q_F(a) = q_F(b), & q_H(b) = q_H(c), \\ q_F(c) = q_F(d), & q_H(d) = q_H(e), \\ \quad \ldots , & \quad \ldots . \end{array} \tag{2.7}$$

The first order condition for utility maximization at the point a is

$$(2.8) \qquad u_F(q_F(a)) \; / \; p_{F1} = u_H(q_H(a)) \; / \; p_{H1} = \lambda_1(a),$$

where (p_{Ft}, p_{Ht}) is a price vector at time t, and $\lambda_t(a)$ is marginal utility of money at equilibrium point a. Similarly, the equations hold at points b, c, ... ,

$$(2.9) \qquad \begin{aligned} u_F(q_F(b)) \; / \; p_{F2} &= u_H(q_H(b)) \; / \; p_{H2} = \lambda_2(b), \\ u_F(q_F(c)) \; / \; p_{F1} &= u_H(q_H(c)) \; / \; p_{H1} = \lambda_1(c), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

In view of the additivity and (2.7) we have

$$(2.10) \qquad \begin{aligned} u_F(q_F(a)) &= u_F(q_F(b)), \qquad u_H(q_H(b)) = u_H(q_H(c)), \\ u_F(q_F(c)) &= u_F(q_F(d)), \qquad u_H(q_H(d)) = u_H(q_H(e)), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

Then we get relationships between the marginal utilities of money at a and b,

$$(2.11) \qquad \lambda_1(a) \; p_{F1} = u_F(q_F(a)) = u_F(q_F(b)) = \lambda_2(b) \; p_{F2} \; ,$$

and therefore

$$(2.12) \qquad \lambda_2(b) = \lambda_1(a) \; p_{F1} \; / \; p_{F2} \; .$$

Similarly, we obtain the equations,

$$\lambda_1(c) = \lambda_2(b)\ p_{H2} / p_{H1} ,$$

(2.13)

$$\lambda_2(d) = \lambda_1(c)\ p_{F1} / p_{F2} ,$$

. . . .

Normalizing as $\lambda_1(a) = 1$ and denoting p_{Ft} / p_{Ht} by ϕ_t, we can get a table for calculating marginal utility from price data;

(2.14)

x	$\lambda_1(x)$	y	$\lambda_2(y)$
a	1		
		b	$1\ p_{F1} / p_{F2}$
c	(ϕ_1 / ϕ_2)		
		d	$(\phi_1 / \phi_2)(p_{F1} / p_{F2})$
e	$(\phi_1 / \phi_2)^2$		
		f	$(\phi_1 / \phi_2)(p_{F1} / p_{F2})^2$

2.6 Since the marginal utility equation (2.4) holds at each point, we can calculate u_F and u_H by the equations,

(2.15)

$$u_F = \lambda\ p_F$$

$$u_H = \lambda\ p_H$$

where λ is given by (2.14).

Thus, given the expansion paths C_1, C_2 and the relative prices ϕ_1, ϕ_2, we first determine the points a, b, c, ... , then calculate the marginal utilities u_F, u_H at these points according to the following tables;

(2.16)

x	$u_F(x)$
$q_F(a) = q_F(b)$	$p_{F1} 1$
$q_F(c) = q_F(d)$	$p_{F1} (\phi_1 / \phi_2)$
$q_F(e) = q_F(f)$	$p_{F1} (\phi_1 / \phi_2)^2$
...	...

(2.17)

y	$u_H(y)$
$q_H(a)$	$p_{H1} 1$
$q_H(b) = q_H(c)$	$p_{H1} (\phi_1 / \phi_2)$
$q_H(d) = q_H(e)$	$p_{H1} (\phi_1 / \phi_2)^2$
...	...

2.7 The measurements, $p_{F1}(\phi_1 / \phi_2)^i$, $p_{H1}(\phi_1 / \phi_2)^i$ are functions of the exogenous prices and are therefore free from errors in the sense of 'shock'.

Estimation problems occur when we determine points a, b, c, ..., or essentially when we fit linear expansion paths C_1, C_2 to bodget data. The fitted linear equations are subject to sampling errors, so are the determined points a, b, c, ...

2.8 We write the fitted linear paths (regression equations by the method of least squares, for instance), for the budget data of time t, as

$$q_H = \alpha_t + \beta_t q_F , \qquad t=1,2. \tag{2.18}$$

Letting $q_F^0 = q_F(a)$, $q_F^1 = q_F(c)$, $q_F^2 = q_F(e)$, and so on, we have

$$\begin{aligned} q_H &= \alpha_1 + \beta_1 q_F^{i+1} , \\ q_H &= \alpha_2 + \beta_2 q_F^{i} , \qquad i=0,1,2, \dots , \end{aligned} \tag{2.19}$$

or

$$q_F^{i+1} = (\alpha_2 - \alpha_1) / \beta_1 + (\beta_2 / \beta_1) q_F^{i} , \qquad i=0,1,2, \dots . \tag{2.20}$$

The solution of the difference equation is, if $\beta_1 \neq \beta_2$,

$$q_F^{i} = (\alpha_2 - \alpha_1)/(\beta_1 - \beta_2) + (q_F^0 - (\alpha_2 - \alpha_1)/(\beta_1 - \beta_2)) \times (\beta_2 / \beta_1)^i , \tag{2.21}$$

or if $\beta_1 = \beta_2$,

(2.22) $$q_F^{\,i} = i(\alpha_2 - \alpha_1) / \beta_1 + q_F^{\,0} .$$

In a similar way, letting $q_H^{\,0} = q_H(a)$, $q_H^{\,1} = q_H(b)$, $q_H^{\,2} = q_H(d)$, and so on, we have the solution, if $\beta_1 \neq \beta_2$,

(2.23) $$q_H^{\,i} = (\beta_1\alpha_2 - \beta_2\alpha_1)/(\beta_1 - \beta_2) + (q_H^{\,0} - (\beta_1\alpha_2 - \beta_2\alpha_1)/(\beta_1 - \beta_2)) \times (\beta_2 / \beta_1)^i ,$$

or if $\beta_1 = \beta_2$,

(2.24) $$q_H^{\,i} = i(\alpha_2 - \alpha_1) + q_H^{\,0} , \qquad i=0,1,2, \ldots .$$

For $q_F^{\,i}$ given by (2.21) or (2.22), the measurement of its marginal utility is

(2.25) $$u_F(q_F^{\,i}) = p_{F1} (\phi_1 / \phi_2)^i ,$$

and for $q_H^{\,i}$ given by (2.23) or (2.24), the utility measurement is

(2.26) $$u_H(q_H^{\,i}) = p_{H1} (\phi_1 / \phi_2)^i .$$

3. Deterministic measurement

3.1 The data on which our analysis in this and the following sections is based is a set of cross-sections of the years 1965 through 1973, Family Income and Expenditure Survey by Bureau of Statistics, Office of the Prime Minister Japan.

The goods F and H are identified as Food and Housing according to the FIES classification. The p_F and p_H are the corresponding price indexes. The q_F and q_H are the quantity indexes derived from the nominal statistics and the price index.

3.2 First fitting regression equations (Engel curves of F and H) to the cross-sectional data of time t by the least squares method, then eliminating the variable of the total expenditure E, we get estimates of coefficients, α_t and β_t, of the expansion path C_t. The estimates and the relative prices for the years 1965 through 1973 are given in Table 1, columns (1) and (5). Later on we call C_t the least squares expansion path.

Among several (9! / 7! 2!) combinations of pair of the relative prices, the pair of ϕ_{1966} and ϕ_{1973} gives the largest difference. We therefore apply Fisher method of Section 2 to the pair of the least squares expansion paths of the years 1966 and 1973.

3.3 Using the method of Section 2, we obtain a measurement of marginal utilities of the goods Food and Housing for the two years, the result being illustrated in Figure 2.

3.4 From the measured curves and the relative price data, we can predict expansion paths, $\hat{C}_t$, for the remaining seven years under different price situation. Comparatively good predicted expansion paths for the years 1965 and 1970 are given in Figure 3.a and Figure 3.b. It will be shown that for the years 1966 and 1973 the prediction $\hat{C}_t$ and observed C_t coincides and that is why we call the method deterministic.

3.5 It appears that the prediction $\hat{C}_{1965}$ based on the measurement from the data of 1966 and 1973 approximates the observed C_{1965} fairly closely. Therefore it is thought reasonable to measure the utility curve from the pair of 1965 and 1966 data. But the least squares expansion paths $C_{1965}$and C_{1966} and the relative prices ϕ_{1965} and ϕ_{1966} turn out to bear a relation like the one in Figure 4. We can not find any regular utility function which yields the expansion paths C_1 and C_2 consistently with utility maximization under the price situations ϕ_1 and ϕ_2.

Among the nine years, we have some pairs of the expansion paths which cross at some point in the observation range. Fisher method dose not work for such cases.

Even leaving aside these pathological (non-integrability) cases, we can not see much uniformity among the measurements from various combinations of C's.

4. Analytical fitting

4.1 If $\beta_1 \neq \beta_2$ and $q_F^{\,0} > (\alpha_2 - \alpha_1)/(\beta_1 - \beta_2)$, the succesive $q_F^{\,i}$'s are given by (2.21) and corresponding values of utilities are given by (2.25). Eliminating the discrete variable i, we obtain, from (2.21) and (2.25),

$$(4.1) \qquad u_F = p_{F1}\,(q_F^{\,0} - (\alpha_2 - \alpha_1)/(\beta_1 - \beta_2))^{-\varepsilon}(q_F - (\alpha_2 - \alpha_1)/(\beta_1 - \beta_2))^{+\varepsilon},$$

where

$$(4.2) \qquad \varepsilon = \log(\phi_1/\phi_2) \,/\, \log(\beta_2/\beta_1).$$

Similarly, we obtain, from (2.23) and (2.26),

$$(4.3) \qquad u_H = p_{H1}(q_H^{\,0} - (\beta_1\alpha_2 - \beta_2\alpha_1)/(\beta_1 - \beta_2))^{-\varepsilon}(q_H - (\beta_1\alpha_2 - \beta_2\alpha_1)/(\beta_1 - \beta_2))^{+\varepsilon}$$

Dividing both sides of (4.1) and (4.3) by $p_{H1}\,\beta_1^{-\varepsilon}(q_F^{\,0} - (\alpha_2 - \alpha_1)/(\beta_1 - \beta_2))^{-\varepsilon}$ and recalling that $q_H^{\,0} = \alpha_1 + \beta_1\, q_F^{\,0}$, we get

$$(4.4) \qquad \begin{aligned} u_F^{*} &= \phi_1\beta_1^{\,\varepsilon}(q_F - (\alpha_2 - \alpha_1)/(\beta_1 - \beta_2))^{\varepsilon}, \\ u_H^{*} &= \qquad (q_H - (\beta_1\alpha_2 - \beta_2\alpha_1)/(\beta_1 - \beta_2))^{\varepsilon}, \end{aligned}$$

which is called the normalized measurement.

4.2 The first order condition under the prices p_{Ft}, p_{Ht} is

$$u^* / p_{Ft} = u_H^* / p_{Ht} , \tag{4.5}$$

which reduces to the equation

$$q_H = a_t + b_T q_F , \tag{4.6}$$

where

$$\begin{aligned} \log b_t &= (\log(\phi_1/\phi_t)/ \log(\phi_1/\phi_2)) \log\beta_2 \\ &\quad - (\log(\phi_2/\phi_t) / \log(\phi_1/\phi_2)) \log\beta_1 , \\ a_t &= ((\beta_1 - b_t)\alpha_2 - (b_t - \beta_2)\alpha_1) / (\beta_1 - \beta_2). \end{aligned} \tag{4.7}$$

The equation (4.7) is the predicted expansion path $\hat{C}_t$ under the price situation p_{Ft}, p_{Ht}, the prediction being based on the measurement from the least squares expansion paths C_1 and C_2. It is seen that the predicted coefficients, $\log b_t$ and a_t, are weighted averages of, respectively, $\log \beta$ and α.

4.3 From (4.6) and (4.7), we see that if $\phi_t = \phi_1$ then $(b_t , a_t) = (\beta_1 , \alpha_1)$, and if $\phi_t = \phi_2$ then $(b_t , a_t) = (\beta_2 , \alpha_2)$.

Thus, in our linear system, the prediction $\hat{C}_t$ by Fisher method exactly coincides to the two least squares expansion paths C_1, C_2 used for the measurement of u_F^*, u_H^*.

In other words, given the data $D_1 = (C_1, \phi_1)$ and D_2 (C_2, ϕ_2), we can construct functions u_F^*, u_H^* such that the equations $u_F^*/p_{F1} = u_H^*/p_{H1}$ and $u_F^*/p_{F2} = u_H^*/p_{H2}$ are the C_1 and C_2. Fisher method is an alogorithm for constructing the u_F^* and u_H^*.

What will happen when the third data D_3 (C_3, ϕ_3) are available? We will have three measurements of u_F^*, u_H^* according to the three pairs, (D_1, D_2), (D_1, D_3) and (D_2, D_3). If the data D_3 satisfies (4.6) and (4.7), then the three measurements must be identical.

4.4 The coefficients a_t and b_t of the prediction $\hat{C}_t$ are given in Table 1. column (2).

4.5 For the case with condition that $\beta_1 \neq \beta_2$ and $q_F^0 <$ $(\alpha_2 - \alpha_1)/(\beta_1 - \beta_2)$, we have the normalized measurement

$$u_F^* = \phi_1\beta_1^{\varepsilon}\,((\alpha_2 - \alpha)/(\beta_1 - \beta_2) - q_F)^{\varepsilon},$$
$$u_H^* = ((\beta_1\alpha_2 - \beta_2\alpha_1)/(\beta_1 - \beta_2) - q_H)^{\varepsilon} \quad (4.8)$$

The prediction equation for this case is also given by (4.6), (4.7).

4.6 For the case with $\beta_1 = \beta_2$, the normalized measurement is given by

(4.9)
$$u_F^* = \phi_2^{\alpha_1/(\alpha_1 - \alpha_2)} \exp\left(\log(\phi_1/\phi_2) q_F / (\alpha_2 - \alpha_1)\right),$$
$$u_H^* = \phi_1^{\alpha_2/(\alpha_1 - \alpha_2)} \exp\left(\log(\phi_1/\phi_2) q_H / (\alpha_2 - \alpha_1)\right).$$

The prediction equation is given as

(4.10)
$$q_H = a_t + b_t q_F ,$$

where

(4.11)
$$b_t = \beta_1 = \beta_2 ,$$
$$a_t = (\log(\phi_1/\phi_t)\alpha_2 - \log(\phi_2/\phi_t)\alpha_1) / \log(\phi_1/\phi_2).$$

5. Statistical measurement

5.1 Under the linearity and additivity, the measurement of Fisher method reduces to the utility fynction (4.4), (4.8) or (4.9). We here reverse the preceding discussion by starting from a specification of marginal utility function.

We reparameterize the function (4.4) as

$$(5.1)\qquad \begin{aligned} u_F^* &= k_F(q_F - l_F)^v \ , \\ u_H^* &= k_H(q_H - l_H)^v \ , \end{aligned}$$

where the parameters k, l, v are to be estimated. The expansion path under relative price ϕ_t is

$$(5.2)\qquad q_H = \alpha_t + \beta_t \, q_F ,$$

where

$$(5.3)\qquad \begin{aligned} \beta_t &= (\phi_t k_F \, / \, k_H)^{1/v} \ , \\ \alpha_t &= l_H - \beta_t \, l_F \ . \end{aligned}$$

5.2 Suppose that we have the data of ϕ_t and the estimated α_t, β_t from cross-sectional budget data at time t. The estimates α_t and β_t are subject to sampling error. From (5.3), we set statistical equations, for the T estimates of α_t and β_t,

$$(5.4)\qquad \begin{aligned} \log \beta_t &= \frac{1}{v} \log k + \frac{1}{v} \log(1/\phi_t), \\ \alpha_t &= l_H + l_F \, (-\beta_t), \qquad t = 1, 2, \ldots, T, \end{aligned}$$

where disturbance terms standing for the sampling errors of α_t β_t are omitted, and $k = k_F/k_H$

The least squares principle applied to (5.4) suggests a set of estimates of the parameters

$$\theta_1 = 1/v, \qquad (5.5)$$
$$\theta_0 = (\log k)/v,$$

as

$$\hat{\theta}_1 = \frac{\sum_{t=1}^{T} (\log(1/\phi_t) - \overline{\log(1/\phi)})(\log\beta_t - \overline{\log\beta})}{\sum_{t=1}^{T} (\log(1/\phi_t) - \overline{\log(1/\phi)})^2} \qquad (5.6)$$

$$\hat{\theta}_0 = \overline{\log\beta} - \hat{\theta}_1 \, \overline{\log(1/\phi)},$$

バーは標本平均値を示す.

where $\bar{x} = \sum x_t/T$.

Since equations (5.4) are simultaneous of a recursive type, we first calculate

$$\hat{\beta}_t = \exp(\hat{\theta}_0 + \hat{\theta}_1 \log(1/\phi_t)), \qquad (5.7)$$

then apply least squares method to the second set of equations of (5.4). This results in the estimates of l_F and l_H,

$$\hat{l}_F = \sum_t (-\hat{\beta}_t + \bar{\hat{\beta}})(\alpha_t - \bar{\alpha}) \Big/ \sum_t (-\hat{\beta}_t + \bar{\hat{\beta}})^2, \qquad (5.8)$$
$$\hat{l}_H = \bar{\alpha} + \hat{l}_F \, \bar{\hat{\beta}}$$

Finally, from (5.5) we derive the estimates of v and k as

$$\text{(5.9)} \qquad \hat{v} = 1/\hat{\theta}_1, \qquad \hat{k} = \exp(\hat{\theta}_0 / \hat{\theta}_1).$$

If we have T (>2) cross-sectional data, α_t and β_t, overall estimation of v, k, l_F and l_H is possible by using the T cross-section expansion paths in terms of α_t, β_t in spite of the deterministic measurement using only two expansion paths.

5.3 If T=2, then the estimates given above become deterministic rather than statistical. Since in this case identities like

$$\text{(5.10)} \qquad \log(1/\phi_1) - \overline{\log(1/\phi)} = (\log(1/\phi_1) - \log(1/\phi_2)) / 2, \qquad \log \beta_1 - \overline{\log \beta} = (\log \beta_1 - \log \beta_2)/2,$$

hold, it follows that

$$\text{(5.11)} \qquad \hat{v} = (\log \phi_1 - \log \phi_2) / (\log \beta_2 - \log \beta_1),$$
$$\hat{k} = \phi_1 \beta_1^{\hat{v}},$$
$$\hat{l}_F = (\alpha_2 - \alpha_1) / (\beta_1 - \beta_2),$$
$$\hat{l}_H = (\beta_1 \alpha_2 - \beta_2 \alpha_1) / (\beta_1 - \beta_2).$$

Thus when T=2, our statistical measurement, $\hat{v}$, $\hat{k}$, $\hat{l}_F$, $\hat{l}_H$, is identical to Fisher measurement under the assumed linear system.

5.4 The information about the nine least squares expansion paths and the relative prices of the years 1965 through 1973, transformed as $\log \beta_t$ and $\log(1/\phi_t)$, is shown in Figure 5. The measurement of Section 3 is obtained by fitting the regression (5.4) deterministically to the two points ($\log \beta_{1966}$, $\log(1/\phi_{1966})$) and ($\log\beta_{1973}$, $\log(1/\phi_{1973})$), therefore the estimate $\hat{v}$ is negative. Whereas fitting the regression to points ($\log\beta_{1965}$, $\log(1/\phi_{1965})$) and ($\log\beta_{1966}$, $\log(1/\phi_{1966})$) gives contradicting positive estimate as easily seen in Figure 5. It is also observed in Figure 5 that the plot ($\log\beta_{1970}$, $\log(1/\phi_{1970})$ is close to the deterministically fitted line, and the prediction $\hat{C}_{1970}$ is close to observed C_{1970}.

5.5 The parallel discussion with the preceding one is possible if we start from a specification;

$$(5.12) \qquad \begin{aligned} u_F{}^* &= k_F(l_F - q_F)^v , \\ u_H{}^* &= k_H(l_H - q_H)^v , \end{aligned}$$

which is a reparameterization of (4.8), or if we start from a specification;

$$(5.13) \qquad \begin{aligned} u_F{}^* &= k_F \exp l_F\, q_F , \\ u_H{}^* &= k_H \exp l_H\, q_H , \end{aligned}$$

a reparameterization of (4.9).

5.6 The plot of $(\alpha_t, -\beta_t)$ shows that the $(\alpha_{1967}, -\beta_{1967})$ is exceptional, we therefore apply the statistical method to the remaining eight years. Resulting regression estimates by the method (5.6) and (5.8) are

$$\text{(5.13)} \quad \begin{aligned} \log \beta_t &= -0.76997 - 2.94125 \log(1/\phi_t), \qquad r = -0.673 \\ \alpha_t &= 267.261 + 5970.210\ (-\hat{\beta}_t), \qquad r = 0.173. \end{aligned}$$

And the reduced utility measurement is

$$\text{(5.14)} \quad \begin{aligned} u_F^* &= 1.2992\ (q_F - 5970)^{-0.33999}, \\ u_H^* &= 1.0\ \ \ \ (q_H - 267)^{-0.33999}. \end{aligned}$$

The prediction based on this utility function is given in Table 1. column (3).

From the eight points in Figure 5 the deterministic measurement is possible in 8! / 6! 2! ways. It is seen, however, that the deterministic method might provide unstable values of $\hat{v}$ including negative and positive ones.

5.7 The correlation coefficients of the regression (5.13) are low, so that the uniformity hypothesis for the utility function during the eight years seems to be rejected. We move on to carry utility measurement by restricting ourselves to the four years from 1969 to 1972. We get regression estimates

$$\log \beta_t = 0.02044 - 11.07738 \log(1/\phi_t), \qquad r = -0.9987,$$

(5.15)

$$\alpha_t = 8481.245 + 28927.166\ (-\hat{\beta}_t), \qquad r = 0.9670,$$

and the reduced utility measurement,

$$u_F{}^* = 0.99816\ (q_F - 28927)^{-0.09027},$$

(5.16)

$$u_H{}^* = 1.0 \quad (q_H - 8481)^{-0.09027}.$$

The prediction by this utility function is given in Table 1. column (4)

6. Concluding remarks

An instance of the history of statistics tells that: After the end of period when many discrepant observations of an object were regarded as inconsistent, we became accustomed to take the mean of a number of obsevations. And statistics endeavoured to show advantages arising by taking the mean of observations.

Fisher's method in its original form yields many discrepant utility measurements when applied to time series of cross section budget data. For such a case we should reduce many measurements into a unique measurement by taking their mean in the way suggested in our analysis or in some other way.

Table 1

	(1)	(2)	外挿 (3)	(4)	(5)
t	α_t	5.4)A式	a_t	5.6)式	ϕ_t
1965	-1619	-1683	-1482	3209	416/486
1966	-1664	-1664	-1420	3887	432/511
1967	-6456	-1663	-1427	3806	435/535
1968	-2832	-1706	-1559	2291	482/555
1969	1153	-1735	-1657	940	511/578
1970	-1895	-1776	-1798	-1373	557/615
1971	-2582	-1800	-1880	-2922	591/644
1972	-2397	-1792	-1852	-2365	613/671
1973	-1843	-1843	-2030	-6225	693/738

t	β_t	b_t		
1965	.2738	.2941	.2931	.1823
1966	.2857	.2857	.2825	.1588
1967	.4121	.2868	.2838	.1616
1968	.3697	.3043	.3058	.2140
1969	.2617	.3172	.3223	.2607
1970	.3361	.3356	.3460	.3407
1971	.3948	.3461	.3597	.3942
1972	.3780	.3425	.3549	.3749
1973	.3652	.3652	.3848	.5084

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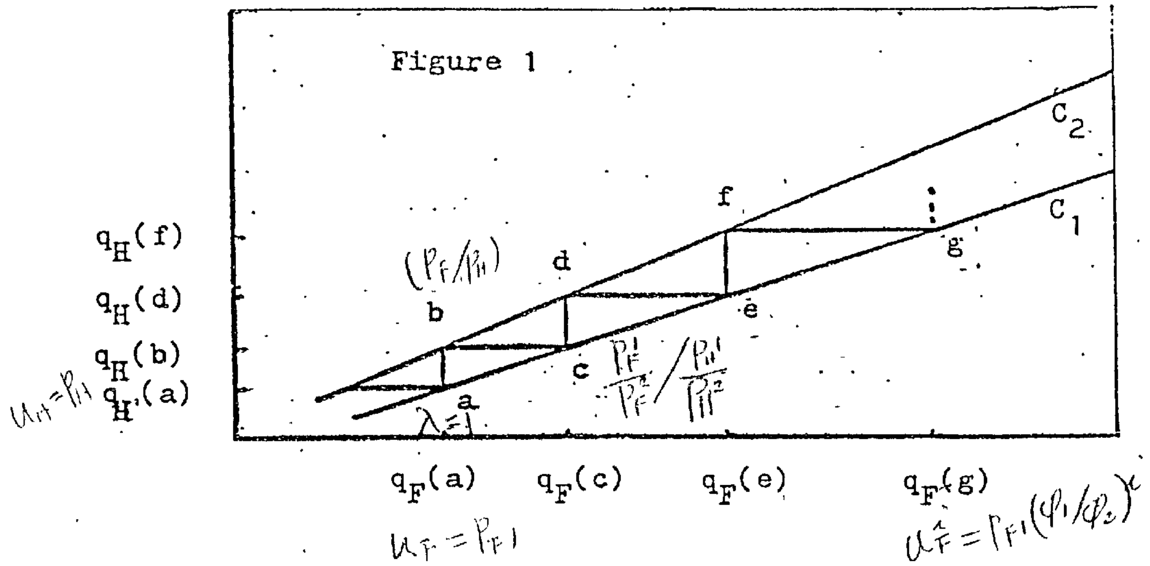


Figure 1
c_2
c_1
f
d
b
a
c
e
g
$q_H(f)$
$q_H(d)$
$q_H(b)$
$q_H(a)$
$q_F(a)$
$q_F(c)$
$q_F(e)$
$q_F(g)$

(4.4)式

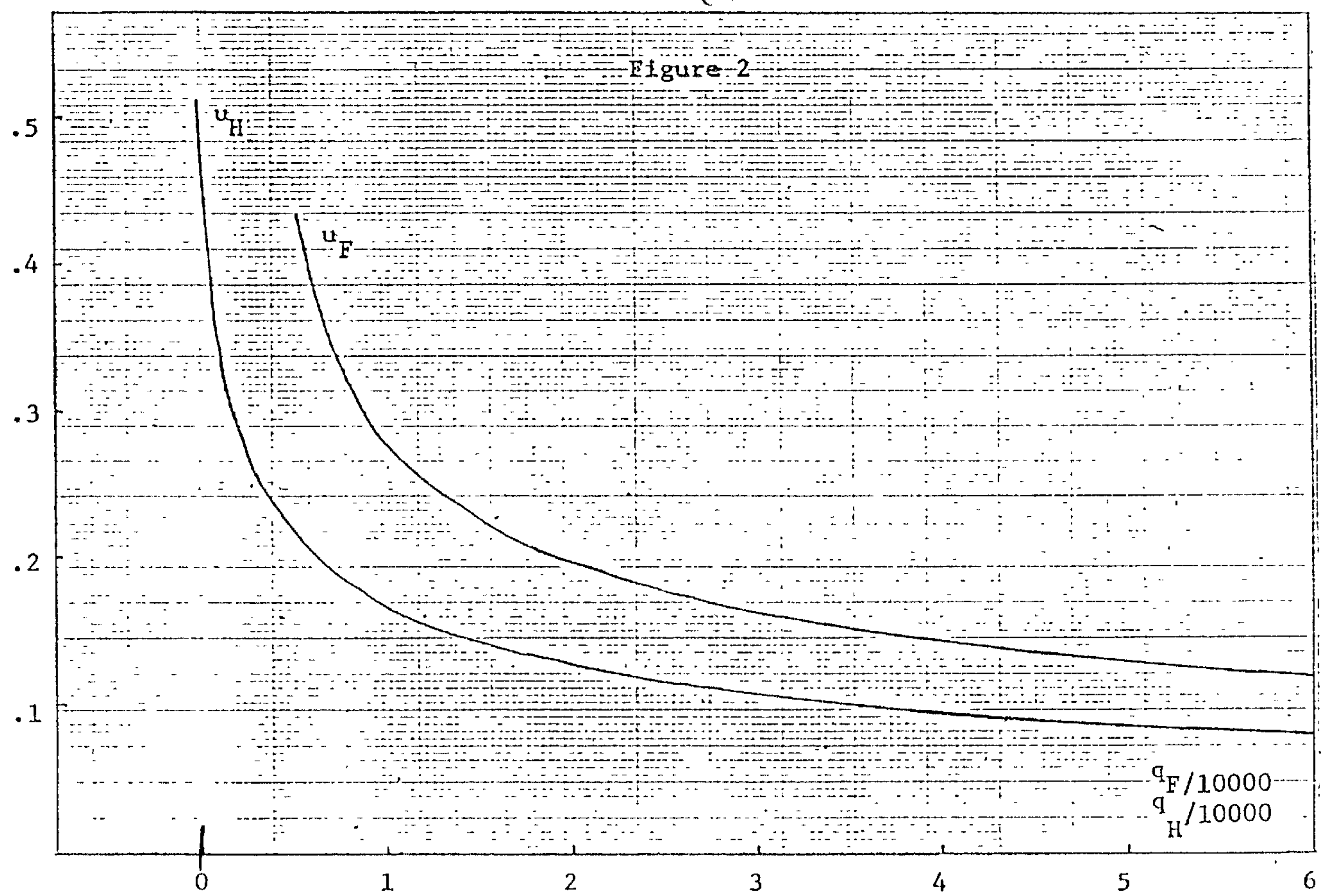
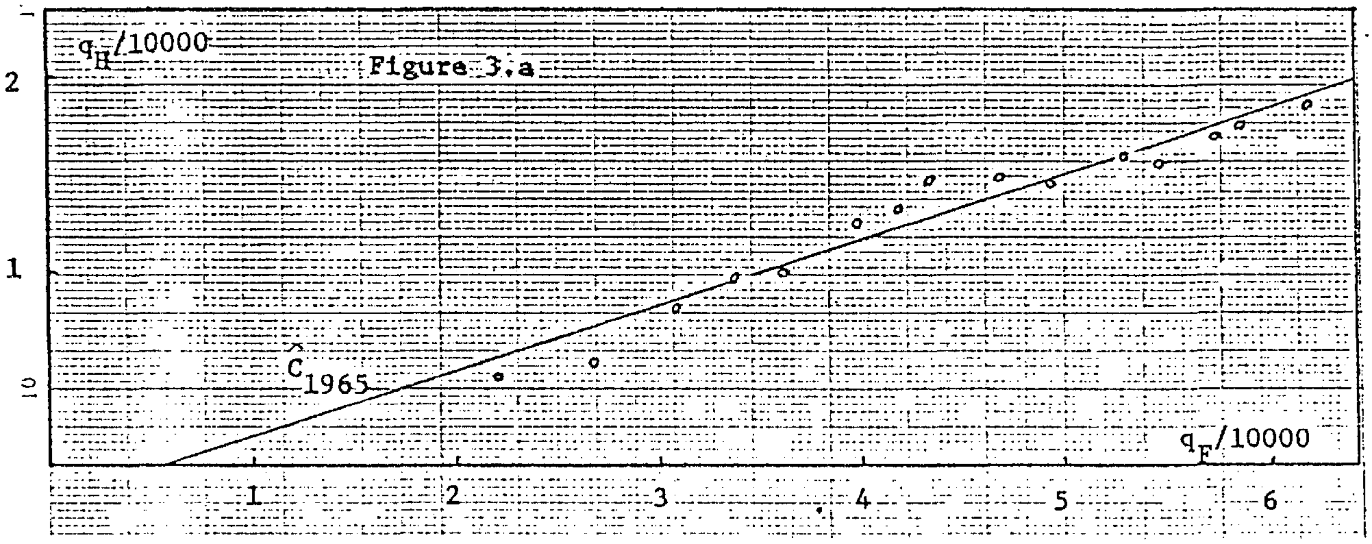
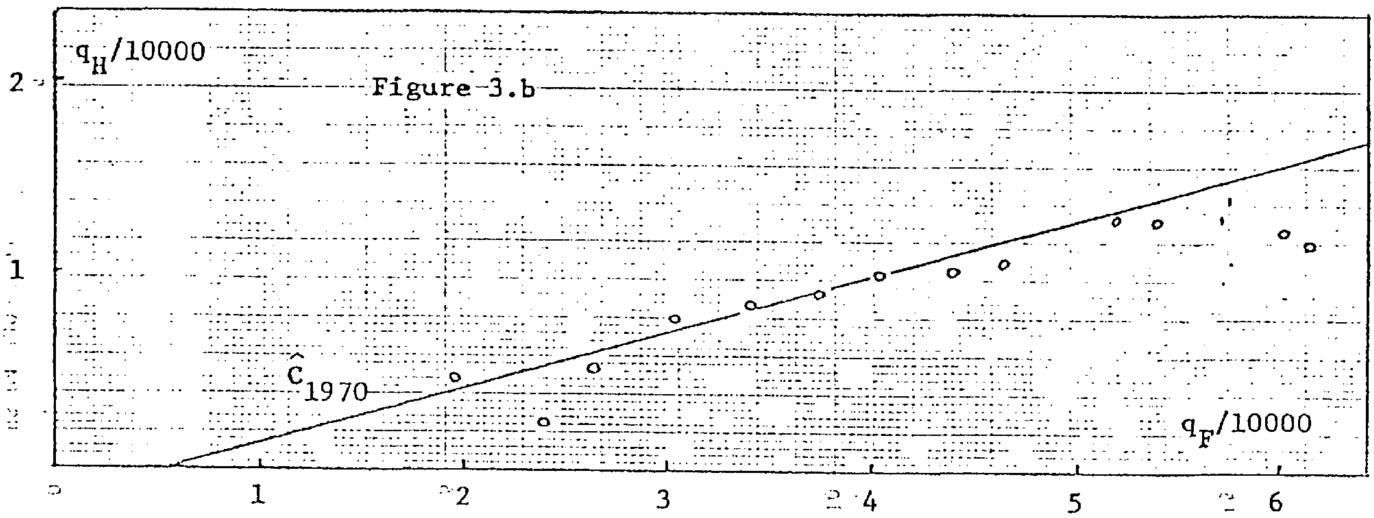


Figure 2
u_H
u_F
.5
.4
.3
.2
.1
0
1
2
3
4
5
6
$q_F/10000$
$q_H/10000$

extrapolation





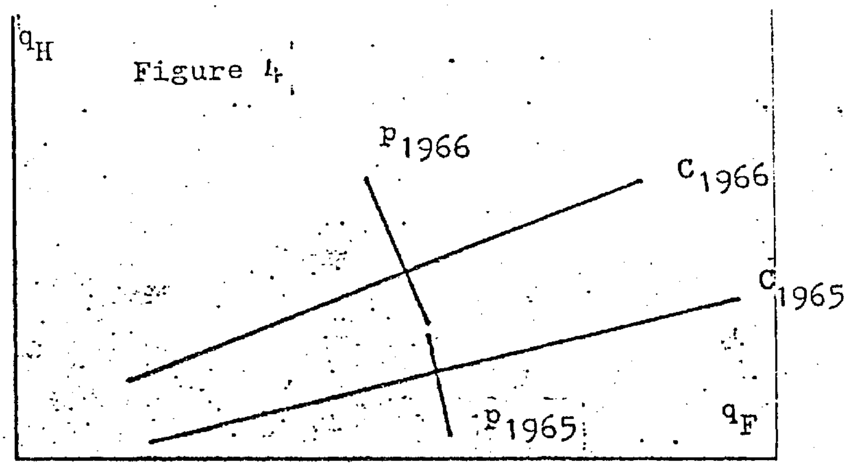
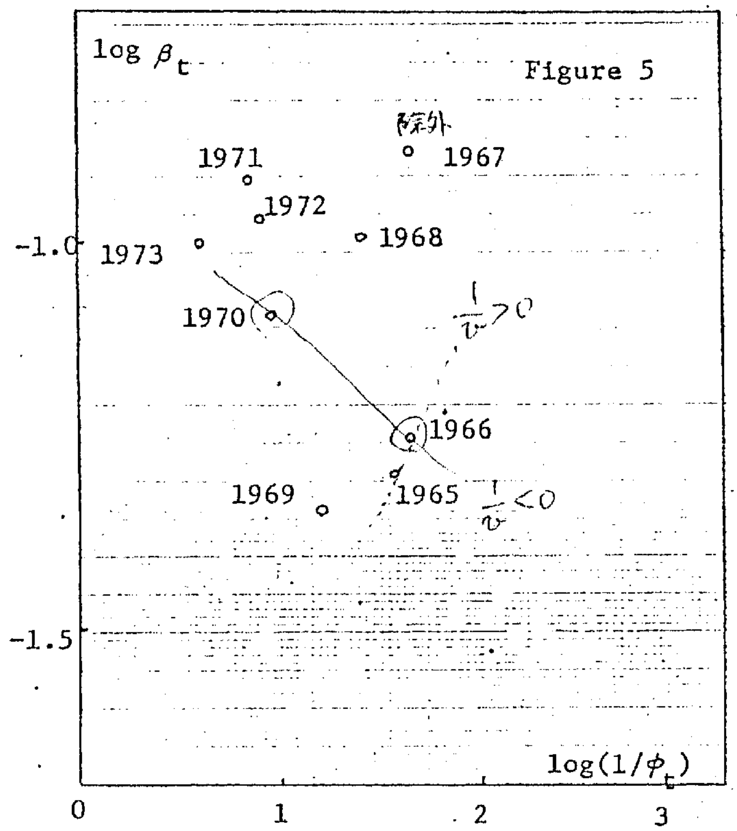

q_H
Figure 4
p1966
c1966
c1965
p1965
q_F

Fig. 5


log β_t
Figure 5
除外
1967
1971
1972
1968
-1.0
1973
1970
1/v > 0
1966
1969
1965
1/v < 0
-1.5
log(1/φ_t)
0
1
2
3