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Convex risk measures on Orlicz spaces: convolution and shortfall

by

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Convex risk measures on Orlicz spaces: convolution and shortfall *

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Abstract

We focus on, throughout this paper, convex risk measures defined on Orlicz spaces. In particular, we investigate basic properties of convolutions defined between a convex risk measure and a convex set, and between two convex risk measures. Moreover, we study shortfall risk measures, which are convex risk measures induced by the shortfall risk. By using results on convolutions, we obtain a robust representation result for shortfall risk measures defined on Orlicz spaces under the assumption that the set of hedging strategies has the sequential compactness in a weak sense. We discuss in addition a construction of an example having the sequential compactness.

1 Introduction

Our aim of this paper is to study some properties of convex risk measures defined on Orlicz spaces. Firstly, we introduce a robust representation theorem which is based on Corollary 28 of Biagini and Frittelli [5]. Moreover, we investigate two types of convolution of convex risk measures on Orlicz spaces. The first type is ones defined between a convex risk measure and a convex set. The other is between two convex risk measures. We shall study their basic properties. In particular, we lead to a sufficient condition under which convolutions become order lower semi-continuous (l.s.c., for short). In addition, we study shortfall risk measures, which are convex risk measures induced by the shortfall risk, and whose definition shall be given in Section 4. By using results on convolutions, we obtain a robust representation result for shortfall risk measures under the assumption that the set of hedging strategies has the sequential compactness in a weak sense. Moreover, we shall construct an example satisfying this additional assumption.

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A convex risk measure is a $(-\infty, +\infty]$-valued proper, monotone, translation invariant and convex functional defined on a linear space of random variables. Moreover, if a convex risk measure is positive homogeneous, then it is called a coherent risk measure. Convex risk measures were undertaken by Föllmer and Schied [11] and Frittelli and Rosazza Gianin [13] independently. In advance of it, coherent risk measures were introduced by Artzer et al. [3]. There are much literature on convex risk measures. Ruszczyński and Shapiro [18] developed robust representation results of convex risk measures on Banach lattices. Kaina and Rüschendorf [14] investigated a theory of convex risk measures on L^p-spaces for $p \geq 1$. Cheridito and Li [6] treated the Orlicz heart case. Moreover, [5] obtained robust representation results for convex risk measures defined on locally convex Fréchet lattices.

First, we shall explain the terminology "Orlicz space". Now, we fix arbitrarily a left-continuous non-decreasing convex non-trivial function $\Phi : \mathbf{R}_+ \to [0, \infty]$ with $\Phi(0) = 0$, where Φ is non-trivial if $\Phi(x) > 0$ for some $x > 0$ and $\Phi(x) < \infty$ for some $x > 0$. Common examples of Φ are $\Phi(x) = x^p$ for $p \geq 1$ and $\Phi(x) = e^x - 1$. Then, we define a space L^Φ of random variables as $L^\Phi := \{X \in L^0 | E[\Phi(c|X|)] < \infty \text{ for some } c > 0\}$. This L^Φ is called an Orlicz space. Likewise, we define an Orlicz heart as $M^\Phi := \{X \in L^0 | E[\Phi(c|X|)] < \infty \text{ for any } c > 0\}$. For example, if Φ is a p-th power function, L^Φ coincides with M^Φ as well as L^p. On the other hand, in the case of an exponential function, say $e^x - 1$ or $e^x - x - 1$, we have $L^p \supset L^\Phi \supset L^\infty$ for any $p \geq 1$, and L^Φ does not consist with M^Φ in general. We introduce some examples of the case where $M^\Phi \subsetneq L^\Phi$.

Example 1.1 In the case where $\Phi(x) = e^x - 1$ or $e^x - x - 1$, if a random variable X follows an exponential distribution with a positive parameter, then $X \in L^\Phi$ but $X \notin M^\Phi$.

Example 1.2 Set $\Phi(x) = e^{x^2} - 1$. Let X be a random variable following a normal distribution. Then $X \in L^\Phi$ but $X \notin M^\Phi$.

Example 1.3 It is natural that an aggregate insurance claim amount follows a compound distribution. See Daykin et al. [7]. We denote by $(N_t)_{t \geq 0}$ the process which describes the number of claims during time period $[0, t]$. We assume that $(N_t)_{t \geq 0}$ is a Poisson process with a positive parameter. The size of the i-th claim is denoted by R_i, which is a nonnegative-valued random variable. We suppose that $(R_i)_{i \geq 1}$ is an i.i.d. sequence which is independent of $(N_t)_{t \geq 0}$. The aggregate insurance claim amount in this model is given by

$$A_t := \begin{cases} \sum_{i=1}^{N_t} R_i, & \text{if } N_t > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Remark that $E[e^{cA_T}] = E[e^{N_T \log M(c)}]$ for any fixed time horizon $T > 0$ and any constant $c > 0$, where $M(c)$ is the moment generating function of R_i, that is, $M(c) := E[e^{cR_i}]$. Taking an exponential type function as Φ, say $\Phi(x) = e^x - 1$, if there exist $c_1 > c_2 > 0$ such that $M(c_1) = \infty$ and $M(c_2) < \infty$, then $A_T \in L^\Phi$, but $A_T \notin M^\Phi$. For instance, if each R_i follows an exponential distribution with parameter $\sigma > 0$, then

$$M(c) = \begin{cases} \frac{\sigma}{\sigma - c}, & \text{if } c < \sigma, \\ \infty, & \text{otherwise.} \end{cases}$$

These two spaces L^Φ and M^Φ have much different properties, although their definitions are very similar. We have to pay attention the fact that L^Φ is not order continuous topology, which is different from M^Φ, equipped with a suitable norm. In [5], they asserted that the order l.s.c. and the C-property which will be defined in Section 2, are needed to obtain robust representations for convex risk measures defined on non-order continuous topologies.

In this paper, we shall investigate basic properties of convolutions of convex risk measures, which was introduced by Delbaen [8] and Barrieu and El Karoui [4] in the framework of L^∞. One of motivations to introduce convolutions lies in the risk allocation problems. For details, see Acciaio [1]. Klöppel and Schweizer [15], [16] investigated a dynamic version of convolutions on L^∞. In this paper, we consider two types of convolution on L^Φ as follows. The first type is defined as, for a convex set $\mathcal{B} \subset L^\Phi$ and a convex risk measure ρ, $\rho \Box \mathcal{B}(X) := \inf_{Y \in \mathcal{B}} \rho(X - Y)$. The second one is defined between two convex risk measures ρ_1 and ρ_2 as $\rho_1 \Box \rho_2(X) := \inf_{Y \in L^\Phi} \{\rho_1(X - Y) + \rho_2(Y)\}$. We shall extend results on these convolutions defined on L^∞ to the Orlicz space setting.

Furthermore, we obtain a robust representation result for shortfall risk measures defined on Orlicz spaces. Some results on convolutions will be very useful to get such a representation. Before stating this matter, we should introduce shortfall risk measures. A shortfall risk measure is a convex risk measure induced by the shortfall risk. The concept of the shortfall risk was undertaken by Föllmer and Leukert [10]. Now, we presume an investor who intends to sell a claim. The investor's attitude toward risk is described by a loss function which is a non-decreasing continuous convex function, and whose negative part sticks on zero. Her shortfall risk, when she selects a certain hedging strategy, is given by the weighted expectation by her loss function of the positive part of the difference between the claim and the value of her hedging strategy at the maturity. If she sells the claim for the so-called super hedging cost, she could eliminate her shortfall risk perfectly by selecting a suitable hedging strategy. It, however, would be too expensive for a buyer to trade in general. She should then sell the claim for a price being less than the super hedging cost. This means that she should live with some shortfall risk. On the other hand, it is natural that she hopes to suppress her shortfall risk less than a certain level, which is called threshold. That is, if a price for which she sells the claim enables her to select a hedging strategy to hold the corresponding shortfall risk to less than or equal to her threshold, then she would accept the price. Then, one problem arises. How much is the least price she can accept? The answer is represented by a convex risk measure which is said a shortfall risk measure. More precisely, the least price of the claim X which the investor can accept is given by $\widehat{\rho}(-X)$, where $\widehat{\rho}$ is the shortfall risk measure determined by the shortfall risk with her loss function and her threshold. The definition of $\widehat{\rho}$ will be given in Section 4.

The first result on shortfall risk measures appeared in [11]. They treated the L^∞ case under the discrete time setting. Arai [2] obtained robust representation results of shortfall risk measures on Orlicz hearts under the continuous time setting. He did not assume the locally boundedness of the asset price process, and dealt with three cases in forms of self-financing admissible portfolios. One is the case where the set of admissible portfolios forms a linear space. The second one includes cone constraint. The third one is the case where admissible portfolios form a predictably convex set. Both two papers [11] and [2] presumed that investors select a self-financing admissible portfolio as a hedging strategy. In other words, in the two papers, the value of a hedging strategy is expressed by a stochastic integral with respect to the asset price process. On the other hand, we take in this paper the set of all attainable claims with zero initial endowment, instead of self-financing admissible portfolios. That is, an investor selects an attainable claim with zero cost as her hedging strategy.

We can decompose a shortfall risk measure $\widehat{\rho}$ into two elements. One is the structure of hedging strategies. The other is preference of the investor, that is, the loss function and the threshold being determined by the investor. Actually, we shall prove that a shortfall risk measure is expressed by the convolution between a convex risk measure ρ_0 induced by the loss function and the threshold, and the convex set of all hedging strategies. Moreover, although we need to the order l.s.c. of $\widehat{\rho}$ so as to obtain its robust representation, it suffices to care only the order l.s.c. of ρ_0 and the sequential compactness of the set of hedging strategies in a weak sense by using results concerned with convolutions. In summary, by grace of results on convolutions, we do not have to investigate the order l.s.c. of the whole $\widehat{\rho}$. Furthermore, the order l.s.c. of ρ_0 will be proved without any additional assumption. As a result, we shall obtain a robust representation for shortfall risk measures on Orlicz spaces under the sequential compactness. In addition, we shall give a sufficient condition under which the sequential compactness holds. More precisely, we form a closed set of hedging strategies as an extension of Xia and Yan [19] in which they constructed an L^p-closed set of stochastic integrals. Moreover, we give a sufficient condition under which the set of hedging strategies has the conditional sequential compactness.

An outline of this paper is as follows: In Section 2, we prepare mathematical preliminaries. Moreover, we shall introduce a robust representation result for convex risk measures defined on Orlicz spaces. This result is strongly depending on Corollary 28 of [5]. Section 3 is devoted to study convolutions. We owe many parts of proofs given in Section 3 to [15]. Results on shortfall risk measures are given in Section 4.

2 Preliminaries

Throughout this paper, we consider an incomplete financial market with maturity $T > 0$ and zero interest rate. Consider a completed probability space

$(\Omega, \mathcal{F}, P; \mathbf{F} = \{\mathcal{F}_t\}_{t\in[0,T]})$, where $\mathbf{F}$ is a filtration satisfying the so-called usual condition, that is, $\mathbf{F}$ is right-continuous, $\mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_0$ contains all null sets of $\mathcal{F}$. Let L^0 be the space of all equivalence classes of $\mathcal{F}$-measurable random variables defined on Ω.

We firstly introduce some terminologies on Orlicz spaces. A left-continuous non-decreasing convex non-trivial function $\Phi : \mathbf{R}_+ \to [0,\infty]$ with $\Phi(0) = 0$ is called an *Orlicz function*, where Φ is non-trivial if $\Phi(x) > 0$ for some $x > 0$ and $\Phi(x) < \infty$ for some $x > 0$. When Φ is an $\mathbf{R}_+$-valued continuous, strictly increasing Orlicz function, we call it a strict Orlicz function in this paper. Note that, for any strict Orlicz function Φ, we have $\Phi(x) \in (0,\infty)$ for any $x > 0$ and $\lim_{x\to\infty} \Phi(x) = \infty$. Moreover, a strict Orlicz function Φ is differentiable a.e. and its left-derivative Φ' satisfies

$$\Phi(x) = \int_0^x \Phi'(u)du.$$

Note that Φ' is left-continuous, and may have at most countably many jumps. Define $I(y) := \inf\{x \in (0,\infty)|\Phi'(x) \geq y\}$, which is called the generalized left-continuous inverse of Φ'. We define $\Psi(y) := \int_0^y I(v)dv$ for $y \geq 0$, which is an Orlicz function and called the conjugate function of Φ.

Remark 1 Any polynomial function starting at 0 whose minimal degree is equal to or greater than 1, and all coefficients are positive, is a strict Orlicz function. For example, cx^p for $c > 0$, $p \geq 1$, $x^2 + 3x^5$ and so forth. Moreover, $e^x - 1$, $e^x - x - 1$, $(x + 1)\log(x + 1) - x$ and $x - \log(x + 1)$ are strict Orlicz functions.

Now, we need the following definitions:

Definition 2.1 For an Orlicz function Φ, we define two spaces of random variables:
Orlicz space : $L^\Phi := \{X \in L^0|E[\Phi(c|X|)] < \infty \text{ for some } c > 0\}$,
Orlicz heart : $M^\Phi := \{X \in L^0|E[\Phi(c|X|)] < \infty \text{ for any } c > 0\}$.
In addition, we define two norms:
Luxemburg norm : $\|X\|_\Phi := \inf\left\{\lambda > 0|E\left[\Phi\left(\left|\frac{X}{\lambda}\right|\right)\right] \leq 1\right\}$,
Orlicz norm : $\|X\|^*_\Phi := \sup\{E[XY]|\ \|Y\|_\Phi \leq 1\}$.

Remark that $M^\Phi \subset L^\Phi$ and both spaces L^Φ and M^Φ are linear. In the case of the lower partial moments $\Phi(x) = x^p/p$ for $p > 1$, the Orlicz space L^Φ and the Orlicz heart M^Φ both are identical with L^p. The conjugate function Ψ in this case is given by $\Psi(x) = x^q/q$, where $q = p/(p - 1)$, and $M^\Psi = L^\Psi = L^q$. In general, if $\limsup_{x\to\infty} \frac{x\Phi'(x)}{\Phi(x)} < \infty$, then M^Φ is identical with L^Φ. For instance, $\Phi(x) = x - \log(x + 1)$ other than the lower partial moments. Otherwise, M^Φ would be a proper subset of L^Φ, for example $\Phi(x) = e^x - 1$. See Examples 1.1 – 1.3. Moreover, if Φ is a strict Orlicz function, the norm dual of $(M^\Phi, \|\cdot\|_\Phi)$ is given by $(L^\Psi, \|\cdot\|^*_\Phi)$. The norm dual of $(L^\Phi, \|\cdot\|_\Phi)$ includes a singular part. This fact would be crucial when we consider convex risk measures on Orlicz spaces.

Note that L^Φ becomes a Banach lattice under the usual pointwise ordering. For more details on Orlicz spaces, see Rao and Ren [17], Edgar and Sucheston [9] and Biagini and Frittelli [5].

Throughout this paper, **we fix a strict Orlicz function** Φ. Now, we define convex risk measures and coherent risk measures on the Orlicz space L^Φ.

Definition 2.2 A functional ρ defined on L^Φ is called a *convex risk measure* on L^Φ if it satisfies the following four conditions:
(1) **Properness** : $\rho(0) \in \mathbf{R}$ and ρ is $(-\infty, \infty]$-valued,
(2) **Monotonicity** : $\rho(X) \geq \rho(Y)$ for any $X, Y \in L^\Phi$ such that $X \leq Y$,
(3) **Translation invariance** : $\rho(X + m) = \rho(X) - m$ for $X \in L^\Phi$ and $m \in \mathbf{R}$,
(4) **Convexity** : $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$ for any $X, Y \in L^\Phi$ and $\lambda \in [0, 1]$.
Moreover, if a convex risk measure ρ satisfies
(5) **Positive homogeneity** : $\rho(\lambda X) = \lambda\rho(X)$ for any $\lambda \geq 0$,
then ρ is called a *coherent risk measure*.

[5] asserts that, when we consider a robust representation of convex risk measures, a significant issue is whether or not the topology on which the convex risk measures are defined has the order continuity. In the case of order continuous, say L^p for $p \in [1, \infty)$ or Orlicz hearts, we do not have to care the order l.s.c. of convex risk measures. On the other hand, in the non-order continuous case, say L^∞ or Orlicz spaces, we have to check it. Indeed, Corollary 28 of [5] products a robust representation for convex risk measures defined on locally convex Fréchet lattices under the order l.s.c. and the C-property. Now, we shall state a robust representation theorem for convex risk measures on L^Φ based on Corollary 28 of [5]. Firstly, we prepare some definitions. A linear topology τ has *C-property*, if a net $\{X_\alpha\}$ converges to X in τ, then there exist a subsequence $\{X_{\alpha_n}\}_{n\geq 1}$ and convex combinations $Y_n \in \mathrm{conv}(X_{\alpha_n}, X_{\alpha_{n+1}}, \ldots)$ such that Y_n is order convergent to X. Note that the topology $(L^\Phi, \sigma(L^\Phi, L^\Psi))$ has the C-property, and, in this case, "$Y_n \to X$ in order" means "$Y_n \to X$ a.s. and there exists a $Y \in L^\Phi$ such that $|Y_n| \leq Y$ for any $n \geq 1$". Let $\mathcal{P}^\Psi$ be the set of all probability measures being absolutely continuous with respect to P and having L^Ψ-density with respect to P, that is, $\mathcal{P}^\Psi := \{Q \ll P | dQ/dP \in L^\Psi\}$. Under the above preparations, we can say a robust representation theorem as follows:

Theorem 2.1 Let ρ be a convex risk measure on L^Φ. We define

$$\alpha_\rho(Q) := \sup_{X \in \mathcal{A}_\rho} E_Q[-X]$$

for any $Q \in \mathcal{P}^\Psi$, where $\mathcal{A}_\rho := \{X \in L^\Phi | \rho(X) \leq 0\}$. If ρ has the order l.s.c., then it is represented as follows:

$$\rho(X) = \sup_{Q \in \mathcal{P}^\Psi} \{E_Q[-X] - \alpha_\rho(Q)\}. \tag{2.1}$$

Proof. Since $\sigma(L^\Phi, L^\Psi)$ has the C-property, Corollary 28 in [5] implies that ρ is expressed by

$$\rho(X) = \sup_{Q \in \mathcal{P}^\Psi} \left\{ E_Q[-X] - \rho^*(-dQ/dP) \right\},$$

where ρ^* is the convex conjugate of ρ. Moreover, we can obtain $\rho^*(-dQ/dP) = \alpha_\rho(Q)$ for any $Q \in \mathcal{P}^\Psi$ by the same way as the proof of Theorem 5 in [11]. $\Box$

Remark 2 The functional α_ρ and the set $\mathcal{A}_\rho$ defined in Theorem 2.1 are called the *penalty function* and the *acceptance set* of ρ, respectively. In order to prove the equivalence of $\rho^*(-dQ/dP)$ and $\alpha_\rho(Q)$, we do not need the order l.s.c. of ρ.

Corollary 2.3 For a coherent risk measure ρ on L^Φ having the order l.s.c., there exists a convex subset $\mathcal{P}' \subset \mathcal{P}^\Psi$ such that $\rho(X) = \sup_{Q \in \mathcal{P}'} E_Q[-X]$.

Proof. See the last assertion of Corollary 28 of [5]. $\Box$

3 Convolution

In this section, we investigate some properties of convolutions between a convex risk measure on L^Φ and a convex subset of L^Φ, and convolutions of two convex risk measures on L^Φ. These types of convolution are undertaken by Delbaen [8] and Barrieu and El Karoui [4]. Results in this section should be regarded as extensions of the L^∞ case. We shall prove that, under some additional assumptions, convolutions on L^Φ have similar properties with the case of L^∞. Some results in this section will be used in the next section. Firstly, we introduce the definitions of convolutions as follows:

Definition 3.1 (a) Suppose that $\mathcal{B} \subset L^\Phi$ is non-empty convex. The *convolution* of a convex risk measure ρ on L^Φ and $\mathcal{B}$ is defined as

$$\rho \Box \mathcal{B}(X) := \inf_{Y \in \mathcal{B}} \rho(X - Y), \text{ for any } X \in L^\Phi.$$

(b) Let ρ_1 and ρ_2 be two convex risk measures on L^Φ. The *convolution* of ρ_1 and ρ_2 is defined as

$$\rho_1 \Box \rho_2(X) := \inf_{Y \in L^\Phi} \{\rho_1(X - Y) + \rho_2(Y)\}, \text{ for any } X \in L^\Phi.$$

We shall prove three propositions in this section. In particular, we obtain sufficient conditions under which convolutions have the order l.s.c. More precisely, we show that the concept of the sequential compactness in a weak sense plays a vital role to ensure the order l.s.c. of a convolution. Many parts of proofs in this section are based on the results of Section 4 in Klöpple and Schweizer [15].

Proposition 3.2 Let ρ be a convex risk measure on L^Φ satisfying

$$\rho(X) < \infty \text{ for any } X \in L^\Phi \text{ such that } E[\Phi(|X|)] < \infty. \tag{3.2}$$

Moreover, let $\mathcal{B} \subset L^\Phi$ be a convex set including 0. If $\rho\Box\mathcal{B}(0) > -\infty$, then the following hold:
(a) $\rho\Box\mathcal{B}$ is a convex risk measure on L^Φ.
(b) If ρ is coherent and $\mathcal{B}$ is cone, then $\rho\Box\mathcal{B}$ is also coherent.
(c) If $\mathcal{B}$ is sequentially compact in $\sigma(L^\Phi, L^\Psi)$ and ρ is order l.s.c., then so is $\rho\Box\mathcal{B}$.

Proof. (a) It is clear that $\rho\Box\mathcal{B}$ has the monotonicity, the translation invariance, and the convexity. We shall prove its properness. First, by $0 \in \mathcal{B}$ and the properness of ρ, we have $\rho\Box\mathcal{B}(0) = \inf_{Y\in\mathcal{B}} \rho(0 - Y) \leq \rho(0) < \infty$, from which $\rho\Box\mathcal{B}(0) \in \mathbf{R}$ follows, since $\rho\Box\mathcal{B}(0) > -\infty$ is assumed.

Next, we prove that $\rho\Box\mathcal{B} > -\infty$. Suppose that there exists an $X \in L^\Phi$ such that $\rho\Box\mathcal{B}(X) = -\infty$. Without loss of generality, we may assume that $X \geq 0$, due to the monotonicity of $\rho\Box\mathcal{B}$. By the convexity of $\rho\Box\mathcal{B}$, we have, for any $\lambda > 0$,

$$\frac{\lambda}{1+\lambda}\rho\Box\mathcal{B}(X) + \frac{1}{1+\lambda}\rho\Box\mathcal{B}(-\lambda X) \geq \rho\Box\mathcal{B}\left(\frac{\lambda X}{1+\lambda} + \frac{-\lambda X}{1+\lambda}\right) = \rho\Box\mathcal{B}(0) > -\infty.$$

Thus, $\rho\Box\mathcal{B}(-\lambda X) = \infty$ holds for any $\lambda > 0$, because $\rho\Box\mathcal{B}(X) = -\infty$. In addition, $0 \in \mathcal{B}$ yields $\rho\Box\mathcal{B}(-\lambda X) = \inf_{Y\in\mathcal{B}} \rho(-\lambda X - Y) \leq \rho(-\lambda X)$, from which $\rho(-\lambda X) = \infty$ follows for any $\lambda > 0$.

Incidentally, $E[\Phi(\lambda|X|)] < \infty$ holds for a sufficient small number $\lambda > 0$, since $X \in L^\Phi$. By the condition (3.2), $\rho(-\lambda X) < \infty$ holds for such a sufficient small number $\lambda > 0$, which contradicts to the fact that $\rho(-\lambda X) = \infty$ for any $\lambda > 0$. Hence, we get $\rho\Box\mathcal{B} > -\infty$.

(b) Remark that $\rho(0) = 0$ since ρ is coherent. We show that $\rho\Box\mathcal{B}(0) = 0$. We have $\rho\Box\mathcal{B}(0) \leq 0$, since $\rho\Box\mathcal{B}(0) = \inf_{Y\in\mathcal{B}} \rho(0-Y) \leq \rho(0) = 0$. Thus, supposing $\rho\Box\mathcal{B}(0) \neq 0$, we have $\rho\Box\mathcal{B}(0) < 0$. This implies the existence of $Y \in \mathcal{B}$ satisfying $\rho(-Y) < 0$. Since $\mathcal{B}$ is cone, we have $\lambda Y \in \mathcal{B}$ for any $\lambda > 0$. Moreover, the positive homogeneity of ρ yields that

$$\rho(-\lambda Y) = \lambda\rho(-Y) \longrightarrow -\infty \text{ as } \lambda \longrightarrow \infty.$$

We have then $\inf_{Y\in\mathcal{B}} \rho(-Y) = -\infty$, namely, $\rho\Box\mathcal{B}(0) = -\infty$, which contradicts to the properness of $\rho\Box\mathcal{B}$. Hence, $\rho\Box\mathcal{B}(0) = 0$ holds.

Next, we have, for any $\lambda > 0$ and any $X \in L^\Phi$,

$$\rho\Box\mathcal{B}(\lambda X) = \inf_{Y\in\mathcal{B}} \rho(\lambda X - Y) = \lambda \inf_{Y\in\mathcal{B}} \rho(X - Y) = \lambda\rho\Box\mathcal{B}(X), \tag{3.3}$$

where $\mathcal{B}$ consists with $\{Y/\lambda | Y \in \mathcal{B}\}$. (3.3) implies that $\rho\Box\mathcal{B}(\lambda X) = \lambda\rho\Box\mathcal{B}(X)$. Consequently, $\rho\Box\mathcal{B}$ has the positive homogeneity.

(c) From the view of Proposition 24 in Biagini and Frittelli [5], ρ has the l.s.c in $\sigma(L^\Phi, L^\Psi)$, and we have only to prove the l.s.c. of $\rho\Box\mathcal{B}$ in $\sigma(L^\Phi, L^\Psi)$. Note that $\sigma(L^\Phi, L^\Psi)$ has the C-property.

Let $(X_n)_{n\geq 1}\subset L^\Phi$ be a sequence converging to some $X\in L^\Phi$ in $\sigma(L^\Phi,L^\Psi)$. Now, for any $n\geq 1$, let $(\widehat{Y}_k^n)_{k\geq 1}\subset\mathcal{B}$ be a minimizing sequence of $\inf_{Y\in\mathcal{B}}\rho(X_n-Y)$. Since $\mathcal{B}$ has the sequential compactness in $\sigma(L^\Phi,L^\Psi)$, for any $n\geq 1$, there exists a $\widehat{Y}^n\in\mathcal{B}$ such that, taking a subsequence if need be,

$$\widehat{Y}_k^n\longrightarrow\widehat{Y}^n \text{ in } \sigma(L^\Phi,L^\Psi) \text{ as } k\to\infty.$$

The l.s.c. of ρ yields that, for any $n\geq 1$, we have

$$\rho(X_n-\widehat{Y}^n)\leq\liminf_{k\to\infty}\rho(X_n-\widehat{Y}_k^n)=\inf_{Y\in\mathcal{B}}\rho(X_n-Y). \tag{3.4}$$

Considering the sequence $(\widehat{Y}^n)_{n\geq 1}\subset\mathcal{B}$ and taking a subsequence again if need be, there exists a $\widehat{Y}\in\mathcal{B}$ such that $\widehat{Y}^n\to\widehat{Y}$ in $\sigma(L^\Phi,L^\Psi)$ as $n\to\infty$. Thus, since $X_n\to X$ in $\sigma(L^\Phi,L^\Psi)$, we have

$$\rho(X-\widehat{Y})\leq\liminf_{n\to\infty}\rho(X_n-\widehat{Y}^n). \tag{3.5}$$

In conclusion, we get from (3.4) and (3.5)

$$\begin{aligned}
\rho\Box\mathcal{B}(X) &= \inf_{Y\in\mathcal{B}}\rho(X-Y)\leq\rho(X-\widehat{Y})\leq\liminf_{n\to\infty}\rho(X_n-\widehat{Y}^n)\\
&\leq \liminf_{n\to\infty}\liminf_{k\to\infty}\rho(X_n-\widehat{Y}_k^n)=\liminf_{n\to\infty}\inf_{Y\in\mathcal{B}}\rho(X_n-Y)\\
&= \liminf_{n\to\infty}\rho\Box\mathcal{B}(X_n),
\end{aligned}$$

which completes the proof of (c). □

Proposition 3.3 Let ρ_i, $i=1,2$, be convex risk measures on L^Φ. Denote their acceptance sets and penalty functions by $\mathcal{A}_i$ and α_i, respectively. Assume that ρ_1 satisfies the condition (3.2), and $\mathcal{A}_2$ includes 0. If $\rho_1\Box\rho_2(0)>-\infty$, then the following hold:
(a) $\rho_1\Box\rho_2$ is a convex risk measure on L^Φ, and represented as

$$\rho_1\Box\rho_2(X)=\rho_1\Box\mathcal{A}_2(X)=\inf_{Y\in\mathcal{B}}\{\rho_1(X-Y)+\rho_2(Y)\}, \text{ for any } X\in L^\Phi,$$

where $\mathcal{B}$ is a subset of L^Φ including $\mathcal{A}_2$.
(b) If ρ_1 and ρ_2 both are coherent, then so is $\rho_1\Box\rho_2$.
(c) The penalty function of $\rho_1\Box\rho_2$ is given by

$$\alpha_{1\Box 2}(Q)=\alpha_1(Q)+\alpha_2(Q) \text{ for any } Q\in\mathcal{P}^\Psi.$$

(d) Let $\mathcal{A}_2$ be sequentially compact in $\sigma(L^\Phi,L^\Psi)$. If ρ_1 is order l.s.c., then so is $\rho_1\Box\rho_2$. Moreover, the acceptance set of $\rho_1\Box\rho_2$ satisfies

$$\mathcal{A}_{1\Box 2}=\overline{\mathcal{A}_1+\mathcal{A}_2}, \tag{3.6}$$

which is the closure of $\mathcal{A}_1+\mathcal{A}_2$ in $\sigma(L^\Phi,L^\Psi)$.

Remark 3 Under the condition of (d), ρ_2 is also order l.s.c. To see it, let $\mathcal{A}_2$ be sequentially compact in $\sigma(L^\Phi, L^\Psi)$, and $(X_n)_{n\geq 1} \subset \mathcal{A}_2$ a sequence converging to some $X \in \mathcal{A}_2$ in $\sigma(L^\Phi, L^\Psi)$. Now, our aim is to show $\rho_2(X) \leq \liminf_{n\to\infty} \rho_2(X_n)$. Thus, we assume that $\rho_2(X) > \liminf_{n\to\infty} \rho_2(X_n)$. This implies that, for a sufficient small number $\varepsilon > 0$, there exist infinitely many ns such that $\rho_2(X) - \varepsilon \geq \rho_2(X_n)$. Hence, we can extract a subsequence $(X_{i_n})_{n\geq 1}$ to satisfy

$$\sup_n \rho_2(X_{i_n}) < \rho_2(X). \tag{3.7}$$

Set $\alpha := \sup_n \rho_2(X_{i_n})$. Note that $\alpha < \infty$ from (3.7). Denoting $X'_n := X_{i_n} + \alpha$ for $n \geq 1$, we have $X'_n \in \mathcal{A}_2$ since $\rho_2(X'_n) = \rho_2(X_{i_n}) - \alpha \leq 0$. Thus, recalling $X'_n \to X + \alpha$ in $\sigma(L^\Phi, L^\Psi)$, $X + \alpha \in \mathcal{A}_2$ holds since $\mathcal{A}_2$ has the sequential compactness. We can say, however, from (3.7) that $\rho_2(X+\alpha) = \rho_2(X) - \alpha > 0$, which means $X + \alpha \notin \mathcal{A}_2$. This is a contradiction. Consequently, ρ_2 is order l.s.c.

Proof of Proposition 3.3. (a) We can prove this assertion by a similar way with the proof of Theorem 4.3 (a) in [15].

(b) By Proposition 4.6 of Föllmer and Schied [12], $\mathcal{A}_2$ is cone. Thus, the assertion (a) and Proposition 3.2 (b) imply the assertion (b).

(c) We can show this equation by the same manner as the proof of (4.5) in Theorem 4.3 of [15].

(d) Although we can prove this assertion by a similar way with the proof of Theorem 4.3 (c) in [15], we need some additional discussion.

Since it is clear that $\rho_1 \Box \rho_2$ is order l.s.c. by Proposition 3.2 (c) and the assertion (a), we have only to prove (3.6). For any $X_i \in \mathcal{A}_i$, $i = 1, 2$, we have

$$\begin{aligned} \rho_1 \Box \rho_2(X_1 + X_2) &= \inf_{Y \in L^\Phi} \{\rho_1(X_1 + X_2 - Y) + \rho_2(Y)\} \\ &\leq \rho_1(X_1 + X_2 - X_2) + \rho_2(X_2) = \rho_1(X_1) + \rho_2(X_2) \leq 0, \end{aligned}$$

which implies $X_1 + X_2 \in \mathcal{A}_{1\Box 2}$. By the proof of Proposition 24 in [5] together with the order l.s.c. of $\rho_1 \Box \rho_2$, $\mathcal{A}_{1\Box 2}$ is $\sigma(L^\Phi, L^\Psi)$ -closed. Thus, we obtain $\mathcal{A}_{1\Box 2} \supset \overline{\mathcal{A}_1 + \mathcal{A}_2}$.

We shall prove the reverse inclusion. By a similar manner with the proof of Theorem 4.3 (c) in [15], we can obtain the following:

$$\inf_{X \in \mathcal{A}_{1\Box 2}} E[ZX] = \inf_{X \in \mathcal{A}_1 + \mathcal{A}_2} E[ZX] = \inf_{X \in \overline{\mathcal{A}_1 + \mathcal{A}_2}} E[ZX] \tag{3.8}$$

for all $Z \in L^\Psi_+$. Now, we suppose that there exists an X' satisfying $X' \in \mathcal{A}_{1\Box 2} \setminus \overline{\mathcal{A}_1 + \mathcal{A}_2}$. The Hahn-Banach theorem implies that there exists a $Z' \in L^\Psi$ such that

$$\inf_{X \in \overline{\mathcal{A}_1 + \mathcal{A}_2}} E[Z'X] > E[Z'X'] > -\infty. \tag{3.9}$$

Next, we prove that $-\overline{\mathcal{A}_1 + \mathcal{A}_2}$ is *solid*, namely,

$$X \in \overline{\mathcal{A}_1 + \mathcal{A}_2}, Y \in L^\Phi, Y \geq X \Longrightarrow Y \in \overline{\mathcal{A}_1 + \mathcal{A}_2}. \tag{3.10}$$

Let $X_1 \in \mathcal{A}_1$ and $X_2 \in \mathcal{A}_2$. Taking a $Y \in L^\Phi$ such that $Y \geq X_1 + X_2$, $Y - X_2 \geq X_1$ holds. Since $\mathcal{A}_i$, $i = 1, 2$ are acceptance sets, $-\mathcal{A}_i$, $i = 1, 2$ are solid by the monotonicity of the convex risk measures. Thus, $Y - X_2 \in \mathcal{A}_1$, and $Y \in \mathcal{A}_1 + \mathcal{A}_2$ because $Y = Y - X_2 + X_2$. Therefore, $-(\mathcal{A}_1 + \mathcal{A}_2)$ is solid. Now, let $X \in \overline{\mathcal{A}_1 + \mathcal{A}_2}$, and $Y \in L^\Phi$ such that $Y \geq X$. We have only to see $Y \in \overline{\mathcal{A}_1 + \mathcal{A}_2}$. Taking a sequence $(X_n)_{n\geq 1} \subset \mathcal{A}_1 + \mathcal{A}_2$ such that $X_n \to X$ in $\sigma(L^\Phi, L^\Psi)$, since $\sigma(L^\Phi, L^\Psi)$ has the C-property, there exist a subsequence $(X_{i_k})_{k\geq 1}$ and a sequence $(W_k)_{k\geq 1}$ such that, each W_k belongs to $\mathrm{conv}(X_{i_k}, X_{i_{k+1}}, \ldots)$ and $W_k \to X$ in order. The convexity of $\mathcal{A}_1 + \mathcal{A}_2$ implies that each W_k belongs to $\mathcal{A}_1 + \mathcal{A}_2$. Since $-(\mathcal{A}_1 + \mathcal{A}_2)$ is solid, we have $W_k \vee Y \in \mathcal{A}_1 + \mathcal{A}_2$ for any $k \geq 1$. Remark that $W_k \vee Y \to X \vee Y = Y$ a.s., and we can take a $\widehat{W} \in L^\Phi$ such that $|W_k| \vee |Y| \leq \widehat{W}$ for each $k \geq 1$. Thus, for any $Z \in L^\Psi$, $E[(W_k \vee Y)Z] \to E[YZ]$, which means that $W_k \vee Y \to Y$ as k tends to ∞ in $\sigma(L^\Phi, L^\Psi)$. Consequently, $Y \in \overline{\mathcal{A}_1 + \mathcal{A}_2}$, that is, $-\overline{\mathcal{A}_1 + \mathcal{A}_2}$ is solid.

We see that the Z' in (3.9) is non-negative. Fix $X \in \overline{\mathcal{A}_1 + \mathcal{A}_2}$ arbitrarily. Setting, for any $c > 0$, $X_c := X + c1_{\{Z'<0\}}$, we have $X_c \in \overline{\mathcal{A}_1 + \mathcal{A}_2}$ since $-\overline{\mathcal{A}_1 + \mathcal{A}_2}$ is solid. Now, assuming $P(Z' < 0) > 0$, we have

$$E[Z'X_c] = E[Z'X] + cE[Z'1_{\{Z'<0\}}] \to -\infty$$

as c tends to ∞, which implies that the LHS of (3.9)$= -\infty$. This is a contradiction. Hence, $P(Z' < 0) = 0$. Since (3.9) holds for some $Z' \in L^\Psi_+$, we have

$$\inf_{X \in \overline{\mathcal{A}_1 + \mathcal{A}_2}} E[Z'X] > \inf_{X \in \mathcal{A}_{1\square 2}} E[Z'X]$$

by $X' \in \mathcal{A}_{1\square 2}$. This contradicts to (3.8). Hence, there is no such an X', that is, $\overline{\mathcal{A}_1 + \mathcal{A}_2} \supset \mathcal{A}_{1\square 2}$ follows. $\square$

Proposition 3.4 Let $\mathcal{B}$ be a convex subset of L^Φ including 0, ρ a convex risk measure on L^Φ satisfying (3.2). Define $\rho^{\mathcal{B}}(X) := \inf\{x \in \mathbf{R} | x + X \in \mathcal{B}\}$ for any $X \in L^\Phi$. Let $\alpha_{\mathcal{B}}$ be defined as, for any $Q \in \mathcal{P}^\Psi$,

$$\alpha_{\mathcal{B}}(Q) = \begin{cases} 0, & \text{if } E_Q[-X] \leq 0 \text{ for any } X \in \mathcal{A}_{\mathcal{B}}, \\ \infty, & \text{otherwise}, \end{cases}$$

where $\mathcal{A}_{\mathcal{B}}$ is the acceptance set of $\rho^{\mathcal{B}}$, that is,

$$\mathcal{A}_{\mathcal{B}} := \{X \in L^\Phi | \rho^{\mathcal{B}}(X) \leq 0\} = \{X \in L^\Phi | x + X \in \mathcal{B} \text{ for any } x > 0\}. \quad (3.11)$$

Assume that $-\mathcal{B}$ is solid, which is defined in (3.10), and $\rho \square \rho^{\mathcal{B}}(0) > -\infty$.
(a) We have

$$\rho \square \mathcal{B} = \rho \square \rho^{\mathcal{B}}.$$

(b) If $\mathcal{B}$ is cone, then the penalty function $\alpha_{\rho\square\mathcal{B}}$ of $\rho\square\mathcal{B}$ is given by

$$\alpha_{\rho\square\mathcal{B}} = \alpha_\rho + \alpha_{\mathcal{B}},$$

where α_ρ is the penalty function of ρ.

Proof. (a) Our proof is based on the proof of Proposition 4.7 in [15]. Since $-\mathcal{B}$ is solid, (3.11) implies that $\mathcal{B} \subset \mathcal{A}_{\mathcal{B}}$. Proposition 3.3 (a) yields that

$$\rho\Box\rho^{\mathcal{B}}(X) = \inf_{Y' \in \mathcal{A}_{\mathcal{B}}} \rho(X - Y') \leq \inf_{Y \in \mathcal{B}} \rho(X - Y) = \rho\Box\mathcal{B}(X).$$

Hence, we have only to show that

$$\inf_{Y' \in \mathcal{A}_{\mathcal{B}}} \rho(X - Y') \geq \inf_{Y \in \mathcal{B}} \rho(X - Y). \tag{3.12}$$

Defining $\mathcal{A}^0_{\mathcal{B}} := \{Y' \in \mathcal{A}_{\mathcal{B}} | \rho^{\mathcal{B}}(Y') = 0\}$, we have

$$\inf_{Y' \in \mathcal{A}_{\mathcal{B}}} \rho(X - Y') = \inf_{Y \in \mathcal{A}^0_{\mathcal{B}}} \rho(X - Y), \tag{3.13}$$

because $\mathcal{A}^0_{\mathcal{B}} \subset \mathcal{A}_{\mathcal{B}}$ and $Y' + \rho^{\mathcal{B}}(Y') \in \mathcal{A}^0_{\mathcal{B}}$ for any $Y' \in \mathcal{A}_{\mathcal{B}}$. Furthermore, for any $Y_0 \in \mathcal{A}^0_{\mathcal{B}}$, we have $Y_0 + \dfrac{1}{n} \in \mathcal{B}$ for any $n \geq 1$. We have then, for any $Y_0 \in \mathcal{A}^0_{\mathcal{B}}$,

$$\inf_{Y \in \mathcal{B}} \rho(X - Y) \leq \rho\left(X - Y_0 - \frac{1}{n}\right) = \rho(X - Y_0) + \frac{1}{n}$$

for any $n \geq 1$, that is,

$$\inf_{Y \in \mathcal{B}} \rho(X - Y) \leq \rho(X - Y_0) \tag{3.14}$$

for any $Y_0 \in \mathcal{A}^0_{\mathcal{B}}$. Taking $\inf_{Y_0 \in \mathcal{A}^0_{\mathcal{B}}}$ on both sides of (3.14), (3.13) implies

$$\inf_{Y \in \mathcal{B}} \rho(X - Y) \leq \inf_{Y_0 \in \mathcal{A}^0_{\mathcal{B}}} \rho(X - Y_0) = \inf_{Y' \in \mathcal{A}_{\mathcal{B}}} \rho(X - Y'),$$

from which (3.12) follows.

(b) By Proposition 3.3 (c), we have $\alpha_{\rho\Box\mathcal{B}} = \alpha_\rho + \alpha_{\rho^{\mathcal{B}}}$, where $\alpha_{\rho^{\mathcal{B}}}$ is the penalty function of $\rho^{\mathcal{B}}$. Thus, we have $\alpha_{\rho^{\mathcal{B}}}(Q) = \sup_{X \in \mathcal{A}_{\mathcal{B}}} E_Q[-X]$ for any $Q \in \mathcal{P}^\Psi$. Since $\mathcal{A}_{\mathcal{B}} = \{X \in L^\Phi | \rho^{\mathcal{B}}(X) \leq 0\} = \{X \in L^\Phi | x + X \in \mathcal{B} \text{ for any } x > 0\}$ and $\mathcal{B}$ is cone, we have, for any fixed $c > 0$,

$$\begin{aligned} X \in \mathcal{A}_{\mathcal{B}} &\iff x + X \in \mathcal{B} \text{ for any } x > 0 \iff c(x + X) \in \mathcal{B} \text{ for any } x > 0 \\ &\iff x + cX \in \mathcal{B} \text{ for any } x > 0 \iff cX \in \mathcal{A}_{\mathcal{B}}. \end{aligned}$$

Hence, for $Q \in \mathcal{P}^\Psi$, if there exists an $X \in \mathcal{A}_{\mathcal{B}}$ such that $E_Q[-X] > 0$, then $\alpha_{\rho^{\mathcal{B}}}(Q) = \infty$. Moreover, we remark that $\alpha_{\rho^{\mathcal{B}}} \geq 0$ since $0 \in \mathcal{B}$. Therefore, for $Q \in \mathcal{P}^\Psi$, if $E_Q[-X] \leq 0$ holds for any $X \in \mathcal{A}_{\mathcal{B}}$, then $\alpha_{\rho^{\mathcal{B}}}(Q) = 0$. That is, we obtain $\alpha_{\rho^{\mathcal{B}}} = \alpha_{\mathcal{B}}$. □

4 Shortfall risk measure

We shall obtain a robust representation theorem for shortfall risk measures defined on L^Φ under the sequential compactness of the set of all hedging strategies in a weak sense. In addition, we shall construct an example satisfying the above sequential compactness.

Let $\mathcal{C}$ be a convex subset of L^Φ including 0. In this section, we regard $\mathcal{C}$ as the set of all attainable claims with zero initial endowment. Furthermore, each element of $\mathcal{C}$ is interpreted as a hedging strategy. We denote by X a contingent claim, which is a payoff at the maturity T. Thus, X is an $\mathcal{F}_T$-measurable random variable. In particular, we presume that X is in L^Φ.

Let l be a function from $\mathbf{R}$ to $\mathbf{R}_+$ satisfying $l(x) = 0$ if $x \leq 0$, and $l(x) = \Phi(x)$ if $x > 0$. Throughout this section, we presume a risk-averse investor who intends to sell the claim X, and whose *loss function* is given by l. When the price of X and the hedging strategy are given by $x \in \mathbf{R}$ and $U \in \mathcal{C}$, respectively, the *shortfall risk* for the seller is expressed by $E[l(-x - U + X)]$. We denote by $\delta > 0$ the *threshold* of the seller. Note that the threshold δ determines the limit of the shortfall risk which she can endure. We define, in addition, a subset of L^Φ as

$$\mathcal{A}_0 := \{X \in L^\Phi | E[l(-X)] \leq \delta\},$$

which is called the acceptance set with level δ. We define, by using $\mathcal{A}_0$, a functional $\widehat{\rho}$ defined on L^Φ as

$$\widehat{\rho}(X) := \inf\{x \in \mathbf{R}| \text{ there exists a } U \in \mathcal{C} \text{ such that } x + U + X \in \mathcal{A}_0\}.$$

We call $\widehat{\rho}$ the *shortfall risk measure*. Note that $\widehat{\rho}(-X)$ would give the least price which the seller can accept. In other words, if the seller sells the claim X for a price more than $\widehat{\rho}(-X)$, then she could find a hedging strategy whose corresponding shortfall risk is less than or equal to the threshold δ. We focus on a robust representation result for $\widehat{\rho}$.

Föllmer and Schied [11] have proved that, roughly speaking, if $\widehat{\rho}$ is defined on L^∞, $\widehat{\rho}$ becomes a convex risk measure, and have obtained a robust representation. Moreover, Arai [2] extend it to the Orlicz heart case to study the problem of good deal bounds. Remark that, although [11] and [2] took a set of stochastic integrals with respect to the asset price process as the set of hedging strategies, we presume that investors select an attainable claim with zero initial cost as their hedging strategy. In this section, we shall try to extend results in [2] to the Orlicz space case. As we have seen in Example 1.3, there are several examples of claims which are included in L^Φ, but not in M^Φ. In particular, when we consider insurance claims, our extension might be very significant. Since Orlicz hearts are order continuous, we do not need to get the order l.s.c. of $\widehat{\rho}$ to obtain its representation. On the other hand, we have to investigate the order l.s.c. of $\widehat{\rho}$ for the Orlicz space case, since Orlicz spaces do not have the order continuity in general.

Hereafter, we look into a sufficient condition for the order l.s.c. of $\widehat{\rho}$. Firstly, we impose the following:

Assumption 4.1 $\widehat{\rho}(0) > -\infty$

By adopting results in Section 3, we could make our problem easier. Actually, the shortfall risk measure $\widehat{\rho}$ is represented by using a convolution. We make sure this fact. Denoting

$$\rho_0(X) := \inf\{x \in \mathbf{R} | x + X \in \mathcal{A}_0\},$$

we have the following results:

Lemma 4.2 ρ_0 is a convex risk measure on L^Φ.

Proof. First, we see the properness of ρ_0. It is clear that $\rho_0(0) \leq 0$. Assuming $\rho_0(X) = -\infty$ for some $X \in L^\Phi$, we have $E[l(x - X)] \leq \delta$ for any $x > 0$. The monotone convergence theorem implies that $\delta \geq \lim_{x\to\infty} E[l(x - X)] = E[\lim_{x\to\infty} l(x - X)] = \infty$. This is a contradiction. Thus, $\rho_0 > -\infty$.

It is easy to check that the set $-\mathcal{A}_0$ is solid, that is

$$X \in \mathcal{A}_0, Y \in L^\Phi, Y \geq X \Longrightarrow Y \in \mathcal{A}_0,$$

and $\mathcal{A}_0$ satisfies

$$\inf\{x \in \mathbf{R} | x \in \mathcal{A}_0\} > -\infty.$$

Thus, Proposition 4.7 (b) of Föllmer and Schied [12] completes the proof of Lemma 4.2. $\square$

Proposition 4.3 $\widehat{\rho} = \rho_0 \square (-\mathcal{C})$.

Proof. Fix $X \in L^\Phi$ arbitrarily. By the definitions of $\widehat{\rho}$ and ρ_0, we have $\varepsilon + \rho_0(X+U) \geq \widehat{\rho}(X)$ for any $\varepsilon > 0$ and any $U \in \mathcal{C}$. Then, we have $\rho_0(X+U) \geq \widehat{\rho}(X)$ for any $U \in \mathcal{C}$, equivalently,

$$\rho_0 \square (-\mathcal{C})(X) \geq \widehat{\rho}(X). \tag{4.15}$$

Fix a $z \in \mathbf{R}$ arbitrarily to satisfy $z < \rho_0(X + U)$ for any $U \in \mathcal{C}$. Supposing $z > \widehat{\rho}(X)$, there exists a $U_0 \in \mathcal{C}$ such that $z + U_0 + X \in \mathcal{A}_0$. By the definition of ρ_0, $z \geq \rho_0(X + U_0)$ holds, this contradicts to the definition of z. Hence, $z \leq \widehat{\rho}(X)$, which implies $\widehat{\rho}(X) \geq \inf_{U\in\mathcal{C}} \rho_0(X + U)$. As a result, we get

$$\widehat{\rho}(X) \geq \rho_0 \square (-\mathcal{C})(X). \tag{4.16}$$

(4.16) together with (4.15) yields Proposition 4.3. $\square$

Lemma 4.4 ρ_0 is order l.s.c.

Proof. From the view of Remark 18 in Biagini and Frittelli [5], it suffices to show the continuity from above. Let $(X_n)_{n\geq 1} \subset L^\Phi$ be a decreasing sequence such that X_n converges a.s. to some $X \in L^\Phi$. The sequence $(\rho_0(X_n))_{n\geq 1}$ is then increasing, and $\rho_0(X_n) \leq \rho_0(X)$ holds for any $n \geq 1$. If $\lim_{n\to\infty} \rho_0(X_n) = \infty$, then $\lim_{n\to\infty} \rho_0(X_n) = \rho_0(X)$ follows. Thus, we assume that $\lim_{n\to\infty} \rho_0(X_n) < \infty$. Fix an $x \in \mathbf{R}$ arbitrarily to satisfy $x > \rho_0(X_n)$ for any $n \geq 1$. By the definition of ρ_0, we have $x + X_n \in \mathcal{A}_0$ for any $n \geq 1$, that is,

$$E[l(-x - X_n)] \leq \delta \text{ for any } n \geq 1.$$

The monotone convergence theorem implies that we get $E[l(-x - X_n)] \to E[l(-x - X)]$, namely, $E[l(-x - X)] \leq \delta$. This implies $x \geq \rho_0(X)$. By the arbitrariness of x, we have $\rho_0(X) = \sup_{n\geq 1} \rho_0(X_n) = \lim_{n\to\infty} \rho_0(X_n)$. □

Combining these results with Proposition 3.2, we can obtain a representation result for $\widehat{\rho}$ under some additional assumptions.

Theorem 4.1 Under Assumption 4.1, if $\mathcal{C}$ is sequentially compact in $\sigma(L^\Phi, L^\Psi)$, then $\widehat{\rho}$ is a $(-\infty, +\infty]$-valued convex risk measure on L^Φ satisfying the following:

$$\widehat{\rho}(X) = \sup_{Q\in\mathcal{P}^\Psi} \left\{ E_Q[-X] - \sup_{X^1\in\mathcal{A}_1} E_Q[-X^1] - \inf_{\lambda>0} \frac{1}{\lambda} \left\{ \delta + E\left[\Psi\left(\lambda \frac{dQ}{dP} \right) \right] \right\} \right\}, \tag{4.17}$$

where $\mathcal{A}_1 := \{X \in L^\Phi |$ there exists a $U \in \mathcal{C}$ such that $X + U \geq 0\}$.

Proof. From the view of Proposition 3.2 (a) together with Lemma 4.2 and Proposition 4.3, we have only to make sure whether ρ_0 satisfies (3.2) in order to prove the first assertion. If X satisfies $E[\Phi(|X|)] < \infty$, then $E[l(-X)] < \infty$. The dominated convergence theorem yields

$$\lim_{x\to\infty} E[l(-x - X)] = E[\lim_{x\to\infty} l(-x - X)] = 0,$$

since $l(-x-X) \leq l(-X) \in L^1$. Hence, for a sufficient large x, we have $E[l(-x-X)] \leq \delta$. As a result, $\rho_0(X) < \infty$ follows.

Propositions 3.2 (c) and 4.3, the sequential compactness of $\mathcal{C}$ and Lemma 4.4 imply that $\widehat{\rho}$ is order l.s.c. Theorem 2.1 yields that $\widehat{\rho}$ has the representation (2.1). By the same manner as the proof of Proposition 3.7 of [2], we obtain the representation (4.17). □

4.1 Construction of $\mathcal{C}$ having the sequential compactness

From the view of Theorem 4.1, $\widehat{\rho}$ has a robust representation if $\mathcal{C}$ has the sequential compactness in $\sigma(L^\Phi, L^\Psi)$. We construct, in this subsection, an example of $\mathcal{C}$ being sequentially compact in $\sigma(L^\Phi, L^\Psi)$.

We consider an incomplete financial market being composed of one riskless asset and d risky assets. The fluctuation of the risky assets is described by an $\mathbf{R}^d$-valued RCLL special semimartingale S, which is possibly non-locally bounded. Instead, we suppose that S is locally in L^Φ in the following sense: there exists a localizing sequence $(\tau^n)_{n\geq 1}$ of stopping times such that, for any $n \geq 1$, the family $\{S_\tau | \tau\text{: stopping time}, \tau \leq \tau^n\}$ is a subset of L^Φ. Now, we construct, by the same manner as Xia and Yan [19], a $\sigma(L^\Phi, L^\Psi)$-closed set of stochastic integrals. Let K^s_Φ be the subspace of L^Φ spanned by the simple stochastic integrals of the form $h^{tr}(S_{\sigma_2} - S_{\sigma_1})$, where $\sigma_1 \leq \sigma_2$ are stopping times such that $\{S_\sigma | \sigma\text{: stopping time}, \sigma \leq \sigma_2\} \subset L^\Phi$ and $h \in L^\infty$ is $\mathcal{F}_{\sigma_1}$-measurable. We denote the following:

$$\mathcal{M}^{\Psi,s} := \{Z \in L^\Psi | E[WZ] = 0 \text{ for any } W \in K^s_\Phi \text{ and } E[Z] = 1\},$$

$$\mathcal{M}^{\Psi,e} := \{Z \in \mathcal{M}^{\Psi,s} | Z > 0 \text{ a.s.}\},$$

and $K^\Phi := \overline{K^s_\Phi}$, which is the closure of K^s_Φ in $\sigma(L^\Phi, L^\Psi)$. We prepare one lemma.

Lemma 4.5 (a) $\mathcal{M}^{\Psi,s} \neq \emptyset \Longrightarrow 1 \notin K^\Phi$.
(b) For any $Z \in L^\Psi$, $Z \in \mathcal{M}^{\Psi,s}$ if and only if $E[Z] = 1$ and $E[WZ] = 0$ for any $W \in K^\Phi$.

Proof. (a) Suppose $\mathcal{M}^{\Psi,s} \neq \emptyset$, and let $Z \in \mathcal{M}^{\Psi,s}$. If $1 \in K^\Phi$, then $E[Z \cdot 1] = 0$, which contradicts to $E[Z] = 1$.

(b) The sufficient part is obvious. For any $W \in K^\Phi$, we can find a sequence $W_n \in K^s_\Phi$ satisfying $W_n \to W$ in $\sigma(L^\Phi, L^\Psi)$. For any $Z \in \mathcal{M}^{\Psi,s}$, we have

$$E[WZ] = E[(W - W_n + W_n)Z] = E[(W - W_n)Z] + E[W_n Z] \to 0$$

This completes the proof. □

The following lemma is proved by the analogy with the proof of Lemma 2.2 of [19].

Lemma 4.6 Assume $\mathcal{M}^{\Psi,s} \neq \emptyset$. We have

$$W \in K^\Phi \Longleftrightarrow W \in L^\Phi \text{ and } E[WZ] = 0 \text{ for any } Z \in \mathcal{M}^{\Psi,s}.$$

Θ^L denotes the set of all S-integrable predictable processes ϑ such that $\int_0^T \vartheta_s dS_s \in L^\Phi$ and $E[\int_0^T \vartheta_s dS_s \cdot Z] = 0$ for any $Z \in \mathcal{M}^{\Psi,s}$. Moreover, we denote $G := \{\int_0^T \vartheta_s dS_s | \vartheta \in \Theta^L\}$. Note that, if $\mathcal{M}^{\Psi,s} \neq \emptyset$, Lemma 4.6 implies $G \subset K^\Phi$. We introduce the following assumption:

Assumption 4.7 $\mathcal{M}^{\Psi,e} \neq \emptyset$.

Now, we can give a proof of the following theorem by the analogy with the proof of Theorem 2.1 in [19].

Theorem 4.2 Under Assumption 4.7, we have $K^\Phi = G$, that is, G is closed in $\sigma(L^\Phi, L^\Psi)$.

Proof. We shall illustrate only parts being different from the proof of Theorem 2.1 in [19]. Fix a probability measure Q whose density belongs to $\mathcal{M}^{\Psi,e}$. Letting $W \in K^{\Phi}$, we have $W \in L^{\Phi}$ and $E[WZ] = 0$ for any $Z \in \mathcal{M}^{\Psi,s}$ by Lemma 4.6. Thus, we have only to show that W is represented as a stochastic integral. Note that there exists a sequence $(W_n)_{n\geq 1} \subset K^s_{\Phi}$ such that $W_n \to W$ in $\sigma(L^{\Phi}, L^{\Psi})$. Since $\sigma(L^{\Phi}, L^{\Psi})$ has the C-property, we can extract a subsequence $(W_{i_k})_{k\geq 1}$ and take convex combinations $Y_k \in \mathrm{conv}(W_{i_k}, W_{i_{k+1}}, \ldots)$ for $k \geq 1$ such that $Y_k \to W$ in order. Hence, we can say $Y_k \to W$ in $L^1(Q)$. Moreover, the set

$$\begin{aligned}\widetilde{G} \;\; := \;\; &\left\{ \int_0^T \vartheta_s dS_s \middle| \vartheta \text{ is an } S\text{-integrable predictable process,} \right.\\ &\left. \qquad \text{the process } \int_0^{\cdot} \vartheta_s dS_s \text{ is } Q\text{-uniformly integrable} \right\}\end{aligned}$$

is closed in $L^1(Q)$. Note that the process S is a local Q-martingale, since S is locally in L^{Φ}. Each W_n belongs to $\widetilde{G}$, thus so is each Y_k, namely, $W \in \widetilde{G}$. □

Now, we impose an additional assumption.

Assumption 4.8 $\lim_{k\to 0} k^{-1}E[\Phi(k|W|)] = 0$ uniformly in $W \in G$.

Under Assumptions 4.7 and 4.8, G is sequentially compact in $\sigma(L^{\Phi}, L^{\Psi})$ by Theorem IV.5.3 of Rao and Ren [17] and Theorem 4.2.

Remark 4 Although, in Theorem IV.5.3 of [17], the function Φ is restricted to be an N-function, we do not have to impose it for our purpose.

Taking $G - A$ as the set $\mathcal{C}$ of all attainable claims with zero initial endowment, where A is a sequentially compact subset of L^{Φ}_{+} in $\sigma(L^{\Phi}, L^{\Psi})$, $\mathcal{C}$ is also sequentially compact. Hence, we can conclude the following:

Theorem 4.3 Under Assumptions 4.1, 4.7 and 4.8, $\widehat{\rho}$ is a $(-\infty, +\infty]$-valued convex risk measure on L^{Φ} satisfying (4.17).

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