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AN ELEMENTARY PROOF OF THE EULER EQUATION IN GROWTH THEORY

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Abstract: In this paper, we derive the Euler equation in a continuous-time macroeconomic dynamic model using only elementary mathematical knowledge. We also provide a proof that the Euler equation and transversality condition are sufficient conditions for the optimal solution. To understand the proofs in this paper, no knowledge of Lebesgue integrals is required. Readers require only some basic mathematical knowledge.

Key words: Continuous-time growth model, Euler equation, transversality condition, du Bois-Reymond's lemma.

JEL Classification Number: C61, A22, E13.

1. INTRODUCTION

In classical continuous-time macroeconomic dynamic models, the Euler equation plays a very important role. However, to the best of our knowledge, the derivation of the Euler equation is not adequately explained in almost all textbooks. For example, Romer (2012) derived the Euler equation by "pretending" that Lagrange's multiplier rule in finite-dimensional space could be applied. Blanchard and Fischer (1989) derived the Euler equation by introducing and applying Pontryagin's maximum principle in an appendix. Similar explanations has been done by Acemoglu (2009) and Barro and Sala-i-Martin (2003). However, deriving Lagrange's multiplier rule in function spaces requires deep knowledge of Banach spaces. Moreover, Pontryagin's maximum principle is nightmarishly difficult to prove. As a result, most undergraduate and graduate students are unable to gain effective knowledge of the techniques for deriving the Euler equation from textbooks.[1]

This paper explains how to derive the Euler equation for a Ramsey-Cass-Koopmans

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[1] Chapter 1 of Ioffe and Tikhomirov (1979) gives a detailed argument of Lagrange's multiplier rule for abstract problems, but one requires a lot of preparation to read and understand this textbook. The simplest proof of Pontryagin's maximum principle that we know of is given by Luenburger (1997), which also requires some basic theorems on Banach spaces.

optimal capital accumulation model.[2] As the primary readers for this paper, we envision undergraduate students in the third year and above, and graduate students in the master course. The only knowledge required is the usual knowledge of differential and integral calculus, a very elementary knowledge of linear algebra, a basic knowledge of the ε-δ method, and a simple knowledge of the properties of concave functions. In particular, as for integrals, only Riemann integrals are treated in this paper. Even readers with no knowledge of Lebesgue integral theory will be able to understand the proofs in this paper.

There are two types of optimal capital accumulation models: decentralized and centralized. Both of which will be treated in this paper. In both models, the Euler equation can be derived using exactly the same method, in which a very simple lemma of linear algebra is crucial. In the centralized model, it is known that the Euler equation and transversality condition are sufficient conditions for the solution, which is also treated in this paper. In the appendix, we have added a discussion of how to derive the Euler equation for more difficult problems.

The assumptions regarding the utility function and the production technology used in this paper are far fewer than in many macroeconomics textbooks. For example, we do not require twice differentiability of the utility function. Although this assumption is popular, it is not necessary at all if we just want to derive the Euler equation. On the other hand, there are some unfamiliar assumptions in the model, such as the definability of the objective function or the real-valuedness of the solution. However, these are essentially necessary for a rigorous discussion in this context, and in this sense, textbooks do not describe the problem correctly.

The remainder of this paper is organized as follows. In Section 2, we describe the centralized capital accumulation model, and explain terms such as solutions and the Euler equation. In Section 3, we present proofs of the Leibniz integral rule and du Bois-Reymond's lemma, and use them to show the main results. Section 4 deals with the decentralized model and again derives the Euler equation. Section 5 is the conclusion, where we discuss in detail what to do for more in-depth study. The appendix contains an example of deriving the Euler equation for a more difficult problem.

2. PRELIMINARIES

2.1. *The Model*

The first model we treat in this paper is as follows.

$$\begin{aligned} \max\ & \int_0^\infty e^{-\rho t}u(c(t))dt, \\ \text{subject to. } & c(t) \ge 0,\ k(t) \ge 0, \\ & c(t) \text{ is continuous}, \\ & k(t) \text{ is continuously differentiable}, \end{aligned} \tag{1}$$

[2] This model was formulated by Ramsey (1928), and modified by Cass (1965) and Koopmans (1965).

$$\int_0^\infty e^{-\rho t}u(c(t))dt \text{ is defined,}$$
$$\dot{k}(t) = f(k(t)) - c(t),$$
$$k(0) = \bar{k},$$

where $\rho > 0$ and $\bar{k} > 0$. Note that, "$\int_0^\infty e^{-\rho t}u(c(t))dt$ is defined" means that this integral is defined in the sense of an improper Riemann integral, in other words, that $\lim_{T\to\infty}\int_0^T e^{-\rho t}u(c(t))dt$ exists, though it may be either $+\infty$ or $-\infty$. Note also that, the symbol $\dot{k}(t)$ denotes the derivative of $k(t)$ at time t: in other words, $\dot{k}(t)$ indicates the speed of increase for the function $k(t)$.

We should explain the equation

$$\dot{k}(t) = f(k(t)) - c(t). \tag{2}$$

Actually, there are three relationships behind this equation. The first relationship is the simplified IS relationship: that is,

$$y(t) = c(t) + i(t),$$

where $c(t)$ is consumption, $i(t)$ is investment, and $y(t)$ is the amount of production at time t. The second relationship is on the production technology:

$$y(t) = g(k(t)),$$

where $k(t)$ is the total amount of capital at time t and $g(k)$ is the production function. The third relationship determines the speed of capital accumulation:

$$\dot{k}(t) = i(t) - dk(t),$$

where $d \geq 0$ represents the capital depreciation rate. Using these three formulas, we obtain

$$\dot{k}(t) = g(k(t)) - dk(t) - c(t),$$

and if we define $f(k) = g(k) - dk$, then we obtain equation (2).

We assume the following.[3]

Assumption U. $u : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ is a continuous and increasing function that is continuously differentiable on $\mathbb{R}_{++}$. Moreover, $u'(c) > 0$ for all $c > 0$.

Assumption F. $g : \mathbb{R}_+ \to \mathbb{R}$ is a continuous and increasing function such that $g(0) = 0$ and g is continuously differentiable on $\mathbb{R}_{++}$.

Assumption C. The functions u, g are concave.

Under Assumption F, we have that $f(0) = 0$, f is continuous, and continuously differentiable on $\mathbb{R}_{++}$. If, in addition, Assumption C holds, then f is also concave. Note that f is not necessarily increasing, because the possibility of $g'(k) < d$ is not excluded.

[3] As usual, we define $\mathbb{R}_+$ as the set of nonnegative numbers and $\mathbb{R}_{++}$ as the set of positive numbers, respectively.

Note also that we do not assume the Inada condition.[4] Although this condition is very popular in this context, we do not require this assumption for our purpose.

2.2. An Admissible Pair and a Solution

Let $(k(t), c(t))$ be a pair of functions from $\mathbb{R}_+$ into $\mathbb{R}_+$. We call these functions an **admissible pair** when all restrictions of (1) are met. In other words, $(k(t), c(t))$ is admissible if 1) $k(t)$ is continuously differentiable, 2) $c(t)$ is continuous, 3) $k(0) = \bar{k}$, 4) the capital accumulation equation (2) holds for all $t \geq 0$, and 5) $\int_0^\infty e^{-\rho t} u(c(t))dt$ can be defined.

Let $(k^*(t), c^*(t))$ be an admissible pair. We call this pair a **solution** to problem (1) if the following two requirements hold.

1. The value
$$\int_0^\infty e^{-\rho t} u(c^*(t))dt$$
is a real number.
2. For every admissible pair $(k(t), c(t))$,
$$\int_0^\infty e^{-\rho t} u(c^*(t))dt \geq \int_0^\infty e^{-\rho t} u(c(t))dt.$$

Note that, by the first requirement, if there exists an admissible pair $(k(t), c(t))$ such that $\int_0^\infty e^{-\rho t} u(c(t))dt = +\infty$, then there is no solution to (1).

Finally, we define the notion of inner solutions. A solution $(k(t), c(t))$ to (1) is said to be an **inner solution** if $k(t) > 0, c(t) > 0$ for all $t \geq 0$.

2.3. The Euler Equation and Transversality Condition

Usually, the Euler equation of problem (1) is written as follows.

$$\dot{c}(t) = (\rho - f'(k(t)))\frac{u'(c(t))}{u''(c(t))}. \tag{3}$$

However, this expression has at least two problems. First, we do not assume the differentiability of $c(t)$. Second, we do not assume the twice differentiability of u. Therefore, we mainly use the following alternative formula:

$$\frac{d}{dt}(u' \circ c)(t) = (\rho - f'(k(t)))(u' \circ c)(t). \tag{4}$$

We say that an admissible pair $(k(t), c(t))$ is a solution to the Euler equation if $k(t) > 0$, $c(t) > 0$, $u'(c(t))$ is continuously differentiable, and (4) holds for all $t \geq 0$.

If u is twice continuously differentiable and $u''(c) \neq 0$ on $\mathbb{R}_{++}$, then the inverse function of u' is also continuously differentiable by the inverse function theorem, and thus, $u'(c(t))$ is continuously differentiable if and only if $c(t)$ is continuously differentiable. In this case, (4) is equivalent to (3). However, at least we do not require the twice differentiability of u in the derivation of the Euler equation.

Next, we mention transversality condition. An admissible pair $(k(t), c(t))$ is said to satisfy transversality condition if $c(t) > 0$ for all $t \geq 0$ and

[4] The Inada condition means that $\lim_{k\to 0} g'(k) = +\infty$ and $\lim_{k\to\infty} g'(k) = 0$.

$$\lim_{T\to\infty} e^{-\rho T}u'(c(T))k(T) = 0. \tag{5}$$

3. RESULTS

3.1. On the Leibniz Integral Rule

The Leibniz integral rule is a famous result in differential calculus, and plays a fundamental role in our proof of main results. However, this rule is not so elementary. Therefore, we provide this rule and its proof rigorously. Readers who are familiar with this result can skip this subsection.

The Leibniz Integral Rule. Suppose that $a < b$ and $f : [a, b]\times U \to \mathbb{R}$ is continuous, where U is an open interval in $\mathbb{R}$. Then, the followings hold.

1) The function
$$g(x) = \int_a^b f(t, x)dt$$
is continuous on U.

2) If the function $\frac{\partial f}{\partial x}(t, x)$ is also continuous, then
$$g'(x) = \int_a^b \frac{\partial f}{\partial x}(t, x)dt.$$

Proof. Choose any $x \in U$ and $\varepsilon > 0$. Because f is continuous, for each $t \in [a, b]$, there exists $\delta(t) > 0$ such that if $(s, y) \in [a, b] \times U$ and $\max\{|s - t|, |y - x|\} < \delta(t)$, then
$$|f(s, y) - f(t, x)| < 2^{-1}(b - a)^{-1}\varepsilon.$$
Let $V(t) = (t-\delta(t), t+\delta(t))$. Because $[a, b]$ is compact, there is a finite set $\{t_1, ..., t_n\}$ such that $[a, b] \subset \cup_{i=1}^n V(t_i)$. Define
$$\delta = \min\{\delta(t_1), ..., \delta(t_n)\}.$$
Suppose that $t \in [a, b]$, $y \in U$ and $|y-x| < \delta$. Then, there exists i such that $|t - t_i| < \delta(t_i)$. Hence,
$$|f(t, y) - f(t, x)| \le |f(t, y) - f(t_i, x)| + |f(t_i, x) - f(t, x)| < (b - a)^{-1}\varepsilon.$$
Therefore, if $y \in U$ and $|y - x| < \delta$, then
$$|g(y) - g(x)| \le \int_a^b |f(t, y) - f(t, x)|dt < \varepsilon,$$
which implies that g is continuous.[5] This completes the proof of 1).

For 2), choose any $x \in U$ and $\varepsilon > 0$. By almost the same arguments as in the above paragraph, we can show that there exists $\delta > 0$ such that

[5] The inequality
$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$$
can easily be shown, and thus we omit its proof.

$$\left|\frac{\partial f}{\partial x}(t, y) - \frac{\partial f}{\partial x}(t, x)\right| < (b-a)^{-1}\varepsilon$$

for any $t \in [a, b]$ and $y \in U$ such that $|y - x| < \delta$. By the mean value theorem,

$$\left|f(t, x+h) - f(t, x) - \frac{\partial f}{\partial x}(t, x)h\right| < (b-a)^{-1}\varepsilon|h|$$

whenever $|h| < \delta$. Therefore, if $|h| < \delta$, then

$$\begin{aligned}&\left|g(x+h) - g(x) - h\int_a^b \frac{\partial f}{\partial x}(t, x)dt\right| \\ &\le \int_a^b \left|f(t, x+h) - f(t, x) - h\frac{\partial f}{\partial x}(t, x)\right| dt \\ &< \varepsilon|h|,\end{aligned}$$

which implies that

$$g'(x) = \int_a^b \frac{\partial f}{\partial x}(t, x)dt.$$

This completes the proof. ■

3.2. On du Bois-Reymond's Lemma

Classically, in the derivation of the Euler equation, the result named du Bois-Reymond's lemma is frequently used.[6] This lemma states the following:

du Bois-Reymond's Lemma. Suppose that $b(t)$ is continuous, and

$$\int_0^T b(t)x(t)dt = 0$$

for every continuous function $x(t)$ such that $x(0) = x(T) = 0$ and

$$\int_0^T x(t)dt = 0.$$

Then, $b(t)$ is a constant function.

In many variational problems, this lemma plays a key role in deriving the Euler equation. We show this result using two lemmas.

Lemma 1. Suppose that V is a vector space, and let f_0 and f_1 be linear functionals on V; that is, each f_i is a function from V into $\mathbb{R}$ such that for every $v_1, v_2 \in V$ and $a_1, a_2 \in \mathbb{R}$, $f_i(a_1v_1 + a_2v_2) = a_1f_i(v_1) + a_2f_i(v_2)$. Define Ker $f_i = \{v \in V | f_i(v) = 0\}$. Then, the following two statements are equivalent.

1) $f_0 = a_1f_1$ for some $a_1 \in \mathbb{R}$.
2) Ker $f_1 \subset$ Ker f_0.

[6] Regarding this surname, Emil Heinrich du Bois-Reymond is quite famous. However, this lemma was found by his brother, Paul David Gustav du Bois-Reymond.

Proof. Clearly, 1) implies 2). Therefore, it suffices to show that 2) implies 1).

If $\mathrm{Ker}\, f_0 = V$, then this claim is obvious. Hence, we assume that there exists $v^* \in V$ such that $f_0(v^*) \neq 0$. Because $\mathrm{Ker}\, f_1 \subset \mathrm{Ker}\, f_0$, we have that $f_1(v^*) \neq 0$. Without loss of generality, we assume that $f_1(v^*) = 1$. Define $f_0(v^*) = a_1$. Choose any $v \in V$. If $f_1(v) = 0$, then $f_0(v) = 0$, and thus $f_0(v) = a_1 f_1(v)$. If $f_1(v) \neq 0$, define $b_1 = f_1(v)$. Then,

$$f_1(v^* - b_1^{-1}v) = f_1(v^*) - b_1^{-1} f_1(v) = 0.$$

Therefore, $v^* - b_1^{-1}v \in \mathrm{Ker}\, f_1 \subset \mathrm{Ker}\, f_0$. Hence,

$$0 = f_0(v^* - b_1^{-1}v) = f_0(v^*) - b_1^{-1} f_0(v),$$

and thus,

$$a_1 = f_0(v^*) = b_1^{-1} f_0(v),$$

which implies that $f_0(v) = a_1 b_1 = a_1 f_1(v)$. In conclusion, we have that for each $v \in V$, $f_0(v) = a_1 f_1(v)$, as desired. This completes the proof. ■

Lemma 2. Let V be the set of all continuous functions $x(t)$ from $[0, T]$ into $\mathbb{R}$ such that $x(0) = x(T) = 0$. Suppose that $b(t)$ is a continuous function such that

$$\int_0^T b(t)x(t)dt = 0$$

for all $x(t) \in V$. Then, $b(t) \equiv 0$.

Proof. Suppose that $b(t) \neq 0$ at $t \in [0, T]$. Without loss of generality, we assume that $0 \neq t \neq T$ and $b(t) > 0$. Choose $\varepsilon > 0$ such that $0 < t - \varepsilon < t + \varepsilon < T$ and for every $s \in [t - \varepsilon, t + \varepsilon]$, $b(s) > b(t)/2$. Define

$$x(s) = \max\{0, 1 - \varepsilon^{-1}|t - s|\}.$$

Then, $x(0) = x(T) = 0$ and

$$\int_0^T b(t)x(t)dt > 0,$$

which is a contradiction. ■

Proof of du Bois-Reymond's Lemma. Let V be the same set as in Lemma 2. Note that V is a linear space. Define

$$\Lambda_0(x(t)) = \int_0^T b(t)x(t)dt,$$

$$\Lambda_1(x(t)) = \int_0^T x(t)dt.$$

By assumption of du Bois-Reymond's lemma, $\mathrm{Ker}\, \Lambda_1 \subset \mathrm{Ker}\, \Lambda_0$ holds true, and thus by 1) of Lemma 1, we have that there exists $a_1 \in \mathbb{R}$ such that

$$\Lambda_0 = a_1 \Lambda_1,$$

which implies that for all $x(t) \in V$,

$$\int_0^T (b(t)-a_1)x(t)dt = 0.$$

By Lemma 2, we have that $b(t)-a_1 \equiv 0$, and thus $b(t) \equiv a_1$. This completes the proof. ■

3.3. Main Result I: Necessity of the Euler Equation

Our first main result is as follows.

Theorem 1. Suppose that Assumptions U and F hold, and let $(k^*(t), c^*(t))$ be an inner solution to (1). Then, this pair is a solution to the Euler equation.

Proof. First, we delete $c(t)$ from (1) by using (2). Because of (2),

$$c(t) = f(k(t)) - \dot{k}(t).$$

Therefore, (1) is equivalent to the following problem:

$$\begin{aligned} \max\ & \int_0^\infty e^{-\rho t}u(f(k(t))-\dot{k}(t))dt, \\ \text{subject to. } & k(t) \text{ is continuously differentiable}, \\ & k(t) \ge 0,\ f(k(t))-\dot{k}(t) \ge 0, \\ & \int_0^\infty e^{-\rho t}u(f(k(t))-\dot{k}(t))dt \text{ is defined}, \\ & k(0) = \bar{k}. \end{aligned} \tag{6}$$

By definition, $k^*(t)$ is a solution to this model and $c^*(t) = f(k^*(t)) - \dot{k}^*(t)$ for all $t \ge 0$.

Next, choose any $T > 0$, and define V as the set of all continuous functions $x : [0,T] \to \mathbb{R}$ such that $x(0) = x(T) = 0$. Choose $x(t) \in V$ such that $\int_0^T x(t)dt = 0$, and define $k(t)$ and $k_s(t)$ as follows:

$$k(t) = \int_0^t x(\tau)d\tau,$$

$$k_s(t) = \begin{cases} k^*(t) + sk(t) & \text{if } 0 \le t \le T, \\ k^*(t) & \text{if } t \ge T. \end{cases}$$

Then, $k_s(t)$ is continuously differentiable. Because $(k^*(t), c^*(t))$ is an inner solution, there exists $\delta > 0$ such that if $|s| < \delta$, then $k_s(t) \ge 0$ and $f(k_s(t)) - \dot{k}_s(t) \ge 0$ for all $t \ge 0$. Define

$$\varphi(s) = \int_0^T e^{-\rho t}u(f(k_s(t)) - \dot{k}_s(t))dt.$$

Because $k^*(t)$ is a solution to (6), $\varphi(0) = \max_{s\in(-\delta,\delta)} \varphi(s)$, and thus $\varphi'(0) = 0$. By the Leibniz integral rule,

$$0 = \varphi'(0) = \int_0^T [e^{-\rho t}u'(c^*(t))f'(k^*(t))k(t) - e^{-\rho t}u'(c^*(t))x(t)]dt. \tag{7}$$

Now, recall the formula of integration by parts:

$$\int_a^b f'(t)g(t)dt = [f(t)g(t)]_a^b - \int_a^b f(t)g'(t)dt.$$

Substituting $-\int_t^T e^{-\rho\tau}u'(c^*(\tau))f'(k^*(\tau))d\tau$ into $f(t)$ and $k(t)$ into $g(t)$, we obtain

$$\begin{aligned}
&\int_0^T e^{-\rho t}u'(c^*(t))f'(k^*(t))k(t)dt \\
&= \left[-\left(\int_t^T e^{-\rho\tau}u'(c^*(\tau))f'(k^*(\tau))d\tau\right)k(t)\right]_0^T \\
&\quad + \int_0^T \left[\int_t^T e^{-\rho\tau}u'(c^*(\tau))f'(k^*(\tau))d\tau\right]x(t)dt \\
&= \int_0^T \left[\int_t^T e^{-\rho\tau}u'(c^*(\tau))f'(k^*(\tau))d\tau\right]x(t)dt.
\end{aligned}$$

Therefore, by using the relation (7), we have that

$$\int_0^T \left[\int_t^T e^{-\rho\tau}u'(c^*(\tau))f'(k^*(\tau))d\tau - e^{-\rho t}u'(c^*(t))\right]x(t)dt = 0.$$

By du Bois-Reymond's lemma, we have that there exists $a \in \mathbb{R}$ such that

$$\int_t^T e^{-\rho\tau}u'(c^*(\tau))f'(k^*(\tau))d\tau = e^{-\rho t}u'(c^*(t)) + a. \tag{8}$$

Therefore,

$$u'(c^*(t)) = e^{\rho t}\left[\int_t^T e^{-\rho\tau}u'(c^*(\tau))f'(k^*(\tau))d\tau - a\right],$$

where the right-hand side is continuously differentiable. Therefore, $u'(c^*(t))$ is continuously differentiable. Differentiating both side of (8), we have that for all $t \in [0, T]$,

$$-\rho e^{-\rho t}u'(c^*(t)) + e^{-\rho t}\frac{d}{dt}(u' \circ c^*)(t) = -e^{-\rho t}u'(c^*(t))f'(k^*(t)).$$

which implies that

$$\frac{d}{dt}(u' \circ c^*)(t) = (\rho - f'(k^*(t)))(u' \circ c^*)(t).$$

This equation is the same as (4). Because $T > 0$ is arbitrary, we have that $(k^*(t), c^*(t))$ is a solution to the Euler equation. This completes the proof. ■

3.4. *Main Result II: Sufficiency of the Euler Equation and Transversality Condiition*

Before arguing our second main result, we make a preparation. Suppose that $L : U \to \mathbb{R}$ is a continuously differentiable and concave function, where $U \subset \mathbb{R}^2$ is convex. Choose any $(x^*, y^*) \in U$. For $(x, y) \in U$, define $d(t) = L((1-t)(x^*, y^*) + t(x, y))$. Then, we can easily check that $d(t)$ is concave, and thus $d(1) - d(0) \le d'(0)$. By the chain rule, we have that

$$L(x, y) - L(x^*, y^*) \le \frac{\partial L}{\partial x}(x^*, y^*)(x - x^*) + \frac{\partial L}{\partial y}(x^*, y^*)(y - y^*). \tag{9}$$

We use this formula in the proof of the following result.

Theorem 2. Suppose that Assumptions U, F, and C hold. Let $(k^*(t), c^*(t))$ be an admissible pair such that $k^*(t) > 0, c^*(t) > 0$ for all $t \geq 0$, and $\int_0^\infty e^{-\rho t}u(c^*(t))dt$ is a real number. If $(k^*(t), c^*(t))$ satisfies the Euler equation and transversality condition, then it is a solution to (1).

Proof. Again, we use the transform

$$c(t) = f(k(t)) - \dot{k}(t)$$

throughout this proof. We define

$$L(x, y) = u(f(x) - y).$$

Then, the function L is continuously differentiable on the set

$$U = \{(x, y) \in \mathbb{R}^2 | x > 0, f(x) - y > 0\}.$$

Let $t \in [0, 1]$ and $(x_1, y_1), (x_2, y_2) \in U$, and define $(x, y) = (1-t)(x_1, y_1)+t(x_2, y_2)$. Then, $x > 0$ and

$$\begin{aligned} f(x) - y &\geq (1-t)f(x_1) + tf(x_2) - (1-t)y_1 - ty_2 \\ &= (1-t)[f(x_1) - y_1] + t[f(x_2) - y_2] > 0, \end{aligned}$$

which indicates that $(x, y) \in U$. Moreover, because u is increasing and concave,

$$\begin{aligned} L(x, y) &= u(f(x) - y) \\ &\geq u((1-t)[f(x_1) - y_1] + t[f(x_2) - y_2]) \\ &\geq (1-t)u(f(x_1) - y_1) + tu(f(x_2) - y_2) \\ &= (1-t)L(x_1, y_1) + tL(x_2, y_2). \end{aligned}$$

Therefore, we have that U is convex and L is concave, and the formula (9) can be applied to L.

Choose any admissible pair $(k(t), c(t))$. Then, for $T > 0$,

$$\begin{aligned} &\int_0^T e^{-\rho t}[u(c(t)) - u(c^*(t))]dt \\ = &\int_0^T e^{-\rho t}[L(k(t), \dot{k}(t)) - L(k^*(t), \dot{k}^*(t))]dt \\ \leq &\int_0^T e^{-\rho t}\left[\frac{\partial L}{\partial x}(k^*(t), \dot{k}^*(t))(k(t) - k^*(t)) + \frac{\partial L}{\partial y}(k^*(t), \dot{k}^*(t))(\dot{k}(t) - \dot{k}^*(t))\right]dt \\ = &\int_0^T e^{-\rho t}[u'(c^*(t))f'(k^*(t))(k(t) - k^*(t)) - u'(c^*(t))(\dot{k}(t) - \dot{k}^*(t))]dt. \end{aligned}$$

Using the Euler equation, we have that

$$\begin{aligned} &e^{-\rho t}u'(c^*(t))f'(k^*(t))(k(t) - k^*(t)) - e^{-\rho t}u'(c^*(t))(\dot{k}(t) - \dot{k}^*(t)) \\ = &\frac{d}{dt}[e^{-\rho t}u'(c^*(t))(k^*(t) - k(t))]. \end{aligned}$$

Because $k^*(0) = k(0) = \bar{k}$ and $k(T) \geq 0$,

$$\begin{aligned}\int_0^T e^{-\rho t}[u(c(t)) - u(c^*(t))]dt &\leq \int_0^T \frac{d}{dt}[e^{-\rho t}u'(c^*(t))(k^*(t) - k(t))]dt \\ &= e^{-\rho T}u'(c^*(T))(k^*(T) - k(T)) \\ &\leq e^{-\rho T}u'(c^*(T))k^*(T) \to 0 \text{ (as } T \to \infty),\end{aligned}$$

where the last line follows from transversality condition. This implies that

$$\int_0^\infty e^{-\rho t}u(c^*(t))dt \geq \int_0^\infty e^{-\rho t}u(c(t))dt,$$

as desired. This completes the proof. ■

4. DECENTRALIZED MODEL

Problem (1) is a sort of centralized model, in which interest rate and wage are absent. However, in many macroeconomic models, the decentralized version of (1) is used and analyzed. In these models, it is thought that Lagrange's multiplier rule for infinite dimensional space is needed to derive the Euler equation. However, this is incorrect. In this section, we try to derive this rule using only our lemmas.

The problem considered in this section is as follows:

$$\begin{aligned}\max\ & \int_0^\infty e^{-\rho t}u(c(t))dt, \\ \text{subject to. } & c(t) \geq 0, \\ & c(t) \text{ is continuous,} \\ & \int_0^\infty e^{-\rho t}u(c(t))dt \text{ is defined,} \\ & \int_0^\infty e^{-R(t)}c(t)dt = \bar{k} + \int_0^\infty e^{-R(t)}w(t)dt,\end{aligned} \tag{10}$$

where $w : \mathbb{R}_+ \to \mathbb{R}_{++}$ and $r : \mathbb{R}_+ \to \mathbb{R}_{++}$ are given continuous functions, and $R(t) = \int_0^t r(\tau)d\tau$. The value $w(t)$ represents wage and $r(t)$ represents interest rate at time t. The last equality represents the consumer's **budget constraint**.

As in section 2, we define the notion of admissible paths. A continuous function $c(t)$ is said to be **admissible** if 1) $c(t) \geq 0$ for all $t \geq 0$, 2) $\int_0^\infty e^{-\rho t}u(c(t))dt$ is defined, and 3) $\int_0^\infty e^{-R(t)}c(t)dt = \bar{k} + \int_0^\infty e^{-R(t)}w(t)dt$. An admissible function $c^*(t)$ is a **solution** to (10) if

$$\int_0^\infty e^{-\rho t}u(c^*(t))dt$$

is a real number, and for any admissible function $c(t)$,

$$\int_0^\infty e^{-\rho t}u(c^*(t))dt \geq \int_0^\infty e^{-\rho t}u(c(t))dt.$$

If $c^*(t)$ is a solution to (10) and $c^*(t) > 0$ for all $t \geq 0$, then it is said to be an **inner solution**.

In this section, the Euler equation is the following:

$$\frac{d}{dt}(u' \circ c)(t) = (\rho - r(t))(u' \circ c)(t). \tag{11}$$

A continuous function $c(t)$ is said to be a **solution** to the Euler equation if $u'(c(t))$ is continuously differentiable and (11) holds for all $t \geq 0$.

Theorem 3. Suppose that Assumption U holds. If $c^*(t)$ is an inner solution to (10), then it is a solution to the Euler equation.

Proof. Fix any $T > 0$. Let V be the set of all continuous functions $x : [0, T] \to \mathbb{R}$ such that $x(0) = x(T) = 0$. Choose any $x(t) \in V$ such that

$$\int_0^T e^{-R(t)} x(t)dt = 0,$$

and define $c_s(t)$ as follows:

$$c_s(t) = \begin{cases} c^*(t) + sx(t) & \text{if } 0 \leq t \leq T, \\ c^*(t) & \text{if } t \geq T. \end{cases}$$

Because $c^*(t)$ is an inner solution, there exists $\delta > 0$ such that if $|s| < \delta$, then $c_s(t) \geq 0$ for all $t \geq 0$. Define

$$\varphi(s) = \int_0^T e^{-\rho t} u(c_s(t))dt.$$

By the same derivation as (7), we have that

$$0 = \varphi'(0) = \int_0^T e^{-\rho t} u'(c^*(t))x(t)dt.$$

Define

$$\Lambda_0(x(t)) = \int_0^T e^{-\rho t} u'(c^*(t))x(t)dt,$$

$$\Lambda_1(x(t)) = \int_0^T e^{-R(t)} x(t)dt.$$

The above arguments say that Ker $\Lambda_1 \subset$ Ker Λ_0. By Lemma 1, there exists $a \in \mathbb{R}$ such that $\Lambda_0 = a\Lambda_1$. By Lemma 2,

$$e^{-\rho t} u'(c^*(t)) = ae^{-R(t)}.$$

Because the left-hand side is positive, we have that $a > 0$, and

$$u'(c^*(t)) = ae^{\rho t - R(t)}.$$

Since the right-hand side is continuously differentiable, we have that $u'(c^*(t))$ is also continuously differentiable. Moreover,

$$\log u'(c^*(t)) = \log a + \rho t - R(t),$$

and differentiating both sides, for every $t \in [0, T]$,

$$\frac{d}{dt}(u' \circ c^*)(t) = (\rho - r(t))(u' \circ c^*)(t),$$

where this equation is the same as (11). Because T is arbitrary, we have that $c^*(t)$ is a solution to the Euler equation. This completes the proof. ■

5. CONCLUSION

This paper has mainly considered the derivation of the Euler equation in a classical continuous-time optimal growth model. Additionally, we have provided a proof of the fact that the Euler equation and transversality condition are sufficient conditions for the optimal solution in concave problems.

In this section, we discuss some matters that are necessary for further study. First, the Euler equation and transversality condition are often treated as "necessary and sufficient conditions" for the solution to a problem. In this paper, however, we have dealt with the fact that "the Euler equation is a necessary condition for the solution" and that "the Euler equation and transversality condition are sufficient conditions for the solution," but we have not discussed whether "transversality condition is a necessary condition for the solution" at all. This was discussed in detail in Kamihigashi (2001), and it seems that transversality condition is a necessary condition for the solution in most cases. However, we could not put this result into an elementary form. In particular, it is very difficult to discuss the above papers while maintaining the continuity requirement of $c(t)$. For this reason, we have decided not to mention it at all in this paper.

Next, there is a problem related to the relaxation of the continuity of $c(t)$ discussed above. In fact, macroeconomic models often allow for the possibility of "jumps" in $c(t)$. In that case, the continuity requirement of $c(t)$ is replaced by that of piecewise continuity or local integrability. Then, the solution to the capital accumulation equation (2) is not necessarily continuously differentiable, and thus the requirement of $k(t)$ changes to piecewise continuous differentiability or absolute continuity. In this case, we inevitably need the dominated convergence theorem to derive the Leibniz integral rule. Additionally, the continuity of $b(t)$ in du Bois-Reymond's lemma is no longer available, and hence other arguments are needed. In any case, without a basic knowledge of Lebesgue integrals, it will be difficult to go further.

Third, there is a question of whether the quirky requirement that "$\int_0^\infty e^{-\rho t} u(c(t))dt$ is defined" can be removed. Actually, this requirement can be removed by considering a slightly different kind of optimality problem called overtaking optimality, and our results can be reproduced by almost the same argument. See Carlson et al. (2011) for detailed arguments. As a matter of fact, Kamihigashi (2001) discusses the transversality condition in this context.

Fourth, there is a problem of corner solutions. In this paper, we have dealt with the Euler equation as a condition for "inner solutions." This paper has nothing to say about corner solutions, i.e., solutions for which $c(t) = 0$ is possible. In this regard, Hosoya (2019) dealt with the requirements of u for excluding corner solutions. However, even in this manuscript, this problem was only partially solved, and it is very difficult to eliminate corner solutions in general. Note that, since the derivation of equation (7) is not possible for corner solutions, the Euler equation is not a necessary condition for such solutions.

Finally, this paper has only dealt with classical optimal growth models, and some readers may have concerns that the same approach may not work for slightly different models that are not classical. For this, we assure you that similar techniques can be used for a number of problems. As an example, the appendix presents a problem in which the objective function u has two variables and there are multiple constraints of the integral equations, and derives the Euler equations by almost the same method as in the proof of Theorem 3. Although this problem is somewhat artificial, it serves as an example of how to apply our techniques to non-classical problems.

A. APPENDIX: MULTIPLE RESTRICTIONS

For proving du Bois-Reymond's lemma, Lemma 1 is sufficient. However, in macroeconomic model, there are problems in which Lemma 1 is insufficient for deriving the Euler equation. In this appendix, we provide a method for solving such problems.

Lemma 3. Suppose that V is a vector space, and let $f_0, f_1, ..., f_n$ be the family of linear functionals on V. Define Ker $f_i = \{v \in V | f_i(v) = 0\}$. Then, the following two statements are equivalent.

1) $f_0 = a_1 f_1 + ... + a_n f_n$ for some $a_1, ..., a_n \in \mathbb{R}$.
2) $\cap_{i=1}^{n}\text{Ker } f_i \subset \text{Ker } f_0$.

Proof. Clearly, 1) implies 2). Therefore, it suffices to show that 2) implies 1). We use mathematical induction on n. The case in which $n = 1$ is just Lemma 1.

Next, choose any $n \geq 2$, and suppose that for $n - 1$, this lemma is correct. If $\cap_{i=1}^{n-1}\text{Ker } f_i \subset \text{Ker } f_0$, then by assumption,

$$f_0 = a_1 f_1 + ... + a_{n-1} f_{n-1} + 0 f_n,$$

and 1) holds. Hence, we assume that $\cap_{i=1}^{n-1}\text{Ker } f_i \not\subset \text{Ker } f_0$. Thus, there exists $v^* \in \cap_{i=1}^{n-1}\text{Ker } f_i$ such that $f_0(v^*) = 1$. Because $\cap_{i=1}^{n}\text{Ker } f_i \subset \text{Ker } f_0$, we have that $f_n(v^*) \neq 0$. Define $b_1 = f_n(v^*)$ and $a_n = b_1^{-1}$. By induction hypothesis, it suffices to show that

$$\cap_{i=1}^{n-1}\text{Ker } f_i \subset \text{Ker } (f_0 - a_n f_n).$$

Choose any $v \in \cap_{i=1}^{n-1}\text{Ker } f_i$. It suffices to show that $f_0(v) = a_n f_n(v)$. If $f_n(v) = 0$, then $f_0(v) = 0$ by 2), and thus $f_0(v) = a_n f_n(v)$. If $f_n(v) = b_2 \neq 0$, then $f_n(v^* - b_1 b_2^{-1} v) = 0$, and thus by 2), $f_0(v^* - b_1 b_2^{-1} v) = 0$. Therefore,

$$1 = f_0(v^*) = b_1 b_2^{-1} f_0(v),$$

and thus,

$$f_0(v) = a_n b_2 = a_n f_n(v),$$

as desired. This completes the proof. ■

We now consider an example in which Lemma 3 can be effectively applied. Consider the following problem.

$$\begin{aligned} \max \quad & \int_0^\infty e^{-\rho t} u(c(t), a(t)) dt, \\ \text{subject to.} \quad & c(t) \ge 0, a(t) \ge 0, \\ & c(t), a(t) \text{ are continuous}, \\ & \int_0^\infty e^{-\rho t} u(c(t), a(t)) dt \text{ is defined}, \\ & \int_0^\infty e^{-R_1(t)} c(t) dt = M, \\ & \int_0^\infty e^{-R_2(t)} a(t) dt = N. \end{aligned} \tag{12}$$

We assume that $u(c, a)$ is continuous on $\mathbb{R}^2_+$ and continuously differentiable on $\mathbb{R}^2_{++}$, and that $R_i(t) = \int_0^t r_i(\tau) d\tau$, where $r_i(t)$ is continuous and positive for all $t \ge 0$. Suppose that $(c^*(t), a^*(t))$ is a solution to (12) such that $c^*(t) > 0$ and $a^*(t) > 0$ for all $t \ge 0$. Now, fix any $T > 0$, and let V be the set of all pairs of continuous functions $(x, y) : [0, T] \to \mathbb{R}^2$ such that $x(0) = x(T) = y(0) = y(T) = 0$. Choose any $(x(t), y(t)) \in V$ such that $\int_0^T e^{-R_1(t)} x(t) dt = 0$ and $\int_0^T e^{-R_2(t)} y(t) dt = 0$. Let

$$g(s) = \int_0^T e^{-\rho t} u(c^*(t) + sx(t), a^*(t) + sy(t)) dt,$$
$$\frac{\partial u}{\partial c}(c^*(t), a^*(t)) = u_c(t), \ \frac{\partial u}{\partial a}(c^*(t), a^*(t)) = u_a(t).$$

Then, by the same arguments as in the proof of Theorem 3, we have that

$$0 = g'(0) = \int_0^T e^{-\rho t} [u_c(t) x(t) + u_a(t) y(t)] dt.$$

Define

$$\Lambda_0(x(t), y(t)) = \int_0^T e^{-\rho t} [u_c(t) x(t) + u_a(t) y(t)] dt,$$
$$\Lambda_1(x(t), y(t)) = \int_0^T e^{-R_1(t)} x(t) dt,$$
$$\Lambda_2(x(t), y(t)) = \int_0^T e^{-R_2(t)} y(t) dt.$$

Then, Lemma 3 implies that $\Lambda_0 = a_1 \Lambda_1 + a_2 \Lambda_2$. If $y(t) \equiv 0$, then

$$\int_0^T e^{-\rho t} u_c(t) x(t) dt = \int_0^T a_1 e^{-R_1(t)} x(t) dt$$

for all continuous function $x : [0, T] \to \mathbb{R}$ such that $x(0) = x(T) = 0$. By Lemma 2, we have that $e^{-\rho t} u_c(t) = a_1 e^{-R_1(t)}$ for all t. By the same reason, we obtain that $e^{-\rho t} u_a(t) = a_2 e^{-R_2(t)}$. To differentiate these equations, we obtain that

$$u_c'(t) = (\rho - r_1(t)) u_c(t), \ u_a'(t) = (\rho - r_2(t)) u_a(t),$$

which is the Euler equation in this model.

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