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COMPLETE CHARACTERIZATION OF DOMAINS OF CHOIC FUNCTIONS FOR RATIONALIZABILITY

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Abstract: In the literature related to choice theory an important problem dealt at length is the rationalizability of choice function of an individual. In the literature a number of choice consistency conditions have been postulated, which guarantee best element and maximal element rationalizability of choice functions. In this paper a set of necessary and sufficient conditions have been derived for the domain to be such that every possible choice function defined over the domain has a best element (maximal element) rationalization. Thus the paper provides complete characterization of partition of domains separately for best and maximal element rationalizable choice functions.

Key words: Rational choice, choice function, domain condition, rationalizability.

JEL Classification Number: D71, D11.

1. INTRODUCTION

In economics the concept of rational choice appears very prominently. It is generally assumed that individuals are rational. By rationality it is meant that individual choice behaviour conforms to two requirements. First requires that individual behaviour is purposive and second requires it to be consistent[1].

The analysis of rational choice behaviour is done using two broad frameworks. In the first framework it is analysed with the help of a well-defined preference relation[2] and individuals are assumed to choose a best alternative whenever the set of best alternatives is non-empty.

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[1] Purposive behaviour requires a rational agent to choose those alternatives that bring him to the best attainable position and consistency requires that the preference ("at least as good as") relation of a rational agent satisfies the strong consistency property to the effect that if alternative x is no less preferred to alternative y, and y to z, then x is no less preferred to z. (Suzumura (1983); Chapter-2, p.19).

[2] By well-defined preference we mean that the preference relation is an ordering.

In the second framework based on revealed preference theory individual choice behaviour is analysed only on the basis of observed choices. This approach of revealed preference theory helps analyse rational choice in the context of set-valued choice function. It is asked in this context whether it is possible to construct a preference relation observing choices under different environments, such that the chosen elements of a set are the same elements as the set of best elements (or no worse elements i.e., maximal elements)[3] of the set according to that preference relation. In the literature of choice theory this problem is known as the rationalizability of choice function. A choice function is said to be best-element (maximal element) rationalizable when the best (maximal) elements according to some preference relation are only chosen from the available sets. This definition of rationality, likewise, reflects the purposive behaviour of an individual and requires individuals to be consistent in their choice behaviours.

The problem of best element rationalizability can be explained with the help of the following example. Suppose we observe the following choices:

EXAMPLE 1. Let X be the set of alternatives and C be the choice function.

$$X = \{x, y, z\};\ C(\{x, y\}) = \{x\}, \quad C(\{x, z\}) = \{x\},$$
$$C(\{z, y\}) = \{y\}, C(\{x, y, z\}) = \{x\}$$

In the above choice environments x is chosen from $\{x, y\}$, $\{x, z\}$ and $\{x, y, z\}$, and y is chosen from $\{z, y\}$. It is immediate that the preference relation $xPyPz$ i.e., 'x is preferred to y preferred to z' represents such choice behaviour. x is the best element of $\{x, y\}$, $\{x, z\}$ and $\{x, y, z\}$ according to the preference relation $xPyPz$ and it is the chosen element of the same sets as well. y is also the best element of $\{z, y\}$ and chosen element too. Similar example also can be constructed to illustrate maximal element rationalizability[4].

The notion of rational choice, however, has been improvised further in the literature to capture different aspect of choice. Gaertner and Xu (2004) tries to incorporate the procedural aspect of choice where the available alternatives are linked to a procedure by which they came into existence. Manzini and Mariotti (2007), Hung Au and Kawai (2011) consider an environment where choices are made sequentially. Under such consideration a decision maker uses more than one preference relations in a fixed order to remove non-preferred alternatives. This procedure sequentially rationalizes the choice function of the decision maker if a unique choice is made for every set belonging to the

[3] Formal definitions of best and maximal elements have been provided in section 2.

[4] Consider the following example: $X = \{x, y, z\}; C(\{x, y\}) = \{x\},\ C(\{x, z\}) = \{x\}, C(\{x, y, z\}) = \{x\}$. We claim that this choice function is M-rationalizable.
Let the preference relation R be : xPy and xPz. Notice, maximal element in $\{x, y\}$ with respect to the above relation is x as no other element in $\{x, y\}$ is preferred to x. y is not maximal according to R as x is preferred to y. Thus $M(\{x, y\}, R) = \{x\}$. Further, for every set, the set of maximal elements with respect to the given preference relation is same as the set of chosen elements:
$M(\{x, y\}, R) = \{x\} = C(\{x, y\})$
$M(\{x, z\}, R) = \{x\} = C(\{x, z\})$
$M(\{x, y, z\}, R) = \{x\} = C(\{x, y, z\})$.

domain. Apesteguia and Ballester (2013) also considers choices by sequential procedure wherein a decision maker makes a choice by ruling out inferior alternatives through binary comparisons in a particular order.

It is worth mentioning in this context that all choice functions are not rationalizable[5]. A number of choice consistency conditions have been introduced in the literature pertaining to different notions of rationalizability owing to Uzawa (1957), Richter (1966), Suzumura (1976, 1983), Arrow (1959), Sen (1970), Bossert et al. (2005), Manzini and Mariotti (2007) and Hung Au and Kawai (2011) and others.

The development of literature in the context of rationalizability of choice functions, as evident from preceding discussion, relates to two classes of choice functions, namely rationalizable and non-rationalizable choice functions. Given this classification of choice functions, the literature investigates the properties of choice functions which make them rationalizable. A number of choice consistency conditions thus have emerged. However, the nature of these consistency conditions is such that they put restrictions on the choice behaviour of an individual and their implications with regard to rationalizability also change as domain of choice function changes[6].

Therefore, a pertinent question arises: what property should a domain satisfy such that any choice function defined over that domain would be rationalizable? This question has very important implication in the context of rational choice. If it is the case that over a domain all choice functions are rationalizable i.e., no matter in whichever way an individual makes his or her choice it always becomes rational then it seems difficult to find any meaningful interpretation of 'purposive behaviour' of an individual in that particular domain, which, as discussed before, is at the core of the notion of rational choice. It is, therefore, important to find a complete characterization of such domains. On one side there is the class of domains over which any choice function is rationalizable and on the other side there is the class of domains over which not all choice functions are rationalizable. The 'purposive behaviour' of an individual would carry meaningful sense in those domains where some choice patterns are rational and some are not. Furthermore, choice consistency conditions which deal with the properties of choice functions restrict choice behaviours of individuals. On the contrary, domain conditions do not

5 Consider following example: $X = \{x, y, z\}$; $C(\{x, y\}) = \{x\}$; $C(\{x, z\}) = \{z\}$; $C(\{z, y\}) = \{y\}$; $C(\{x, y, z\}) = \{x\}$.
For this choice function C, x is the only element chosen from $\{x, y, z\}$. So any preference relation which rationalizes this choice function must have the following preference: 'x is at least as good as x', 'x is at least as good as y' and 'x is at least as good as z'. This implies that x belongs to the set of best elements of the set $\{x, z\}$ according to the said preference relation; but x does not belong to the choice set of $\{x, z\}$. z is the only element chosen from the set $\{x, z\}$. Therefore the choice function C is not best element rationalizable. Likewise, this choice function is not maximal element rationalizable. Notice, $\{z\} = C(\{x, z\})$ implies zPx but $x \in C(\{x, y, z\})$.

6 Take Arrow's Axiom (Arrow (1959)) for instance, under full domain (a collection of *all* nonempty finite subsets of the set of alternatives) it is necessary and sufficient for a choice function to have an ordering rationalization. When domain of choice function becomes general (a nonempty collection of nonempty subsets of the set of alternatives) Arrow's Axiom fails to be sufficient for ordering rationalization.

impose any such restrictions[7].

In this paper we shall introduce a set of domain conditions-$C.1$, $C.2$ and show that $C.1$ is necessary and sufficient for a domain over which all choice functions have best element rationalization, which in turn provides complete characterization of domain for best element rationalizability and $C.1$, $C.2$ together is necessary and sufficient for a domain over which all choice functions have maximal element rationalization, which in turn provides complete characterization of domain for maximal element rationalizability.

This paper is divided into four sections. Second section contains basic notations and definitions which have been used in the subsequent sections. Sections three and four provide the characterization results. Section five concludes the paper.

2. NOTATIONS AND DEFINITIONS

Let X be a non-empty finite set of alternatives and 2^X be the power set of X. For a set S, $\#S$ denotes the cardinality of the set S. Let D be a nonempty collection of nonempty subsets of X, $D \subseteq 2^X - \{\emptyset\}$. A choice function C is a mapping from D to $2^X - \{\emptyset\}$, $C : D \mapsto 2^X - \{\emptyset\}$ such that $C(S) \subseteq S$ for all $S \in D$. In succeeding sections we denote D to be the domain of choice function. For any statement 'A', '$\sim A$' denotes negation of the statement. For statements A and B, '$A \wedge B$' and '$A \vee B$' denote conjunction (and) and disjunction (or) of two statements respectively.

Let R be a binary relation defined over X. We would often express $(x, y) \in R$ as xRy and R may be interpreted as 'at least as good as' relation. xRy, therefore, may be read as 'x is at least as good as y'. Let $I(R)$ and $P(R)$ denote symmetric and asymmetric parts of R respectively.
$(\forall x, y \in S)(xI(R)y \leftrightarrow xRy \wedge yRx)$
$(\forall x, y \in S)(xP(R)y \leftrightarrow xRy \wedge \sim yRx)$.
Given the interpretation of R, $xI(R)y$ and $xP(R)y$ may be read as 'x is indifferent to y' and 'x is preferred to y' respectively.
R defined on S is said to be asymmetric *iff* $(\forall\, x, y \in S)(xRy \rightarrow \sim yRx)$.

Given a choice function C on D, define binary relation R_c

$$R_c = \{(x, y) \in X \times X |\ (\exists S \in D)(x \in C(S) \wedge y \in S)\}\,.$$

x is said to be a greatest (best) element in a set S with respect to a binary relation R *iff* $(\forall y \in S)(xRy)$ i.e., x is best in S if and only if it is at least as good as all the elements in S. Let $G(S, R)$ denote the set of greatest (best) elements of a set S with respect to R.

We say that a choice function C is greatest (best) element rationalizable (henceforth G-rationalization) *iff* there exists a binary relation R defined over the set of alternatives such that for every set in the domain the choice set is equal to the set of greatest elements of the set with respect to R, i.e.,

[7] See Pal (2017) for a discussion on the implication of domain conditions for rationalizability of choice function.

$$(\exists R \subseteq X \times X)(\forall\, S \in D)(C(S) = G(S, R))\,.$$

EXAMPLE 2. Let $X = \{x_o, x_1, x_2\}$;$D = \{\{x_o, x_2\}, \{x_o, x_1\}, \{x_o, x_1, x_2\}\}$.Let $S_1 = \{x_o, x_2\}$, $S_2 = \{x_o, x_1\}$, $S_3 = \{x_o, x_1, x_2\}$. Let C be a choice function defined over D in the following way: $C(S_1) = \{x_o\}$, $C(S_2) = \{x_1\}$, $C(S_3) = \{x_1\}$. Now consider the following binary relation:

$$R = \{(x_o, x_o), (x_1, x_1), (x_1, x_o), (x_1, x_2), (x_o, x_2)\}$$
$$G(S_1, R) = \{x_o\} = C(S_1)$$
$$G(S_2, R) = \{x_1\} = C(S_2)$$
$$G(S_3, R) = \{x_1\} = C(S_3)\,.$$

So the choice function C is G-rationalizable.

x is said to be a maximal element of S with respect to R *iff* $(\forall y \in S)(\sim yPx)$ i.e., x is maximal element in S if and only if there is no element in S which is preferred to x. Let $M(S, R)$ denote the set of maximal elements of the set S with respect to binary relation R.

A choice function is maximal-element rationalizable (henceforth M-rationalization) *iff* there exists a binary relation R defined over the set of alternatives such that for every set in the domain choice set is equal to the set of maximal elements of that set with respect to binary relation R, i.e.,

$$(\exists R \subseteq X \times X)(\forall\, S \in D)(C(S) = M(S, R))$$

EXAMPLE 3. Let $X = \{x_o, x_1, x_2\}$; $D = \{\{x_2\}, \{x_o, x_1\}, \{x_o, x_1, x_2\}\}$. Let $S_1 = \{x_2\}$, $S_2 = \{x_o, x_1\}$, $S_3 = \{x_o, x_1, x_2\}$. Let C be a choice function defined over D in the following way: $C(S_1) = \{x_2\}$, $C(S_2) = \{x_o\}$, $C(S_3) = \{x_o\}$. Now consider the following binary relation:

$$R = \{(x_o, x_1), (x_o, x_2)\}$$
$$M(S_1, R) = \{x_2\} = C(S_1)$$
$$M(S_2, R) = \{x_o\} = C(S_2)$$
$$M(S_3, R) = \{x_o\} = C(S_3)\,.$$

So the choice function C is M-rationalizable.

We introduce following notation:

Define D_X as follows:

$$D_X = \{\ \{x\} \mid x \in X\};$$
$$D_{2X} = \{\ S \mid S \subseteq X \wedge 1 \leq \#S \leq 2\}\,.$$

D_{2X} is the collection of singleton and doubleton sets. Naturally, $D_X \subseteq D_{2X}$.

3. G-RATIONALIZABILITY AND DOMAIN CONDITION

We introduce domain condition $C.1$ which we prove to be necessary and sufficient for a domain over which every choice function has a G-rationalization.

$C.1$: $\forall S \in D - D_X$, $\forall x \in S$, it should not be the case that $\forall y \in S - \{x\}, \exists S' \in D - \{S\}$ such that $\{x, y\} \subseteq S'$.

Condition $C.1$ requires that for any set S having at least two alternatives and any element x belonging to S it is not the case that for every element y in S, distinct from x, there exists a set S' different from S such that x, y belong to that set S'. The intuition behind the condition is as follows. If condition $C.1$ is violated i.e., there exists a set S and an element x belonging to S and for every element y not equal to x there exists a set S' different from S such that x, y belong to that set S', then it is possible to construct a choice function which would induce R_c according to which x would be a best element in S but x would not belong to the choice set of S. Condition $C.1$ prevents such cases.

An example may help illustrate this condition. Consider a domain- $D = \{\{x, y, z, w\}, \{x, y, w\}, \{x, z\}\}$. Domain D clearly violates condition $C.1$ as there exist a set $\{x, y, z, w\}$ in D and an element $x \in \{x, y, z, w\}$ such that x has association with rest of the elements of the sets in other sets belonging to the domain, namely- $\{x, y, w\}$ and $\{x, z\}$. Given such domain it is easy to construct a choice function which is not rationaliazable. Let $C(\{x, y, z, w\}) = \{y\}, C(\{x, y, w\}) = \{x\}, C(\{x, z\}) = \{x\}$. For this choice function, x is at least as good as x, y, z and w as x is chosen from $\{x, y, w\}$ and $\{x, z\}$ but x is not a chosen element in $\{x, y, z, w\}$. Hence the choice function is not rationalizable. Following theorem shows the necessity and sufficiency of conditon $C.1$.

THEOREM 1. *Every choice function defined over D is G-rationalizable iff D satisfies condition* $C.1$.

Proof. Suppose D violates the condition $C.1$, i.e.,

$$\exists S_1 \in D - D_X, \exists x \in S_1 \text{ such that } \forall y \in S_1 - \{x\},$$
$$\text{there exists } S' \in D - \{S_1\} \text{ such that } \{x, y\} \subseteq S'. \tag{1}$$

Let $S_1 = \{x, y_2, y_3, \ldots, y_n\}$. (1) $\rightarrow$ $(\exists S_2, S_3, S_4, \ldots, S_n \in D)(\{x, y_2\} \subseteq S_2$ and $\{x, y_3\} \subseteq S_3$ and $\{x, y_4\} \subseteq S_4$ and ... and $\{x, y_n\} \subseteq S_n)$. Note that $S_2, S_3, S_4, \ldots, S_n$ are not necessarily distinct.

Now, consider the following choice function:

$$\begin{aligned} \hat{C}(S_1) &= \{y_2\} \\ \hat{C}(S_2) &= \{x\} \\ \hat{C}(S_3) &= \{x\} \\ &\vdots \\ \hat{C}(S_n) &= \{x\} \quad and \end{aligned}$$
$$(\forall S \in D - \{S_1, S_2, \ldots, S_n\})(C(S) = S)$$

This choice function defined on D is not rationalizable. Since x belongs to the choice set of each of the sets $S_2, S_3, \ldots, S_n$ and $y_2, y_3, \ldots, y_n$ belong to $S_2, S_3, \ldots, S_n$ respectively, any preference relation R which rationalizes the choice function C must have the following preference: $xRx, xRy_2, xRy_3, \ldots, xRy_n$. This implies that x belongs to the set of best elements of the set S_1; but x does not belong to the choice set of S_1. Therefore, this choice function C is not rationalizable.

Let D satisfy condition $C.1$. We show that R_c rationalizes every choice function defined over D. Let C be any choice function defined over D. We Show: $C(S) = G(S, R_c)$

Let $x \in C(S)$

$$\rightarrow (\forall y \in S)(xR_c y)$$
$$\rightarrow x \in G(S, R_c)$$

Let $x \in G(S, R_c)$. Suppose $x \notin C(S)$

$$\rightarrow (\exists y \in S)(y \in C(S))$$
$$\rightarrow \{x, y\} \subseteq S$$
$$\rightarrow S \text{ contains atleast two distinct elements.} \tag{2}$$

Let $S = \{x, y, x_3, \ldots, x_n\}$

$$\begin{aligned} & x \in G(S, R_c) \rightarrow (\forall w \in S)(xR_c w) \\ & \rightarrow (\exists S_2, S_3, S_4, \ldots, S_n \in D)^{8}[(x \in C(S_2) \text{ and } y \in S_2) \\ & \text{and } (x \in C(S_3) \text{ and } x_3 \in S_3) \\ & \text{and } \ldots \text{ and } (x \in C(S_n) \text{ and } x_n \in S_n)] \\ & \rightarrow (\exists S_2, S_3, S_4, \ldots, S_n \in D)(\{x, y\} \subseteq S_2 \text{ and } \{x, x_3\} \subseteq S_3 \\ & \text{and } \{x, x_4\} \subseteq S_4 \text{ and } \ldots \text{ and } \{x, x_n\} \subseteq S_n) \end{aligned} \tag{3}$$

It is clear that $S_2, S_3, S_4, \ldots, S_n$ are distinct from S.

(2) and (3) imply violation of condition $C.1$.

$$\therefore x \in C(S).$$

4. M-RATIONALIZABILITY AND DOMAIN CONDITION

The notion of maximal element rationalization is important in the context of rational choice. It is argued that rational behavior which mostly represents the maximizing behavior of an individual does not necessarily correspond to choosing best elements of a set[9]. In fact, the general discipline of maximization does not necessarily invoke the concept of choosing best element always; it only requires choice set to be the set of alternatives which are no worse than others, that is precisely the set of maximal elements. Sometimes interpreting rational choice by choosing best elements always

[8] $S_2, S_3, S_4, \ldots, S_n$ are not necessarily distinct.

[9] See: Sen (1997), Sugden (1985), Schwartz (1972).

may run into serious decision problem[10].

Likewise, it may well happen that because of limited information available on some alternatives or due to 'unsolved value conflict' among some alternatives a preference relation over the set of alternatives turns out to be unconnected. Under such circumstances best element may not exist in some sets with respect to that preference relation. Hence, representation of maximizing behaviour of an individual by choosing the set of best elements may face a serious drawback. Choice of maximal elements, on the other hand, may carry meaningful sense in some circumstances where the absence of best elements fails to represent the maximizing behaviour of an individual.

Below we provide a characterization of domains for M-rationalizability. We introduce condition $C.2$ and prove that $C.2$ and $C.1$ together is necessary and sufficient for a domain over which every choice function has a M-rationalization. Henceforth in this section rationalizability would mean M-rationalizability.

$C.2$: $\forall S \in D - D_{2X}, \forall T \subset S$, it should not be the case that:
(i) $\#(S - T) \geq 2$ and,
(ii) $\exists x \in S - T$ such that: $[\forall k \in S - T, \exists V \in D - \{S\}(\{x, k\} \subseteq V)$ and $\forall z \in T, \forall w \in S - \{x\}, \exists B \in D - \{S\}(\{w, z\} \subseteq B)]$

Condition $C.2$ requires that for any set S having at least three alternatives and any subset of S, T (say), it is not the case that $S - T$ would have at least two elements and there exists an element x in $S - T$ such that for every element k in $S - T$ there exists a set V distinct from S, which contains x, k; and for any element z, w in T and S respectively, distinct from x, there exists a set B different from S such that it contains z, w.

An example may help understand the condition $C.2$. Consider a domain $D = \{\{x, y, z, w\}, \{z, w\}, \{x, y, w\}\}$. This domain clearly violates condition $C.2$. Supposing $S = \{x, y, z, w\}$ and $T = \{x, y\}$ there exists an element $z \in S - T$ such that $\{z, w\} \in D$ and for every element in T and every element in $S - \{z\}$ there exists a set in the domain containing both the elements namely- $\{x, y, w\}$.

Given that D violates $C.2$ it is now easy to construct a choice function that is not rationalizable. Let $C(\{x, y, z, w\}) = \{w\}$, $C(\{x, y, w\}) = \{x, y, w\}$, $C(\{z, w\}) = \{z\}$. Any preference relation that rationalizes this choice function must display w not to be preferred to z as $C(\{z, w\}) = \{z\}$. Likewise due to $C(\{x, y, w\}) = \{x, y, w\}$ no element from $\{x, y, w\}$ is preferred over other in the set. It, therefore, follows that in the set $\{x, y, z, w\}$ z has to be preferred to x and y in order to generate the choice set $C(\{x, y, z, w\}) = \{w\}$. Thus, in the set $\{x, y, z, w\}$ no element is preferred to z. Hence z should belong to $C(\{x, y, z, w\})$ but it does not. Above choice function, therefore, is not rationalizable.

It is to be noted that in the context of M-rationalizability if an element in a set is not chosen then there has to exist a different element in the same set which is preferred

[10] The following quotation from Sen (1997) reflects such concern.
"... the tale of the donkey that dithered so long in deciding which of the two haystacks x or y was better, that it died of starvation."

to that element i.e., the former is defeated in preference by the latter. For a choice function to be M-raionalizable it is, therefore, necessary that every unchosen element has an association with some other element in the set that defeats it. Furthermore, if an element defeats other element in a set then under no circumstances the defeated element should be chosen in presence of the element that defeats it. This association can be expressed by a single valued function defined below.

$$\text{For } S \in D \text{ and } (S - C(S) \neq \emptyset), \text{ define: } f_S : S - C(S) \mapsto S,$$
$$\text{such that } (\forall (x, y) \in f_S)(\sim xR_c y \wedge x \neq y)$$

$(x, y) \in f_S$ i.e., $f_S(x) = y$[11] may be interpreted as y defeats x. Notice, $\sim xR_c y$ ensures that the defeated element x is not chosen in presence of y.

Let $D = \{S_1, S_2, S_3, \dots, S_k\}$ and $\bar{D} \subseteq D$ contain all the sets from D such that for every set, S, $C(S) \neq S$. Let $\bar{D} = \{S_1, S_2, S_3, \dots, S_n\}$; clearly, $(\forall S \in \bar{D})(S - C(S) \neq \emptyset)$. Let F be a class of functions defined above with respect to $\bar{D}$:

$$F = \{f_{S_1}, f_{S_2}, f_{S_3}, \dots, f_{S_n}\}$$

Define a binary relation R'

$$R' = \{(x, y) \in X \times X \mid \sim (xR_c y \vee yR_c x) \wedge (\exists S \in D)(f_S(y) = x) \wedge (\forall T \in D)(x, y \in T \rightarrow f_T(x) \neq y)\}$$

Now consider,

$$\bar{R} = R_c \cup R'$$

Above definitions are illustrated with an example below:

EXAMPLE 4. Let $D = \{\{x, y, z, w\}, \{z, w\}, \{x, y\}\}$.
$C(\{x, y, z, w\}) = \{w\}$, $C(\{x, y\}) = \{x\}$, $C(\{z, w\}) = \{z, w\}$.
$\bar{D} = \{\{x, y, z, w\}, \{x, y\}\}$.
Let $S_1 = \{x, y, z, w\}$, $S_2 = \{x, y\}$ and $S_3 = \{z, w\}$.
$R_c = \{(x, x), (w, w), (z, z), (x, y), (z, w), (w, x), (w, y), (w, z)\}$.
$F = \{f_{S_1}, f_{S_2}\}$.
Let, $f_{S_1} : f_{S_1}(x) = w$, $f_{S_1}(y) = x$, $f_{S_1}(z) = y$
$f_{S_2} : f_{S_2}(y) = x$
$R' = \{(y, z)\}$
$\bar{R} = \{(x, x), (w, w), (z, z), (x, y), (z, w), (w, x), (w, y), (w, z), (y, z)\}$.

PROPOSITION 1. *Conditions $C.1$ and $C.2$ are independent and none of them alone is sufficient for a domain to ensure rationalizability for all choice functions.*

Proof. Consider two domains of choice function namely D_1 and D_2 given below. Let $D_1 = \{\{x, y\}, \{x, y, z\}\}$ and $D_2 = \{\{x, y, z, w\}, \{y, z, a\}, \{y, w, b\}, \{x, z, c\}, \{x, w, d\}\}$. It can be verified, while D_1 satisfies $C.2$ and violates $C.1$, D_2 satisfies $C.1$ and violates $C.2$. In D_1, for $x \in \{x, y\}$, x, y belong to $\{x, y, z\}$, thus it violates $C.1$. To show that $C.2$ is satisfied, consider $T \subset \{x, y, z\}$ such that $\#(\{x, y, z\} - T) \geq 2$.

[11] Since any function can be written in form of a binary relation, $(x, y) \in f_S$ and $f_S(x) = y$, therefore, have the same interpretation.

T, therefore, can be $\{x\}$, $\{y\}$, $\{z\}$ or $\emptyset$. It can be easily checked that for any T, the requirements of $C.2$ are satisfied. To show D_2 violates $C.2$ consider $S = \{x, y, z, w\}$ and $T = \{x\}$. Now, for $y \in S - T = \{y, z, w\}$ there exist sets in the domain namely $\{y, z, a\}$,$\{y, w, b\}$ where $y, z \in \{y, z, a\}$ and $y, w \in \{y, w, b\}$. Again for every element in T and every element in $S - \{y\}$ there exists a set in the domain containing both the elements namely-$\{x, z, c\}$, $\{x, w, d\}$. This is a violation of condition $C.2$. Additionally, it can be verified that D_2 satisfies $C.1$.

Notice, none of the domains ensure that all choice functions defined over it are rationalizable. Define choice functions $\hat{C}$ and C on D_1 and D_2 respectively as follows:

$$\hat{C}(\{x, y, z\}) = \{x, y\}, \hat{C}(\{x, y\}) = \{y\}; \text{ and}$$
$$C(\{x, y, z, w\}) = \{z\};\ C(\{y, z, a\}) = \{y, z\};$$
$$C(\{y, w, b\}) = \{y, w\};\ C(\{x, z, c\}) = \{x, z\};\ C(\{x, w, d\}) = \{x, w\}$$

Choice functions $\hat{C}$ and C are not rationalizable. For choice function $\hat{C}$, $\hat{C}(\{x, y, z\}) = \{x, y\}$ implies that y is not preferred to x. Thus, x should belong to $\hat{C}\{x, y\}$ but it does not. Again, for the choice function C, $C(\{y, z, a\}) = \{y, z\}$ and $C(\{y, w, b\}) = \{y, w\}$ imply that z, w are not preferred to y. $C(\{x, z, c\}) = \{x, z\}$, $C(\{x, w, d\}) = \{x, w\}$ imply w, z is not preferred to x. Notice, $x, y \notin C(\{x, y, z, w\})$ as $C(\{x, y, z, w\}) = \{z\}$ and w, z are not preferred to x and y. This implies that for choice function C to be rationalizable there have to be some elements which are preferred to x and y respectively. If x is preferred to y then there is no element in $\{x, y, z, w\}$ that is preferred to x, which in turn imply that x should belong to $C(\{x, y, z, w\})$ but it does not. So is true for y. Both the choice functions, therefore, are not rationalizable. It follows that neither $C.1$ nor $C.2$ is sufficient to guarantee rationalizability of all choice functions.

We now show that conditions $C.1$ and $C.2$ together is necessary and sufficient for domain to be such that all choice functions defined over it are rationalizable.

LEMMA 1. *Let choice function C be defined over D. Let $\exists S \in D$ and $C(S) \neq S$. If D satisfies condition $C.1$ then there exists a class of functions F.*

Proof. Let choice function C be defined over D. Let $\exists S \in D$ and $C(S) \neq S$.
Suppose, there does not exist any F. This implies the following,
$\exists S \in D, \exists x \in S,$ such that $x \in S - C(S)$ and $(\forall y \in S - \{x\})(x R_c y)$.
This again implies by definition of R_c that- $\forall y \in S - \{x\}, \exists S' \in D - \{S\}(\{x, y\} \subseteq S')$.
This is a violation of condition $C.1$.

Lemma 1 ensures that for any set S in the domain if $C(S) \neq S$, then f_S exists. It may be noticed that f_S is not unique. Consider, $D = \{\{x, y, z, w\}, \{x, y, w\}, \{z, w\}\}$. $C(\{x, y, z, w\}) = \{w\}, C(\{x, y, w\}) = \{w\}, C(\{z, w\}) = \{z, w\}$. $\bar{D} = \{\{x, y, z, w\}, \{x, y, w\}\}$. Let $S_1 = \{x, y, z, w\}$, $S_2 = \{x, y, w\}$ and $S_3 = \{z, w\}$ $F = \{f_{S_1}, f_{S_2}\}$

f_{S_1}, f_{S_2} are not unique.

Consider the following two cases:

Case (i)

$$f_{S_1} : f_{S_1}(x) = y,\ f_{S_1}(y) = w,\ f_{S_1}(z) = y$$
$$f_{S_2} : f_{S_2}(y) = x,\ f_{S_2}(x) = w$$

and,

Case (ii)

$$f_{S_1} : f_{S_1}(x) = w,\ f_{S_1}(y) = w,\ f_{S_1}(z) = y$$
$$f_{S_2} : f_{S_2}(y) = w,\ f_{S_2}(x) = w\,.$$

Since, as discussed before, $f_S(x) = y$ has the interpretation that y defeats the unchosen element x in S, in order for a choice function to be rationalizable it should not be the case that in any other set, T, x defeats y i.e., we must not have $f_T(y) = x$. Notice, f_{S_1} and f_{S_2} as defined in case (i) do not satisfy this property but they satisfy under the case (ii).

In lemma 2 we shall show, if the domain of choice function satisfies $C.1$ and $C.2$ then it is possible to construct $f_S \in F$ such that it satisfies the above property.

LEMMA 2. *Let choice function C be defined over D which satisfies $C.1$ and $C.2$. Let $\exists S \in D$ and $x \in S - C(S)$, then it is possible to find $z \in S$ such that $(f_S(x) = z \wedge (\forall K \in D)(f_K(z) \neq x))$.*

Proof. Let $\exists S \in D$ and $x \in S - C(S)$. By lemma 1, f_S exists. Suppose $f_S(x) = y \wedge y \in C(S)$. Then yR_cx and hence $(\forall K \in D)(f_K(y) \neq x)$ (by definition of f_S). Suppose, it is not possible to find $y \in C(S)$ such that $f_S(x) = y$ i.e., $(\forall y \in C(S))(f_S(x) \neq y)$. This implies $(\forall y \in C(S))(xR_cy)$.

$$Let(S - (C(S) \cup \{x\})) = \{w_1, w_2, \ldots, w_n\} \tag{4}$$

Without loss of generality, suppose $f_S(x) = w_1$. If $(\forall K \in D)(f_K(w_1) \neq x)$ then we are through. Suppose $(\exists K_1 \in D)(f_{K_1}(w_1) = x)$. $(f_{K_1}(w_1) = x) \to (\forall z \in K_1 - \{x\})(w_1R_cz)$. Because, if $(\exists z \in K_1 - \{x\})(\sim w_1R_cz)$ then it is possible to have

$$f_{K_1}(w_1) = z \text{ and hence } f_{K_1}(w_1) \neq x\,. \tag{5}$$

Analogous argument holds for $w_2, w_3, \ldots, w_{n-1}$ and finally, if $f_S(x) = w_n$ and $(\forall K \in D)(f_K(w_n) \neq x)$ then we are through. Suppose $(\exists K_n \in D)(f_{K_n}(w_n) = x)$.

$$(f_{K_n}(w_n) = x) \to (\forall z \in K_n - \{x\})(w_nR_cz))\,. \tag{6}$$

Let $T = \{w_1, w_2, \ldots, w_n\}$ and $(\forall w_i \in T)(w_i \in K_i)$. Now we have the following cases:

Case (i): $(\forall K \in \{K_1, K_2, \ldots, K_n\})(S = K)$

Case (ii): $(\exists K \in \{K_1, K_2, \ldots, K_n\})(S \neq K)$.

Case (i): Let $(\forall K \in \{K_1, K_2, \ldots, K_n\})(S = K)$

Now, (4) $\wedge$ (5) $\wedge$ (6) $\to$ $(\forall k \in S - T)(xR_ck) \wedge (\forall z \in T)(\forall w \in S - \{x\})(zR_cw)$ $\to$ $[[(\forall k \in S - T)(\exists V \in D - \{S\})(\{x, k\} \subseteq V)] \wedge [(\forall z \in T)(\forall w \in S - \{x\})(\exists B \in D - \{S\})(\{w, z\} \subseteq B)]]$ $\to$ a violation of condition $C.2$.

Case (ii): $\exists K \in \{K_1, K_2, \ldots, K_n\})(S \neq K)$

Let $w_k \in \{w_1, w_2, \ldots, w_n\} \wedge w_k \in K$. Now, (4) $\wedge$ (5) $\wedge$ (6) $\rightarrow$ $(\forall z \in K - \{x\})(w_k R_c z) \rightarrow (\forall z \in K - \{x\})(\exists B \in D - \{K\})(w_k, z \in B)$. Again, $w_k, x \in S \wedge S \neq K$ $\rightarrow$ violation of condition $C.1$.

PROPOSITION 2. *The binary relation R' is asymmetric.*

Proof. Proof is straight forward and follows from the definition.

THEOREM 2. *Any choice function defined over D is M-rationalizable iff D satisfies conditions $C.1$ and $C.2$.*

Proof. Suppose D violates the condition $C.1$, i.e.,

$$\exists S_1 \in D - D_X, \exists x \in S_1 \text{ such that } \forall y \in S_1 - \{x\},$$
$$\text{there exists } S' \in D - \{S_1\} \text{ such that } \{x, y\} \subseteq S' . \tag{7}$$

Let $S_1 = \{x, y_2, y_3, \ldots, y_n\}$.
(7) $\rightarrow$ $(\exists S_2, S_3, S_4, \ldots, S_n \in D)$[12]$(\{x, y_2\} \subseteq S_2 \wedge \{x, y_3\} \subseteq S_3 \wedge \{x, y_4\} \subseteq S_4 \wedge, \ldots, \wedge \{x, y_n\} \subseteq S_n)$
Now, consider the following choice function:

$$\begin{gathered}\hat{C}(S_1) = \{y_2\}\\ \hat{C}(S_2) = \{x\}\\ \hat{C}(S_3) = \{x\}\\ \vdots\\ \hat{C}(S_n) = \{x\} \quad \textit{and}\\ (\forall S \in D - \{S_1, S_2, \ldots, S_n\})(\hat{C}(S) = S)\end{gathered}$$

This choice function defined on D is not rationalizable. Since x is chosen from $S_2, S_3, S_4, \ldots, S_n$ no element is preferred to x. Therefore, x should belong to S_1 but it does not. The reason for this kind of choice function not to be rationalizable has been discussed with an example at length in proposition 1.

Suppose D violates the condition $C.2$, i.e., $\exists S \in D - D_{2X}, \exists T \subset S,$ such that:
(i) $\#(S - T) \geq 2$ and,
(ii) $\exists x \in S - T$ such that: $[\forall k \in S - T, \exists V \in D - \{S\}(\{x, k\} \subseteq V)$ and $\forall z \in T, \forall w \in S - \{x\}, \exists B \in D - \{S\}(\{w, z\} \subseteq B)]$.
Let $S = \{x, y, y_3, y_4, \ldots, y_n\} \wedge \#(S - T) \geq 2 \wedge x, y \in (S - T)$.
Now Consider a choice function $\hat{C}$ such that:

$$\hat{C}(S) = \{y\}$$
$$(\forall k \in S - T)(\exists V \in D - \{S\})(\{x, k\} \subseteq V \wedge \hat{C}(V) = \{x\})$$
$$(\forall z \in T)(\forall w \in S - \{x\})(\exists B \in D - \{S\})(\{w, z\} \subseteq B \wedge \hat{C}(B) = \{w, z\}) .$$

This choice function defined on D is not rationalizable. The reason for this kind of

[12] $S_2, S_3, S_4, \ldots, S_n$ are not necessarily distinct.

choice function not to be rationalizable has been discussed with an example at length in proposition 1.

Let D satisfy condition $C.1$ and $C.2$. Let C be any choice function defined over D. Now consider, $\bar{R} = R_c \cup R^{'}$ We show that $\bar{R}$ rationalizes the choice function C i.e., we show that $C(S) = M(S, \bar{R})$. Let $x \in C(S) \rightarrow (\forall y \in S)(xR_c y)$. $(\forall y \in S)(xR_c y) \rightarrow \sim yP(R_c)x$ and $\sim yP(R^{'})x$ (by definitions of R_c and $R^{'}$) $\rightarrow x \in M(S, \bar{R})$

Let $x \in M(S, \bar{R})$.

Suppose $x \notin C(S)$.

Since $x \in S - C(S)$, by lemma 2, it is possible that

$$(\exists z \in S)(f_S(x) = z \wedge (\forall K \in D)(f_K(z) \neq x)) . \tag{8}$$

Furthermore, since we have now $f_S(x) = z$, $xR_c z$ is not possible i.e., $\sim xR_c z$ by definition of f_S.

Suppose $zR_c x$.

We then have $\sim xR_c z \wedge zR_c x$ i.e., $zP(R_c)x$ and hence $zP(\bar{R})x$.

In view of the fact $x \in M(S, \bar{R})$ this is not possible.

We, therefore, have $\sim xR_c z \wedge \sim zR_c x$. (9)

Again, $((f_S(x) = z) \wedge (8) \wedge (9)) \rightarrow zR^{'}x$.

By proposition 2, $R^{'}$ is asymmetric and hence $zP(R^{'})x$.

$\rightarrow zP(\bar{R})x$

Notice, $x \in M(S, \bar{R}) \rightarrow \sim zP(\bar{R})x$.

A contradiction.

Therefore, $x \in C(S)$.

5. CONCLUSION

In the literature of choice theory the problem of rationalizability has been addressed solely on the basis of imposing conditions on the choice behaviour, generally known as choice consistency conditions. These conditions are of the nature that if a choice function satisfies these conditions it becomes possible to construct a preference relation which generates the choice sets. Eventually we see that in the literature a number of choice consistency conditions have emerged which provide characterization of rationalizable choice functions.

Unlike the approach followed in the literature in connection to the problem of rationalizability, this paper adopts a different approach to this problem. This approach deals with this problem without invoking any choice consistency condition and thereby does not put any restriction on the choice behaviour. It deals with the domain of choice function. The question it asks is: what condition does a domain need to satisfy in order to make every choice function defined on it rationalizble? The paper provides domain conditions for both maximal element and best element rationalizable choice functions, which in turn provide complete characterization of a partition of domains for the said classes for choice functions. On one hand there is a class of domains over which any choice function is G-rationalizable (M-rationalizable) and there is a class of domains over which not all choice functions are G-rationalizable (M-rationalizable) on the other.

The merit of considering domain conditions lies in many respects. First, as said before, choice consistency conditions deal with the properties of choice functions and restrict choice behaviours of individuals. On the contrary, domain conditions do not impose any constraint on the 'act of choice'. It ensures that no matter whichever way an individual makes choices, the choice function always becomes rational.

Second, the 'purposive behaviour' of an individual seems to carry meaningful sense in those domains where some choice patterns are rational and some are not. If over a domain all choice patterns are rational then it becomes difficult to identify any objective in the choice behaviour.

Third, we observe a clear partition among the whole range of choice functions namely- rationalizable and non-rationalizable choice functions. The existing literature has characterized this partition by invoking choice consistency conditions. Unlike partitioning class of choice functions, the domain condition provides a partition of domains for rationalizability.

Finally, in social choice theory, domain condition plays an important role. Many paradoxes that exist in the literature are resolved through domain conditions. The paradox of voting does not arise under a domain that restricts the preferences of individuals to single-peakedness. Likewise, the inconsistency in liberal rights also disappears if domain is suitably restricted[13]. Since rationalizability is considered to be a desirable property for a choice function in social choice theory, domain conditions here also provide a new set of conditions which ensure the rationalizability of all choice functions.

REFERENCES

Apesteguia, J., Ballester, M.A., Choice by Sequential Procedures. Games and Economic Behavior 77, 90–99 (2013).

Arrow, K. J., Rational Choice Functions and Orderings. Economica, New Series, 26, 102, 121–127 (1959).

Bossert, W. Sprumont, Y. and Suzumura, K., Maximal-Element Rationalizability, Theory and Decision, 58 325–350 (2005).

Gaertner, W., Xu, Y., Procedural Choice. Economic Theory, 24, 335–349 (2004).

Hung, A.P., Kawai, K., J., Sequentially Rationalizable Choice with Transitive Rationales. Games and Economic Behavior, 73, 608–614 (2011). Manzini, P., Mariotti, M., Sequentially Rationalizable Choice. Amer. Econ. Rev. 97, 1824–1839 (2007).

Pal, Debabrata, Rationalizability of Choice Functions: Domain Conditions" Economics Bulletin, 37, 3, 1911–1917 (2017).

Richter, M. K., Revealed Preference Theory. Econometrica, 34, 3, 635–645 (1966).

Samuelson, P. A., A Note on the Pure Theory of Consumer's Behaviour. Economica, New Series, 5, 17, 61-71 (1938).

Sen, A. K., Collective Choice and Social Welfare. San Francisco: Holden-Day (1970).

Sen, A. K., Maximization and the Act of Choice, Econometrica, 65, 745–779 (1997).

Schwartz, T., Rationality and the Myth of the Maximum, Nous, 6, 97–117 (1972).

Sugden, R., Why be Consistent? A Critical Analysis of Consistency Requirements in Choice Theory, 52 (1985).

Suzumura, K., Rational Choice and Revealed Preference, Review of Economic Studies, 43, 149–58 (1976).

Suzumura, K., On the Consistency of Libertarian Claims, The Review of Economic Studies, 45, 329–342

[13] See: Suzumura (1978).

(1978).
Suzumura, K., Rational Choice, Collective Decisions and Social Welfare. Cambridge University Press, New York (1983).
Uzawa, H., Note on Preference and Axioms of Choice, Ann Inst Stat Math, 8, 35–40 (1957).