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MULTIDIMENSIONAL LORENZ DOMINANCE: A DEFINITION AND AN EXAMPLE

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Abstract: This paper seeks to extend the unidimensional notion of Lorenz dominance to the multidimensional framework in the context of distributions of well-being in an economy. It formulates a definition of a multidimensional Lorenz dominance relation (MLDR) that incorporates two generalizations of the well-known Pigou-Dalton transfer condition as well as a condition relating to the sensitivity of the dominance relation to the correlations between distributions of the different attributes. The paper notes that the existing literature does not seem to contain an example of a relation that is an MLDR as per the definition developed here. It seeks to provide one.

Key words: Multidimensional Lorenz dominance, uniform majorization, Pigou-Dalton bundle principle, correlation increasing majorization.

JEL Classification Number: D63.

1. INTRODUCTION

It is by now generally recognized that well-being of an individual depends not only on his or her income but also on other attributes (such as, education, health etc.). Therefore, the methods of measuring inequality in the distribution of well-being among the individuals in an economy need to be extended from the unidimensional to the multidimensional context.

As in the case of a single dimension, the method of comparing between the levels of inequality of alternative multidimensional distributions may take the approach of constructing a complete ordering over the set of distributions by proposing a scalar-valued *inequality index*.

However, since different inequality indices may lead to different complete orderings

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of the distributions, attention may also be given to the task of constructing orderings which may be partial but which would be more readily acceptable in some intuitive sense. In single-attribute theory the most widely used partial ordering of this type is the Lorenz partial ordering: a distribution **x** Lorenz dominates another distribution **y** if the Lorenz curve for **x** does not lie below that for **y** at any point and lies above it at, at least, one point. **x** is then interpreted to be a more desirable distribution than **y**. In the multidimensional case a main task under this approach is to extend the notion of Lorenz dominance to the multi-attribute context.

Although the economic theory of multidimensional inequality measurement as a whole is a relatively new field of research, within this field the first of the two approaches mentioned above has by now led to a sizable literature containing important contributions. For reviews see, for instance, Savaglio (2006) and Weymark (2006).

The second approach, however, seems be a relatively neglected area. In this paper we shall be concerned with multidimensional Lorenz dominance. We shall, however, confine ourselves to the case of empirical distributions.

In this context it is convenient to describe the allocations of the different attributes to the different individuals by a matrix. We shall suppose that in a *distribution matrix* each column refers to an attribute and each row to an individual. The entries represent the allocations. If X and Y are two distribution matrices, the question under what conditions X is to be considered to Lorenz dominate Y does not seem to have an obvious and unique answer. Various suggestions have, however, been made. (See, for instance, Arnold (2008) and Koshevoy and Mosler (2007). For reviews see Savaglio (2006) and Trannoy (2006)).

This paper formulates a definition of a multidimensional Lorenz dominance relation (MLDR) on the set of distribution matrices. The definition uses, apart from other requirements, two different (and independent) generalizations of the Pigou-Dalton transfer condition of unidimensional theory viz. the Uniform Majorization (**UM**) condition due to Kolm (1977) and the Pigou-Dalton Bundle Principle (**PDBP**) introduced in Fleurbaey and Trannoy (2003).

Both **UM** and **PDBP** were originally introduced in the literature as conditions on equity-sensitive social evaluation functions. Later they have also been stated as conditions on multidimensional inequality indices. The inequality index version of each of these conditions takes the following form: Letting I(X) denote the value of an inequality index I for any distribution matrix X, each of the conditions requires that if the distribution matrix Y is obtained from X by subjecting it to a specified type of transformation, then I(X) <I(Y). However, since the conditions require *any* inequality index to behave in the specified way, intuitively it seems reasonable to adapt these to the present context by restating them to require that X Lorenz dominates Y if Y is obtained from Y in the specified manner.

We shall also desire an MLDR to satisfy an additional condition which seems to be intuitively reasonable, to wit, the condition of Correlation Increasing Majorization (**CIM**) introduced in the economic literature by Tsui (1999). We shall, again adapt it from the inequality index context for our purposes. The essential idea behind this condition is that greater correlation among the columns of the distribution matrix increases multidimensional inequality, however such inequality may be measured.

The existing literature does not seem to contain an example of an MLDR satisfying all of the conditions considered here. We seek to close this gap by suggesting such an MLDR.

Section 2 below introduces the notations and develops a definition of an MLDR. Section 3 reviews the literature to search in vain for a relation which is an MLDR as per our definition. Section 4 proposes a specific binary relation on the set of distribution matrices and proves that it is an MLDR. Section 5 concludes the paper.

2. NOTATIONS, DEFINITIONS ETC.

Consider an economy with n individuals whose levels of well-being are determined by the amounts of $m$attributes that are allocated to them. Allocations are assumed to be non-negative. $M = \{1, 2, \ldots, m\}$ and $N = \{1, 2, \ldots, n\}$ will denote the set of attributes and the set of individuals respectively. Since we shall be concerned with inequality among the standards of living of the individuals, we assume that $n \geq 2$. However, we assume that while m is exogenously fixed, n is allowed to be any positive integer. This allows inequality comparisons to be made across populations of different sizes.

By a *distribution matrix* X we shall mean an $n \times m$ *non-negative* matrix whose (p-th row, j-th column) term, x_p^j, is the amount of attribute j allocated to individual p for all j in M and for all p in N. Thus, a distribution matrix describes a pattern of allocations of the attributes in the economy. For a distribution matrix X, $\mathbf{x}_p$ will denote its p-th row and $\mathbf{x}^j$ its $j-$th column.

It is assumed that in any distribution matrix the sum of each column is positive i.e. for every attribute there is a positive total amount to be distributed among the individuals.

Thus the domain of matrices under consideration is: $\mathbf{X} = \{X \, \varepsilon \, \Re^{nm}_{+} : \mu(\mathbf{x}^j) > 0,\ j$ $1.2, \ldots, m$, and $n \geq 2\}$ for some given and fixed positive integer m where, for any vector $\mathbf{y}$, $\mu(\mathbf{y})$ denotes its arithmetic mean.

We shall be concerned with inequality dominance. In the case where there is a single attribute ($m = 1$), the standard notion of inequality dominance is that of Lorenz dominance. For any non-negative distribution vector $\mathbf{x}$ specifying the allocations of the attribute to the $n$individuals, let $\mathbf{x}^{\sim}$ denote the rearrangements of $\mathbf{x}$ in non-decreasing order and let $\mu(\mathbf{x})$ denote the arithmetic mean of $\mathbf{x}$. As per the standard Gastwirth (1971) definition of a *Lorenz curve* applied to the case of a discrete distribution, the Lorenz curve of $\mathbf{x}$ is the curve in the unit square obtained by joining the $(n + 1)$ points $(0, 0)$ and $(k/n,\ (1/n) \sum_{i=1}^{k} x_i^{\sim} / \mu(\mathbf{x}))$, $k = 1, 2, \ldots, n$ by line segments, $x_i^{\sim}$ being the i-th component of $x^{\sim}$.

For the distribution vector $\mathbf{x}$, the mapping from [0, 1] into [0, 1] described by the Lorenz curve of $\mathbf{x}$ is denoted by L_x. For all distribution vectors, $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{x}$ *Lorenz dominates* $\mathbf{y}$ if and only if $L_x(p) \geq L_y(p)$ for all p in [0, 1]. It *strictly Lorenz dominates*

y if, in addition, $L_x(p) > L_y(p)$ for some p in [0, 1].

We shall denote the unidimensional *Lorenz dominance* relation on the set of all non-negative distribution vectors by L: for all distribution vectors **x** and **y**, **x** L**y** if and only if **x** Lorenz dominates **y**. Clearly, L is a quasi-ordering. P will denote the *strict Lorenz dominance* relation: **x** P **y** if and only if **x** strictly Lorenz dominates **y.** P coincides with the asymmetric component of L. The symmetric component of L will be denoted by I. For all distribution vectors **x** and **y**, **x** I **y** if and only if the Lorenz curve of **x** coincides with that of **y** (i.e. $\mathbf{x} = \mathbf{y}$ or **x** is a permutation of **y**).

Lorenz dominance is closely related to the notion of Pigou-Dalton (PD) transfers. If the attribute in question is income, a PD transfer is an income transfer from a richer to a poorer person by an amount less than their initial income difference. The following three statements are equivalent (Hardy, Littlewood and Polya (1952) and Marshall and Olkin (1979, Ch.1)): (1) **x** strictly Lorenz dominates **y**; (2) **x** *Pigou-Dalton majorizes* **y** i.e. **x** is obtained from **y** by a *finite* sequence of PD transfers; and (3) $\mathbf{x} = \mathrm{B}\mathbf{y}$ for some bistochastic matrix B which is not a permutation matrix. (A bistochastic matrix is a non-negative matrix in which each row as well as each column sums to 1.)

In this paper we are interested in obtaining a multidimensional version of the notion of Lorenz dominance. For this purpose we first define a weak inequality dominance relation, D, on **X**. For all X and Y in **X**, if X DY, this will be interpreted to mean that relative inequality in the distribution of overall well-being in the pattern of allocations described by X is not more than that in the pattern described by Y, whatever may be the specific method of measuring the degree of overall inequality. D_{P} and D_{I} will denote the asymmetric and the symmetric components of D respectively, i.e., for all X and Y in **X**, X D_{P} Y if and only if [X DY and $\neg$(Y DX)] ; and X D_{I} Y if and only if [X DY and Y DX].

We shall impose a number of conditions on D. We start with some basic conditions which are *not* related to equity considerations.

Ratio-Scale Invariance (RSI): For all $n \times m$ matrices X in **X** and for all diagonal matrices Λ with positive entries along the main diagonal, X $D_{\mathrm{I}}(\mathrm{X}\Lambda)$.

Restricted Continuity (RCONT): Let $\mathbf{X}_{\mathbf{2}\times m}$ denote the subset of **X** consisting of distribution matrices with 2 rows. The restriction of D to $\mathbf{X}_{\mathbf{2}\times m}$ is continuous i.e. for any X in $\mathbf{X}_{\mathbf{2}\times m}$, the sets $\{\mathrm{Y}\ \varepsilon\ \mathbf{X}_{\mathbf{2}\times m}\colon \mathrm{Y}\ D_{\mathrm{P}}\ \mathrm{X}\}$and $\{\mathrm{Y}\ \varepsilon\ \mathbf{X}_{\mathbf{2}\ \times m} : \mathrm{X}\ D_{\mathrm{P}}\ \mathrm{Y}\}$are open.

Quasi-ordering (QORD): D is a quasi-ordering i.e. it is a reflexive and transitive relation on **X** but is not necessarily complete. Thus, (i) for all X in **X**, X D X; and (ii) for all X , Y and Z in **X**, if X D Y and Y D Z, then X D Z. However, it is not necessarily the case that, for all X and Y in **X**, either X D Y or Y D X.

Anonymity (ANON): If X and Y in **X** are such that Y is obtained by a permutation of the rows of X, then X D_{I} Y.

Population Replication Invariance (PRI): For all X and Y in **X** such that Y is obtained by a k-fold replication of the population in X for some positive integer k i.e., for all p in$N$,

$$\mathbf{x}_{\mathrm{p}} = \mathbf{y}_{\mathrm{p}} = \mathbf{y}_{\mathrm{n+p}} = \dots\dots = \mathbf{y}_{\mathrm{n(k-1)+p}},$$

X D_{I} Y.

One of the first issues that arise in any multidimensional analysis is that of *commensurability* of the attributes. Commensurability requires that the attributes are measured in the same or, at least, similar (for instance, monetary) units. Since this may not be true of the original data, we make the entries in the distribution matrices independent of the scales of measurement of the different attributes. Imposing the condition of **RSI** is one way of doing this. It requires that if each column of a distribution matrix is multiplied by a positive constant (possibly different for the different columns), the matrix obtained is 'equivalent' to the original matrix in terms of the weak inequality dominance relation D. The requirement also tallies with the fact that in this paper we shall be concerned with *relative* inequality.

It is known that unidimensional Lorenz dominance is not a continuous relation. If **x**, **y** and **z** are such that **x** strictly Lorenz dominates **y** and if **z** is "close" to **x**, **z** need not strictly Lorenz dominate **y**. (Consider the case where the Lorenz curve for **x** is not below that for **y** at any point and is above it at some (but *not* all) points. Consider now the case where **z** is obtained by a "small" perturbation of **x** which perturbs the Lorenz curve for **x** at one of its common points with the curve for **y**. The Lorenz curves for **x** and **z** then may intersect even though those for **x** and **y** do not.)

However, there is a special case where L would be continuous, to wit, the case where there are only 2 individuals. In this case the Lorenz curves for any two distributions **x** and **y** which are not permutations of one another cannot intersect. Lorenz dominance is continuous in this case. **RCONT** extends this notion to the multidimensional case.

It may be noted that a matrix in $\mathbf{X}_{\mathbf{2}\times m}$ can be thought of as a vector in $\Re^{2m}{}_{+}$. A subset of $\mathbf{X}_{\mathbf{2}\times m}$ is, therefore, open if the corresponding set of vectors is open in $\Re^{2m}{}_{+}$.

While a relation on a set is called an *ordering* if it is reflexive, complete and transitive, **QORD** dos not insist on completeness though it implies reflexivity and transitivity. It is, thus, a weaker requirement. **ANON** requires that the labelling of the individuals in the economy should be inconsequential. **PRI** implies that in any distribution matrix it is the proportion of the population (rather than the absolute number of individuals) getting a particular allocation of an attribute that is important.

We now turn to equity considerations. The literature on multidimensional inequality contains generalizations of the concept of Pigou-Dalton majorization. One of the most widely used among such generalization is the concept of Uniform Majorization UM). (See Kolm (1977).) For all $n \times m$ matrices X and Y in **X**, Y is said to uniformly majorize X if Y $\neq$ X and Y $=$ BX for some bistochastic matrix B which is not a permutation matrix. Since Y $=$ BX implies, $\mathrm{y^i} = \mathrm{Bx^i}$ for all i in M, $\mathrm{y^i}$ Pigou-Dalton majorizes $\mathrm{x^i}$ for each i in M; and since the same matrix B is used to majorize all the columns of X, the majorization is said to be *uniform* across the attributes. A variant of this type of majorization is w-majorization formulated in Savaglio (2011) where B is required to be a row-stochastic (but not necessarily a bistochastic) matrix.

Kolm (1977) used UM to formulate an axiom regarding an equity-sensitive social evaluation function. According to this axiom, for all X and Y, if Y uniformly majorizes X, then the society considers Y to be superior to X from the distributional point of view. In the present framework we do not use a social evaluation function. However, axioms similar to the ones mentioned above can be formulated in terms of multidimensional indices of inequality. Take, for instance, the concept of UM. Let f be a mapping of $\mathbf{X}$ into the real line. If f is to be an index of multidimensional inequality, it is to satisfy the following axiom (called the axiom of UM): for all X and Y in $\mathbf{X}$, if Y is a UM of X, then $f(\mathrm{Y}) < f(\mathrm{X})$.

We wish to formulate a condition under which a distribution matrix can reasonably be said to *dominate* another. However, in analogy with the unidimensional case, the statement that Y dominates X may be interpreted to mean that, according to *any* reasonable measure of multidimensional inequality, Y would have a lower degree of inequality than X. Hence, the generalizations of the Pigou-Dalton majorization can be used to formulate suitable conditions of inequality dominance in the multidimensional context. This type of adaptation of the axiom of UM leads to the following condition on D.

Uniform Majorization (UM):: For all X and Y in $\mathbf{X}$ such that Y is a UM of X, Y D_{P}X.

The recent literature on inequality has, however, pointed out a number of inadequacies of the axiom of UM. First, all attributes may not be transferable in principle. (What, for instance, do we mean by transferring educational attainments or health status?) Secondly, even when all of these are transferable, there seem to be cases in which a transfer is non-uniform across the attributes and yet there seem to be reasonable grounds for hypothesizing that it leads to an unambiguously superior state of distribution.. UM does not cover these cases. For a more detailed discussion on these two issues see, for instance, Lasso de la Vega, Urrutia and Amaia de Sarachu (2010).

In this paper in order to take these considerations into account we shall use the Pigou-Dalton Bundle Principle (PDBP) introduced by Fleurbaey and Trannoy (2003) in the context of the normative theory of inequality. (See Lasso de la Vega et. al. (2010) for an innovative use of PDBP for the purpose of deriving a multidimensional inequality index.)

Consider the case where the amounts of the attributes that are transferred are allowed to differ between attributes and are not restricted to be non-zero for all attributes. It is, however, assumed (i) that transfers from an individual q to an individual p are allowed only if q is unambiguously richer than p (i.e. q has more of every attribute than p) and (ii) that transfers preserve the relative ranks, in each dimension, of the two individuals whose allocations are altered.

Definition 2.1: For all X and Y in $\mathbf{X}$, Y is said to be derived from X by a **Pigou-Dalton Bundle Transfer (PDBT)** if there exist p and q in N such that

(i) $\mathbf{x}_{\mathrm{q}} > \mathbf{x}_{\mathrm{p}}$;

(ii) $\mathbf{y}_{\mathrm{q}} = \mathbf{x}_{\mathrm{q}} - \mathbf{d}$ and $\mathbf{y}_{\mathrm{p}} = \mathbf{x}_{\mathrm{p}} + \mathbf{d}$ for some $\mathbf{d}$ in $\Re^{\mathrm{m}}{}_{+}$ such that $\mathbf{d} \neq 0$.

(iii) $\mathbf{y}_{\mathrm{r}} = \mathbf{x}_{\mathrm{r}}$ for all r in $N - \{p, q\}$;

(iv) $\mathbf{y}_{\mathrm{q}} \geq \mathbf{y}_{\mathrm{p}}$.

Part (i) of Definition 2.1 states that individual q is unambiguously richer than individual p in the initial allocation matrix X. Part (ii) requires that non-negative amounts of the different attributes are transferred from individual q to individual p. The amounts or the proportions of the transfers need not be the same for all attributes. Neither is it required that some amounts of *all* attributes must be transferred i.e. it is recognized that some attributes may, by their nature, be non-transferable. It is required, however, that the transfer is non-trivial i.e. some amount of at least one attribute is transferred. Part (iii) states that all individuals other than p and $q$are unaffected by the transfer. Part (iv) states that after the transfer q remains unambiguously at least as well off as p.

As an illustration consider the case in which $n = 3$, $m = 2$, $\mathrm{X} = \begin{pmatrix} 10 & 9 \\ 2 & 8 \\ 7 & 6 \end{pmatrix}$ and $\mathrm{Y} = \begin{pmatrix} 8 & 9 \\ 4 & 8 \\ 7 & 6 \end{pmatrix}$. In X individual 1 is unambiguously richer than individual 2. Y is obtained from X by transferring 2 units of the first attribute from individual 1 to individual 2. This is a PDBT since, as is easily checked, all parts of Definition 2.1 are satisfied.

We impose the following condition on the dominance relation D.

Pigou-Dalton Bundle Principle (PDBP): For all X and Y in **X** such that Y is obtained from X by a finite sequence of PDBT's, Y D_{P} X.

All inequality dominance relations are, by definition, concerned with equity considerations. Some basic aspects of such considerations are captured by generalizations of the Pigou-Dalton transfer principle such as **PDBP** and **UM.** In multi-attribute theory, however, there is another aspect of the matter. It is related to the pattern of inter-relation among the distributions of the attributes and its effect on multidimensional inequality.

For all X in **X** and for all p, q in N, let $\mathrm{x_p} \wedge \mathrm{x_q}$ denote the vector $\{\min(\mathrm{x_p^1}, \mathrm{x_q^1}), \min(\mathrm{x_p^2}, \mathrm{x_q^2}), ..., \min(\mathrm{x_p^m}, \mathrm{x_q^m})\}$and $\mathrm{x_p} \vee \mathrm{x_q}$ the vector $\{\max(\mathrm{x_p^1}, \mathrm{x_q^1}), \max(\mathrm{x_p^2}, \mathrm{x_q^2}), ..., \max(\mathrm{x_p^m}, \mathrm{x_q^m})\}$.

Definition 2.2: For all X and Y in **X** such that X is not equal to Y or a row permutation of Y, X is said to be obtained from Y by a Correlation Increasing Transfer (CIT) if there exist p and $q$in N such that

(i) $\mathrm{x_p} = \mathrm{y_p} \wedge \mathrm{y_q}$;

(ii) $\mathrm{x_q} = \mathrm{y_p} \vee y_{\mathrm{q}}$; and

(iii) $\mathrm{x_r} = \mathrm{y_r}$ for all $r$in $N- \{p, q\}$.

We shall desire L^{M} to satisfy the following condition:

Correlation Increasing Majorisation (CIM): For all X and Y in **X** such that Y is obtained from X by a finite sequence of CIT's, X D_{P} Y.

The basic idea behind **CIM** is that greater correlation among the different columns of the distribution matrix implies greater inequality, irrespective of how inequality is measured. It was introduced in the economic literature by Tsui (1999) in the context of inequality measurement. In the statistical literature it was proposed by Boland and

Proschan (1988). The concept of CIT on which it is based was studied in Atkinson and Bourguignon (1982) and in Epstein and Tanny (1980).

The acceptability of a condition depends on its intuitive plausibility. **CIM** seems to have a strong intuitive appeal. Consider, for instance, the following example. Let $n = 2 = m$.

Let Y $= \begin{pmatrix} 9 & 7 \\ 6 & 3 \end{pmatrix}$ and X $= \begin{pmatrix} 9 & 3 \\ 6 & 7 \end{pmatrix}$. Y is obtained by a switch of the entries in the second column of X.It is easily checked that this is a CIT. If it is now asked whether we should consider X to Lorenz dominate Y (i.e. whether X should be judged to display a lower degree of equality as per any measure of inequality), there seems to be intuitive grounds for an affirmative answer. In X individual 1 has a higher allocation of attribute 1 than individual 2. But this is at least partially compensated for by the fact that w.r.t. attribute 2 it is individual 2 who has a lower allocation. In Y, however, the effect of the lower allocation of attribute 1 to allocation 2 is compounded by the fact that individual 2 faces the same predicament w.r.t. attribute 2 i.e. there is a compounding of inequalities across the attributes.

We are now ready to state the definition of a multidimensional inequality dominance relation.

Definition 2.3: A multidimensional inequality dominance relation (**MIDR**), D, is a binary relation on **X** satisfying **RSI**, **RCONT**, **QORD**, **ANON**, **PRI**, **UM**, **PDBP** and **CIM**.

Since we are interested in obtaining a generalization of the unidimensional Lorenz dominance relation, L, it is natural to require that the dominance relation reduces to L if there is just one attribute.

Definition 2.4: A multidimensional Lorenz dominance relation (**MLDR**), L^{M}, is an MIDR on **X** such that $L^{\mathrm{M}} = L$ if $m = 1$.

The antisymmetric and symmetric components of an MLDR, L^{M}, will be denoted by P^{M} and I^{M} respectively.

In this paper we look for a binary relation on **X** which is an MLDR as per Definition 2.4.

3. "CANDIDATE" LORENZ DOMINANCE RELATIONS

The existing literature contains a number of specific suggestions regarding the construction of MLDR's. In this Section we review some of these suggestions and assess their acceptability in terms of the conditions stated in Section 2.

Examples of Suggested MLDR's:

(1) Directional Lorenz Majorization (L^1): L^1 is the binary relation on **X** such that, for all X and Y in **X**, X L^1 Y if and only if X is a *directional Lorenz majorization* of Y i.e. (Xw) L(Yw) for all w in $\Re^m$.

(2) Lorenz Majorization by Non-negative Weights (L^2): L^2 is such that, for all X and Y in **X**, X L^2 Y if and only if X is a *majorization* of *Y by non-negative weights* i.e. (Xw) L(Yw) for all w in the set of non-negative m- dimensional real vectors $\Re^m{}_+$.

(3) Lorenz Majorization by Positive Weights (L^3): L^3 is such that, for all X and Y in **X**, X L^3 Y if and only if X is a *majorization* of Y *by positive weights* i.e. (Xw) L(Yw) for all w in the set of positive m-dimensional real vectors $\Re^m_{++}$.

(4) Columnwise Lorenz Majorization (L^4): For all X and Y in **X**, X L^4 Y if and only if X is a *columnwise majorization* of Y i.e. $\mathbf{x}^{\mathrm{i}}$ L $\mathbf{y}^{\mathrm{i}}$ for all i in M i.e. (Xw) L(Yw) for all m-vectors w such that $\mathrm{w_i} = 1$ for some i in M and $\mathrm{w_j} = 0$ for all j in M such that $j \neq i$.

For L^1, L^2 and L^3 see, for instance, Bhandari (1988), Joe and Verducci (1993), Kolm (1977), List (1999) and Koshevoy and Mosler (2007). L^1, L^2 and L^4 are also mentioned by Arnold (2008) among 'candidate definitions' of multi-attribute Lorenz dominance.

(5) Majorization of Lorenz Zonoids (L^{Z}): While the MLDR's illustrated in Examples 1 through 4 are suggested multi-attribute analogues of Lorenz dominance in the single-attribute case, they do not suggest a multi-attribute Lorenz curve. It would seem that a more satisfactory approach would be to proceed in more direct analogy with the single-attribute case i.e. to first suggest an extension of the concept of Lorenz curve to the case of multiple attributes and then to define Lorenz dominance for this case in terms of dominance relations between the generalized curves for different distribution matrices. Because of the mathematical difficulties inherent in this approach, progress along these lines has been slow. Arnold (1983) and Taguchi (1972) were among the early attempts in this direction. In recent statistical literature a more satisfactory definition of such a multi-attribute analogue has emerged. See Koshevoy (1995) for the case of empirical distributions and Koshevoy and Mosler (2007) and Mosler (2002) for extension to the case of random variables and other developments.

(More recently, Sarabia and Jorda (2013) have used the definition proposed in Arnold (1983) to obtain closed expressions for bivariate Lorenz curves. However, their formulation involves specific assumptions regarding the underlying bivariate distributions.)

The Koshevoy-Mosler approach is based on the notion of a *Lorenz zonoid*. First define the *lift zonoid* of an $n \times m$ matrix X, Z(X) (say), as the Minkowski sum of the n line segments $[0_{m+1}, ((1/n), (\mathrm{x_p}/n)]$, $p = 1, 2, \ldots, n$, in $\Re^{m+1}$. It is a convex set. The *Lorenz zonoid* of a distribution matrix X, Z*(X) (say), is then defined as the lift zonoid of X* where, for all X, X* is the scaled version of X i.e. the matrix obtained by dividing each entry in X by the arithmetic mean of the column containing it. Thus, for all X, Z*(X) = Z(X*). For details see the references cited above.

Koshevoy and Mosler (2007) introduced the following *strict dominance* relation: for all $n \times m$ matrices in X and Y, X strictly Lorenz dominates Y if and only if Z*(X) $\subset$ Z*(Y).

As shown by the authors, for all $n \times m$ matrices X and Y, Z*(X) $\subset_{Z*}$(Y) if and only if (X*w) P(Y*w) for all w in $\Re^m$. We shall denote this strict dominance relation by P^{Z}. It is easily seen that P^{Z} does not coincide with the asymmetric component of (P^1, say) of L^1 but is more restrictive i.e. $P^{\mathrm{Z}} \subset P^1$.

Hence, we can obtain a "candidate" MLDR by constructing a quasi-ordering whose asymmetric component would coincide with P^{Z}. We shall consider the relation L^{Z}

defined as follows. $L^{Z} = P^{Z} \cup I^{Z}$ where I^{Z} is the relation on **X** such that, for all X and Y in **X**, X I^{Z} Y if and only if Z*(X) = Z*(Y).

(6) Majorization of Extended Lorenz Zonoids (L^{eZ}): Koshevoy and Mosler (2007) also introduced the concept of the *extended Lorenz zonoid.* Define the *extended lift zonoid* of a distribution matrix X, eZ (X), as the lift zonoid augmented by all points that are below a point in the lift zonoid Z(X) w.r.t. the first coordinate and above the point w.r.t. the other $m$coordinates:

eZ(X) = $\{(\mathrm{v}_0, \mathrm{v}_1, \ldots\ldots, \mathrm{v}_m) : \mathrm{v}_0 \leq \mathrm{z}_0, \mathrm{v}_\mathrm{j} \geq \mathrm{z}_\mathrm{j}, j = 1, 2, \ldots\ldots, m$, for some $(\mathrm{z}_0, \mathrm{z}_1, \ldots\ldots, \mathrm{z}_m) \varepsilon$ Z(X)$\}$.

The *extended Lorenz Zonoid* of a distribution matrix X, eZ*X), is the extended lift zonoid of the scaled version of X i.e. eZ*(X) = eZ (X*).

In similarity with the case of Lorenz zonoids a strict dominance relation P^{eZ} (say) can be defined in terms of strict set set inclusion of extended Lorenz zonoids: for all admissible X and Y, X P^{eZ} Y if and only if eZ*(X) $\subset$ eZ*(Y). However, it has been shown that, for all X and Y, eZ*(X) $\subset$ eZ*(Y) if and only if (X*w) P(Y*w) for all w in $\Re^m_+$. (Thus, P^{eZ} is *not* the asymmetric component (P^2, say) of L^2.)

We shall consider the acceptability, as MLDR, of a relation L^{eZ} on **X** whose asymmetric component would coincide with P^{eZ}. We define L^{eZ} to be $P^{eZ} \cup I^{eZ}$ where I^{eZ} is the relation on **X** for which, for all X and Y in **X**, X I^{eZ} Y if and only if eZ*(X) = eZ*(Y).

(7) Majorization by data-driven weights (L^{w}): The idea of using data-driven weights for the purpose of majorization of distribution matrices has also been pursued in the literature. The following criterion was suggested in Banerjee (2014).

This dominance criterion, applies only on the space of distribution matrices in which no attribute is perfectly equally distributed. For any pair (X, Y) of $n \times m$ distribution matrices of this type, first define the pair (X_0, Y_0) as follows: If X and Y are such that

(i) $\mu(\mathbf{x}^{\mathrm{j}}) = \mu(\mathbf{y}^{\mathrm{j}})$ for all j in M; and

(ii) for some non-empty subset N' of N, $\mathbf{x}_{\mathrm{p}} = \mathbf{y}_{\mathrm{p}}$ for all p in N',

then X_0 and Y_0 are the $(n-n') \times m$ matrices (where n' is the cardinality of N') obtained from X and Y respectively by deleting the common rows. In all other cases (X_0, Y_0) = (X, Y).

Now, for any matrix X, let X^ denote its *comonotonization* (i.e. the comonotonic matrix obtained by rearranging, if necessary, the entries in each column of X) and let C(X) denote the covariance matrix of X.

The suggested criterion L^{w} is defined to be such that, for all admissible distribution matrices X and Y, X L^{w} Y if and only if $[(\mathrm{X}^*_0)\ \mathrm{w}(\mathrm{X}^*_0)]\ L\ [(\mathrm{Y}^*_0)\mathrm{w}(\mathrm{Y}^*_0)]$ where, for all X in **X**, $\mathrm{w}(\mathrm{X}^*_0)$ is the first eigen vector (i.e. the eigen vector associated with the maximal eigen value) of $\mathrm{C}[(\mathrm{X}^*_0)$^]. □

However, we show below that none of the binary relations on **X** mentioned above is an MLDR as per Definition 2.4 of Section 2. For this purpose it suffices to show that each of these relations violates at least one of the requirements of the definition.

All of the seven relations satisfy **QORD**, **ANON** and **PRI**. Moreover, all of them coincide with the unidimensional Lorenz dominance relation when $m = 1$. However, L^3 violates **RSI**. For instance, consider the case where $n = 2 = m$, X $= \begin{pmatrix} 4 & 0 \\ 2 & 4 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. If I^3 denotes the symmetric component of L^3, **RSI** requires that X I^3 (XΛ) $= \begin{pmatrix} 8 & 0 \\ 4 & 4 \end{pmatrix}$i.e. it requires that, for all positive w, Xw either equals or is a permutation of (XΛ)w. However, if w $= \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$, Xw $= \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and (XΛ)w $= \begin{pmatrix} 4 \\ 4 \end{pmatrix}$, violating the requirement. The same argument shows that **RSI** is also violated by L^1 and L^2.

It may be noted, however, that slightly restated versions of these relations would avoid this problem. Redefine L^3 in terms of the *scaled* versions of the matrices: for all X and Y in **X**, X L^3 Y if and only if (X*w) L (Y*w) for all positive w. In what follows all references to L^3 will assume that it has been so redefined. L^1 and L^2 will also be assumed to have been similarly restated. (All of the other relations mentioned above satisfy **RSI**.)

However, L^1, L^2, L^3 and L^4 violate **RCONT**. Consider L^3first. Consider, for instance, the case where $n = 2 = m$, X $= \begin{pmatrix} 6 & 3 \\ 6 & 3 \end{pmatrix}$ and Y $= \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$. Then X* $= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and Y* $= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. It is easily seen that X L^3 Y but $\neg$[X L^3 Y]. Thus, X P^3 Y where P^3 is the asymmetric component of L^3. Consider now the matrix Z $= \begin{pmatrix} 6+d & 3 \\ 6 & 3 \end{pmatrix}$where d is an arbitrarily small positive number. **RCONT** then requires that Z P^3 Y. However, Z* $= \begin{pmatrix} 1+k & 1 \\ 1-k & 1 \end{pmatrix}$ where $0 < k = [d/(1+d)] < 1$. If, now, w $= \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$, Z*w $= \begin{pmatrix} 1+(k/2) \\ 1-(k/2) \end{pmatrix}$while Y*w $= \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus, we have: $\neg$[Z P^3 Y]. The same example serves to show that L^1 and L^2 also fail to satisfy this condition. To show the violation of **RCONT** by L^4, let P^4 denote its asymmetric component. Let X and Z be the same as before and let W $= \begin{pmatrix} 4 & 6 \\ 4 & 0 \end{pmatrix}$ so that W* $= \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$. It can be checked X P^4 W but $\neg$[Z P^4 W].

On the other hand, L^{Z} and L^{eZ} fail to satisfy **UM.** For instance, let X and Y be such that $X^* = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $Y^* = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Note that $X^* = BY^*$ where B is the bistochastic matrix $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. Hence, if (restated) L^{eZ} is to satisfy **UM**, it is required that X P^{eZ} Y (where P^{eZ} is the asymmetric component of L^{eZ}) i.e. that [(X*w) P (Y*w)] for all $w \geq 0$. However, if $w_1 = ½= w_2$, $Xw = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = Yw$, contradicting **UM**. It follows that L^Z would also violate this condition.

Moreover, the same is true of L^w. To show this, let $X = \begin{pmatrix} 3 & 2 \\ 1 & 6 \end{pmatrix}$ and Y $= \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}$. X and Y then belong to the domain on which L^w is defined. However, $X = QY$ where Q is the bistochastic matrix $\begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$. **UM**, therefore, requires X P^w Y where P^w is the asymmetric component of L^w. Noting that X and Y do not have a common row, we require that [(X*w(X*)) L^w (Y*w(Y*))] and ¬[(Y*w(Y*)) L^w (X*w(X*))] where, for all admissible X, w(X*) is as specified in the definition of L^w above. With that specification, however, it can be checked that w(X*) $= \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = w(Y^*)$. Since $X^* = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix}$ and $Y^* = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, we have: $X^*w(X^*) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = Y^*w(Y^*)$. Therefore, **UM** is contradicted.

This completes the demonstration of the fact that none of the seven binary relations on **X** reviewed above satisfies all of the definitional requirements of an MLDR.

4. A MULTIDIMENSIONAL LORENZ DOMINANCE RELATION

In this Section we suggest a binary relation on the set of distribution matrices and show that it is an MLDR as per Definition 2.4. The suggested relation is a modification of the relation L^4 (stated in the previous Section), the dominance relation based on columnwise majorization. The modification is based on the observation that L^4 satisfies all requirements of Definition 2.4 excepting **CIM**. In other words, if there was no need to take account of the interdependence between the distributions of the different attributes of standard of living, L^4 would provide a reasonable notion of inequality dominance: X is unambiguously more unequal than Y if $\mathbf{x}^j$ is unambiguously more unequal than $\mathbf{y}^j$ for all j in M.

Intuitively, in the special case of *independence* of the different attribute distributions the over-all degree of inequality in the economy could be taken to be a function of the unidimensional inequalities of the different attributes. In general, however, the contribution of an attribute toward the over-all degree of inequality (however measured) cannot be taken to be given by its "direct"(or "own") contribution. The indirect effects of the attribute through its interactions with the other attributes are to be taken into account. One aspect of this requirement is captured by the condition of **CIM** stated in Section 2. (Essentially, this aspect relates to the point that the magnitude of the interaction effect should be sensitive to rank correlations between the distributions of the attributes.) However, the issue is more general. For instance, even when $\mathbf{x}^{\mathrm{i}}$ and $\mathbf{x}^{\mathrm{j}}$ are comonotonic, the magnitude of the effect of interaction between inequalities in the distributions of the i-th and the j-th attributes should be allowed to change when $\mathbf{x}^{\mathrm{i}}$changes to, say, $\mathbf{y}^{\mathrm{i}}$ but $\mathbf{y}^{\mathrm{i}}$and $\mathbf{x}^{\mathrm{j}}$ are, again, comonotonic.

For simplicity, however, we shall make *three* assumptions in this context. First, it is assumed that, for all X in **X** and for all i and j in M, the indirect contribution of the distribution of the i-th attribute to over-all inequality through its interaction with the distribution of the j-th attribute is indicated by the inequality of the distribution given by the simple arithmetic mean of $\mathbf{x}^{\mathrm{i}}$ and $\mathbf{x}^{\mathrm{j}}$. This assumption incorporates some intuitively plausible features of such interdependence. For instance, the effect on over-all inequality of the "interaction" of an attribute with itself coincides with its "own" contribution. Moreover, the interaction effects would be symmetric: (On the other hand, this assumption restricts the effects of the interaction between the distributions $\mathbf{x}^{\mathrm{i}}$ and $\mathbf{x}^{\mathrm{j}}$ to be independent of $\mathbf{x}^{\mathrm{k}}$ for all k in M such that $i \neq k \neq j$.).

Secondly, the total contribution of an attribute to over-all inequality is assumed to be given by the arithmetic mean of the contributions made by it through its interaction with all the attributes.

In order to obtain a unit-free procedure, however, we shall, again, state our proposed criterion in term of the scaled versions of the distribution matrices.

For all j in M and for all X in **X**, let the vector $[(1\,/\,m)\sum_{k=1}^{m}(\mathbf{x}^{*\mathrm{j}} + \mathbf{x}^{*\mathrm{k}})\,/\,2]$ be denoted by $\mathrm{A}^{\mathrm{j}}(\mathrm{X}^*)$. Under the assumptions stated above $\mathrm{A}^{\mathrm{j}}(\mathrm{X}^*)$ can be interpreted as the allocation vector of the j-th attribute in the distribution matrix X augmented or modified so that the inequality of the augmented vector would reflect the total contribution of the attribute in this matrix.

The proposed dominance criterion is obtained by suggesting that if, for all j in M, the total contribution to inequality made by j in the distribution matrix X is not less than that in the matrix Y, however inequality may be measured, (i.e. if $\mathrm{A}^{\mathrm{j}}(\mathrm{X}^*)$ weakly Lorenz dominates $\mathrm{A}^{\mathrm{j}}(\mathrm{Y}^*)$ for all j in M), then X weakly Lorenz dominates Y.

Accordingly, consider the following binary relation, L^*, on **X**.

Definition 4.1: L^* is such that, for all X and Y in **X**, X L^* Y if and only if, for all j in M,

$$[(1/m)\sum_{k=1}^{m}(\mathbf{x}^{*\mathrm{j}} + \mathbf{x}^{*\mathrm{k}})/2]\;L[(1/m)\sum_{k=1}^{m}(\mathbf{y}^{*\mathrm{j}} + \mathbf{y}^{*\mathrm{k}})/2]\,.$$

P^* and I^* will denote the asymmetric and the symmetric components of L^* respectively.

Proposition 4.1: L^* is an MLDR as per Definition 2.4.

Proof: L^* satisfies **RSI** by construction. It can be checked that it satisfies **RCONT** as a simple consequence of the fact that L satisfies the unidimensional version of this condition.

QORD, again, is easily checked. To check **ANON** let X and Y in **X** be such that Y is a row permutation of X. Then $\mu(\mathbf{x}^{\mathrm{j}}) = \mu(\mathbf{y}^{\mathrm{j}})$ for all j in M; and, for all j and k in M, $[(\mathbf{x}^{*\mathrm{j}} + \mathbf{x}^{*\mathrm{k}})/2]$ is a permutation of $[(\mathbf{y}^{*\mathrm{j}} + \mathbf{y}^{*\mathrm{k}})/2]$. Therefore, $\mathrm{A}^{\mathrm{j}}(\mathrm{X}^*)$ is a permutation of $\mathrm{A}^{\mathrm{j}}(\mathrm{Y}^*)$. Since L satisfies Anonymity, it follows that, X I^* Y. To see that L^* satisfies **PRI**, let X and Y in **X** be such that Y is obtained by a q-fold population replication of X for a positive integer q. For all j and k in M, $[(\mathbf{y}^{*\mathrm{j}} + \mathbf{y}^{*\mathrm{k}})/2]$ is now a q-fold replication of $[(\mathbf{x}^{*\mathrm{j}} + \mathbf{x}^{*\mathrm{k}})/2]$. The result now follows from the fact that L satisfies Population Invariance.

To show that L^* satisfies **UM**, let X and Y in **X** be such that X = BY where B is a bistochastic (but not a permutation or an identity) matrix. Then X* = BY*. Therefore, for all j in M, $\mathbf{x}^{*\mathrm{j}} = \mathrm{B}\mathbf{y}^{*\mathrm{j}}$ and, for at least one j in M, $\mathbf{x}^{*\mathrm{j}} \neq \mathbf{y}^{*\mathrm{j}}$. Hence, for all j and k in M, $[(\mathbf{x}^{*\mathrm{j}} + \mathbf{x}^{*\mathrm{k}})/2] = \mathrm{B}[(\mathbf{y}^{*\mathrm{j}} + \mathbf{y}^{*\mathrm{k}})/2]$ i.e. $[\mathbf{x}^{*\mathrm{j}} + \mathbf{x}^{*\mathrm{k}})/2]$ Pigou-Dalton majorizes $[(\mathbf{y}^{*\mathrm{j}} + \mathbf{y}^{*\mathrm{k}})/2]$. It can be checked that under these circumstances, for all j in M, $\mathrm{A}^{\mathrm{j}}(\mathrm{X}^*) = \mathrm{BA}^{\mathrm{j}}(\mathrm{Y}^*)$ i.e. $\mathrm{A}^{\mathrm{j}}(\mathrm{X}^*)$ P $\mathrm{A}^{\mathrm{j}}(\mathrm{Y}^*)$. Thus, X P^* Y.

To prove that L^* satisfies **PDBP** (i.e. to show that if X and Y in **X** are such that Y is obtained from X by a *finite* sequence of PDBT's, then Y P^* X), first suppose that Y is obtained from X by a *single* PDBT. Recall that, according to Definition 2.1 of PDBT, this implies that there exist q and p in N such that $\mathbf{x}_{\mathrm{q}} > \mathbf{x}_{\mathrm{p}}$ and that positive amounts of one or more attributes are transferred from individual q to individual $p$subject to the restriction that $\mathbf{y}_{\mathrm{q}} \geq \mathbf{y}_{\mathrm{p}}$.

Let j in M be such that the PDBT in question does not involve transfers of the j-th attribute. If the same is true of k in M, then $[(\mathbf{y}^{*\mathrm{j}} + \mathbf{y}^{*\mathrm{k}})/2]$ equals $[(\mathbf{x}^{*\mathbf{j}} + \mathbf{x}^{*\mathrm{k}})/2]$, irrespective of whether $k = j$ or not. If, however, $k \neq j$and if a positive amount of the k-th attribute is transferred, then the former vector is a Pigou-Dalton majorization of the latter. On the other hand, if a positive amount of attribute j is transferred, then this is true irrespective of whether any amount of the k-th attribute has been transferred and irrespective, again, of whether $j = k$ or not. Moreover, by definition, the PDBT involves transfer of at least one of the attributes. It can be checked that these statements imply that, for all j in M, $\mathrm{A}^{\mathrm{j}}(\mathrm{Y}^*)$ is a Pigou-Dalton majorization of $\mathrm{A}^{\mathrm{j}}(\mathrm{X}^*)$. Thus, $[\mathrm{A}^{\mathrm{j}}(\mathrm{Y}^*)]$ P $[\mathrm{A}^{\mathrm{j}}(\mathrm{X}^*)]$ for all j in M. Hence, Y P^* X.

If, now, Y is obtained from X by a finite sequence of PDBT's but not necessarily by a single such transfer, the same conclusion is reached by virtue of **QORD**.

To check **CIM** let X and Y in **X** be such that Y is obtained from a finite sequence of CIT's but is not a row permutation of X. To show that X P^* Y, by **QORD,** again, it suffices to prove this for the case where Y is obtained from X by a single CIT. In this case Y* is obtained from X* by a CIT. Definition 2.2 of a CIT can be used to show that, for any given j in M, one of the following two statements is true for any k in M:

(i) $[(\mathbf{x}^{*\mathrm{j}} + \mathbf{x}^{*\mathrm{k}}) / 2] = [(\mathbf{y}^{*\mathrm{j}} + \mathbf{y}^{*\mathrm{k}}) / 2]$;

(ii) $[(\mathbf{x}^{*\mathrm{j}} + \mathbf{x}^{*\mathrm{k}}) / 2]$ is a Pigou-Dalton majorization of $[(\mathbf{x}^{*\mathrm{j}} + \mathbf{x}^{*\mathrm{k}}) / 2]$.

Moreover, since X is neither equal to Y nor a row permutation of Y, (ii) is true for at least one k in M. These statements can be used to show that, for all j in M, $A^j(X^*)$ is a Pigou-Dalton majorization of $A^j(Y^*)$. Definition 4.1, therefore, implies the desired conclusion.

This completes the proof of the fact that L^* is an MIDR as per Definition 2.3. Since $L^* = L$ if $m = 1$, it is an MLDR as per Definition 2.4. □

5. CONCLUSION

In this paper we have sought to formulate a definition of an MLDR on the set of distribution matrices by using a number of conditions which seem to reflect the basic requirements of such a relation. It is seen, however, that none of the relations that have so far been proposed in the literature in this context is an MLDR as per the definition developed here. The question, therefore, arises as to whether such a relation exists. We have sought to give an affirmative answer to the question by proposing a new dominance relation.

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