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PRICE RIGIDITY AND USE OF MONEY

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Abstract: We consider an economy in which the price mechanism in competitive markets partially malfunctions; namely some prices do not move in response to discrepancy between supply and demand. We formalize this case as a model where some prices are flexible as usual whereas others are fixed. On the assumption that every agent is a price taker, we show that even in this situation general equilibria do exist, but at the cost of two important properties; namely, decisive relative prices and determinacy of equilibrium. On one hand, ineffectiveness of relative prices naturally induces use of money. We show, on the other hand, that money is of great use to solve the problem of indeterminacy of equilibrium. Specifically, money proves generically to yield the local determinacy of equilibrium under some condition, which also implies that money substantially affects a real economy.

Key words: Flex-fix prices, relative and absolute prices, indeterminacy of equilibrium, money, local uniqueness of equilibrium.

JEL Classification Number: C62, D51.

1. INTRODUCTION

Needless to say, our economy is crucially dependent upon a competitive market system that is usually formalized as a model in which all commodities are traded in perfectly competitive markets. In a perfectly competitive market, a price of a good is assumed to change smoothly in response to discrepancy between demand and supply for the good. In reality, however, this flexibility of a price does not always hold for every good. We actually see many prices more or less rigid. From a realistic viewpoint, there are various reasons causing price rigidity, ranging from institutional ones (eg. parity price, rent control etc.) to technological ones (menu cost, search cost, switching cost

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I am grateful to an anonymous referee for his/her valuable comments on my original paper. Thanks to those comments, I believe the paper was much improved. But, needless to say, I am responsible for all errors that may still remain in the paper.

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etc., see Levy (2007)). In this paper, reflecting the above realistic view, we consider an economy with a market failure in the sense of price rigidity. In order to articulate the malfunction of the price mechanism, we adopt a model in which flexible and rigid prices coexist. Specifically we assume that each good belongs to *either* the fix price market *or* the flexible price market. We do not refer to the determination of the fix prices and only take them as given. Thus, we need some specific adjuster for those markets with fix prices. Otherwise, we would always suffer from disequilibrium in those markets[1]. Following the literature dealing with this kind of price rigidity, we consider a rationing scheme for those markets. Thus, in our model, the flexible price markets are based on a normal competitive price mechanism whereas the fix price markets are administered through a rationing scheme. It is worth noting that every agent is assumed to be a price taker for both types of markets. In other words, we assume that every agent directly or indirectly does not pay attention to others' behavior at all, which is crucially different from the classic literature such as Drèze (1975) and Bénassy (1975).

In our model presented below it is shown without any particular difficulty that there exist equilibria. This implies that the competitive price mechanism has a robust property with respect to the existence of equilibrium in the sense that even with a partial malfunction of the price mechanism we can always retain the equilibrium. It, however, turns out that we have a different kind of difficulty at the same time. Specifically, confining ourselves to relative prices does not make sense in our model while the equilibrium absolute prices constitute a continuum, which implies indeterminacy of equilibrium. What is worse, each equilibrium price system yields a different resource allocation, that is to say, we have indeterminacy of equilibrium *in real term*. This difficulty is shown to be caused by the fact that in our model the demand function of each agent loses the homogeneity in flexible prices while Walras' law holds, which gives discrepancy between the number of equations and the number of variables (=the number of flexible prices) in the equation system expressing the equilibrium condition.

In order to handle this difficulty, we note a role of money. It is common in practice to neglect money in the standard competitive equilibrium model. It is because without money we are allowed to have specific relative prices that equalize the quantity demanded with the quantity supplied for every commodity. Money, if introduced in this situation, will only determine the absolute level of prices without any effect on the real side of an economy. In our model including fix price markets, however, we have to consider absolute prices instead of relative prices as described above. Moreover, these absolute prices are allowed to take any value between 0 and $+\infty$. In these circumstances, a certain amount of money is naturally necessitated not only because some unit of account is required but also because some limit is needed to determine the level of absolute prices. Once money is introduced, we should think of the medium of exchange as well as the unit of account for a role of money. Then, it turns out that

[1] If a fix price market has no adjustment mechanism of its own, flexible prices have to adjust fix price markets as well as their own markets. Then, the equilibrium condition leads to an excess in the number of equations over the number of variables, typically resulting in no equilibrium. Thus, we consider the rationing scheme as an adjustment mechanism in a fix price market.

these two functions of money enable us to successfully cope with the difficulty above mentioned. Specifically, by introducing money into our model in an appropriate way of reflecting these two functions, we succeed in generically proving the local determinacy of the equilibrium. Incidentally, it is worth noting that the third role of money, namely the store of value, should not be admitted in our model because of its purely static framework.

Finally, we should refer to another view on an economy with price rigidity which has been taken by some researchers who are interested in a role of government particularly in underdeveloping countries. Vasil'ev and Wiesmeth (2008) have investigated an economy consisting of two types of market similar to ours. But in their mixed-market economy, each commodity has both the fix price market and the flexible price market. The feature of their model consists in the role of the government in the fix price market, where the government controls every aspect of the rationing, fixing a price and assigning a quota for each agent. The reason why they work with this specific modeling is that they have in mind the transition economies of the NIS and some other countries (Vasil'ev and Wiesmeth (2008), p.132). In the literature we see many authors adopt this sort of model (Makarov et al. (1995), Vasil'ev (1996, 1999), Sidorov and Vasil'ev (1997), van der Laan et al. (2000)). Contrary to ours, their technical difficulty lies in the proof of existence of equilibrium, which is, technically speaking, caused by the nonlinearity of the income function with respect to flexible prices (Vasil'ev and Wiesmeth, *op.cit.*). More importantly, money is dispensable in their arguments.

In section 2, after an exposition about the specific structure of our mixed-market model, we show a difficulty in defining an equilibrium in that model. Then we propose a suitable equilibrium concept and demonstrate its existence. However, we also refer to indeterminacy of equilibrium that is inevitable. At the end, we give a numerical example that will help us understand the problem we face. In section 3, after showing necessity of money we demonstrate that through introduction of money we can succeed in overcoming our problem. Specifically, we can show the determinacy of equilibrium prices from the generic viewpoint. In the last section, after referring to the relevant literature, we summarize our findings and stress the important working of money that has been overlooked.

2. MIXED-MARKET ECONOMY AND EXISTENCE OF EQUILIBRIA

In our mixed-market economy, we have to consider both flexible price markets and fix price markets at the same time. Since we consider a partial malfunction of the competitive price mechanism, it is legitimate to assume the market structures of these two types to be equal except for the price rigidity and the rationing scheme in a fix price market. It follows that how a fix price is determined is consciously left unnoticed. To keep matters simple, we confine ourselves to a pure exchange economy.

Thus, our model consists of I agents, n goods of flexible prices and m goods of fix prices. Each agent, indexed by i, is characterized by his/her own utility function u^i and initial endowments $\boldsymbol{\omega}^i \, (= (\omega^i_1, \ldots, \omega^i_n, \omega^i_{n+1}, \ldots, \omega^i_{n+m}))$ on which we make following assumptions.

ASSUMPTION 1. Each utility function u^i $(i = 1, \ldots, I)$ satisfies the following conditions:

1. $u^i : \boldsymbol{R}_{+}^{n+m} \to \boldsymbol{R}$ is continuous, monotone and strictly quasi-concave.
2. if $U^i(\bar{\boldsymbol{x}}) = \{\boldsymbol{x} \in \boldsymbol{R}_{+}^{n+m} \mid u^i(\boldsymbol{x}) \geq u^i(\bar{\boldsymbol{x}})\}$, then $U^i(\bar{\boldsymbol{x}}) \subset \boldsymbol{R}_{++}^{n+m}$ for each $\bar{\boldsymbol{x}} \in \boldsymbol{R}_{++}^{n+m}$.

ASSUMPTION 2. $\boldsymbol{\omega}^i \in \boldsymbol{R}_{++}^{n+m}$ for all i

In the following, we use the subscript c to signify flexible price goods since a flexible price market may be identified with a competitive market. Thus, $\boldsymbol{p}^c$ denotes a flexible (competitive)-price vector. On the other hand, $\bar{\boldsymbol{p}}$ is a fix price vector, where $\bar{\boldsymbol{p}}$ is arbitrarily given and fixed. For these vectors, we assume the following condition.

ASSUMPTION 3. $\boldsymbol{p}^c \in \boldsymbol{R}_{++}^{n}$ and $\bar{\boldsymbol{p}} \in \boldsymbol{R}_{++}^{m}$

For the sake of notation, we write a consumption vector $\boldsymbol{x}^i$ of agent i as $(\boldsymbol{x}_c^i, \boldsymbol{x}_f^i)$ where the subscript f is supposed to denote fix price goods. Componentwisely, $\boldsymbol{x}_c^i = (x_{c1}^i, \ldots, x_{cn}^i)$ $(resp.\ \boldsymbol{x}_f^i = (x_{f1}^i, \ldots, x_{fm}^i))$, $i = 1, \ldots, I$. For the initial endowments, the similar notation is applied; that is, $\boldsymbol{\omega}^i = (\boldsymbol{\omega}_c^i, \boldsymbol{\omega}_f^i)$. For simplicity, we call flexible(i.e., competitive)-price goods c-goods and also call fix price goods f-goods in the following.

In order to define rationing schemes, we consider the net demand for f-goods, that is, $\boldsymbol{x}_f^i - \boldsymbol{\omega}_f^i$ which is denoted by $\tilde{\boldsymbol{x}}_f^i$, $i = 1, \ldots, I$. Then, we adopt a rationing function a là Bénassy (Bénassy (1975, 1982, 2002)) as a rationing system.

DEFINITION 1. A system of rationing schemes for each agent $i (\in \{1, \ldots, I\})$ is described by a map $F^i : \boldsymbol{R}^{mI} \to \boldsymbol{R}^m$ such that $\sum_i^I F^i(\tilde{\boldsymbol{x}}_f^1, \ldots, \tilde{\boldsymbol{x}}_f^I) = \boldsymbol{0}$ where the arguments of each component function F_j^i are $(\tilde{x}_{fj}^1, \ldots, \tilde{x}_{fj}^I)$, $j = 1, \ldots, m$.

Now we turn to the definition of equilibrium of the mixed-market economy with the rationing scheme.

Given a mixed-market economy specified by $\mathcal{E}(\bar{\boldsymbol{p}}, \{u^i, \boldsymbol{\omega}^i, F^i\}_i)$, it seems plausible to think of an allocation $(\{\boldsymbol{x}_c^{i*},\ \boldsymbol{x}_f^{i*}\}_i)$ and a price vector $\boldsymbol{p}^{c*}$ satisfying the following conditions as an equilibrium.

(i) for every i, $(\boldsymbol{x}_c^{i*},\ \boldsymbol{x}_f^{i*})$ maximizes $u^i(\boldsymbol{x}_c^i, \boldsymbol{x}_f^i)$ in the budget set $\{(\boldsymbol{x}_c^i, \boldsymbol{x}_f^i) \in \boldsymbol{R}_+^n \times \boldsymbol{R}_+^m \mid \boldsymbol{p}^{c*}\boldsymbol{x}_c^i + \bar{\boldsymbol{p}}\boldsymbol{x}_f^i = \boldsymbol{p}^{c*}\boldsymbol{\omega}_c^i + \bar{\boldsymbol{p}}\boldsymbol{\omega}_f^i\}$

(ii) $\sum_i^I \boldsymbol{x}_c^{i*} = \sum_i^I \boldsymbol{\omega}_c^i$

(iii) $F^i(\boldsymbol{x}_f^{1*} - \boldsymbol{\omega}_f^1, \ldots, \boldsymbol{x}_f^{I*} - \boldsymbol{\omega}_f^I) = \boldsymbol{x}_f^{i*} - \boldsymbol{\omega}_f^i,\ i = 1, \ldots, I$

It is, however, easily seen that an equilibrium provided above is generally unobtainable because $\boldsymbol{x}_c^{i*}$ and $\boldsymbol{x}_f^{i*}$ as a function in $\boldsymbol{p}^c$ derived from (i) cannot meet (ii) and (iii) at the same time (the number of unknowns (n) is much less than that of equations (n+Im)). Thus, we need to modify the notion of equilibrium in a mixed-market economy, though a resultant equilibrium is at most second best. We present a desirable equilibrium in the following.

We first consider the behavior of each agent facing two types of goods. As we have stated before, a fix price market is assumed to be competitive except for the price rigidity and the rationing scheme, which implies that every agent is a price taker even for the fix price markets[2]. However, a fix price market is equipped with a decisive rationing scheme through which every agent is given a certain tradable quantity which will most likely be different from the quantity he/she is willing to trade. Thus, a rational agent would be conscious that if he/she declares his/her demand or supply for all goods at the same time, the consequent trade may violate his/her budget constraint. This consideration leads us to formalize the behavior of an agent through two steps.

Step 1.

The optimization problem an agent i faces is obviously expressed as follows ($i = 1, \ldots, I$):

$$\begin{aligned} \max \quad & u^i(\boldsymbol{x}^i_c, \boldsymbol{x}^i_f) \\ s.t. \quad & \boldsymbol{p}^c \boldsymbol{x}^i_c + \bar{\boldsymbol{p}} \boldsymbol{x}^i_f = \boldsymbol{p}^c \boldsymbol{\omega}^i_c + \bar{\boldsymbol{p}} \boldsymbol{\omega}^i_f \end{aligned}$$

Agent i is aware that the simultaneous offer of all the solutions of this problem to all markets may result in nonfulfilment of his/her budget. Thus, it matters to him/her which is first among c and f goods. If c-goods precede f-goods, the price mechanism will eventually yield an equilibrium in c-goods markets, determining a specific price system. Then, on the basis of the price system agent i determinates the quantities demanded or supplied of f-goods which must undergo a rationing scheme. However, the consequent rationed quantities are not guaranteed to meet his/her final budget constraint. Therefore, it is legitimate to give priority to f-goods over c-goods. Let us consider his/her demand for f-goods that is given by the solution of this problem, which can be described as follows.

$$\boldsymbol{x}^i_f = \boldsymbol{x}^i_f(\boldsymbol{p}^c)$$

Let $\tilde{\boldsymbol{x}}^i_f(\boldsymbol{p}^c)$ denote $\boldsymbol{x}^i_f(\boldsymbol{p}^c) - \boldsymbol{\omega}^i_f,\ i = 1, \ldots, I$. Then, for this net demand each i is rationed with $F^i(\tilde{\boldsymbol{x}}^1_f(\boldsymbol{p}^c), \ldots, \tilde{\boldsymbol{x}}^I_f(\boldsymbol{p}^c))$ which is denoted by $\hat{\boldsymbol{x}}^i_f(\boldsymbol{p}^c),\ i = 1, \ldots, I$. As we have stated above, through the competitiveness of all the markets, every agent does not pay any attention to others' demand in fix price markets as well as flexible price markets. Thus, we may properly assume that each i takes $\hat{\boldsymbol{x}}^i_f(\boldsymbol{p}^c)$ as given.

Step 2.

After dealing with f-goods, each agent turns to the determination of the demands for c-goods. It follows that each i ought to be concerned with the following problem.

$$\begin{aligned} \max \quad & u^i(\boldsymbol{x}^i_c, \hat{\boldsymbol{x}}^i_f(\boldsymbol{p}^c) + \boldsymbol{\omega}^i_f) \\ s.t. \quad & \boldsymbol{p}^c \boldsymbol{x}^i_c = \boldsymbol{p}^c \boldsymbol{\omega}^i_c - w^i(\boldsymbol{p}^c) \end{aligned}$$

[2] Thus, we may assume that each agent is not aware of the strategic interrelation through the rationing and that he/she does not behave strategically. It is worth noting that Bénassy's 'quantity tâtonnement' known as an adjustment process in fix price markets is inapplicable here because his process is crucially dependent on the so called perceived constraints which are formed by each agent's estimation of others' quantities demanded and supplied (Bénassy, *op. cit.*).

where $w^i(\boldsymbol{p}^c) = \bar{\boldsymbol{p}}\hat{\boldsymbol{x}}^i_f(\boldsymbol{p}^c),\ i = 1, \ldots, I$.

Let the demand function for c-goods derived by the solution of this problem be $\hat{\boldsymbol{x}}^i_c(\boldsymbol{p}^c),\ i = 1, \ldots, I$. It is worth noting that this is different from the solution for c-goods in Step 1. The latter solution, after the rationing, does not necessarily meet the budget constraint while this solution $\hat{\boldsymbol{x}}^i_c(\boldsymbol{p}^c)$ does meet it.

Now that we have formalized the behavior of each agent, we are in a position to define an equilibrium of our model. The definition of an equilibrium should be solely based on $\hat{\boldsymbol{x}}^i_c(\boldsymbol{p}^c),\ i = 1, \ldots, I$ obtained above since the influence of fix price markets has already been incorporated into them.

DEFINITION 2. An equilibrium for a mixed-market economy $\mathcal{E}(\bar{\boldsymbol{p}}, \{u^i, \boldsymbol{\omega}^i, F^i\}_i)$ is characterized by a price vector $\boldsymbol{p}^c$ such that $\sum_i^I \hat{\boldsymbol{x}}^i_c(\boldsymbol{p}^c) = \sum_i^I \boldsymbol{\omega}^i_c$

In order for the equilibrium to be well defined, we need an additional assumption on a rationing scheme.

ASSUMPTION 4. For all i, F^i_j $(j = 1, \ldots, m)$ meets following conditions.

1. continuity in all arguments.
2. nondecrease in $\tilde{x}^i_{fj}$.
3. nonincrease in the other arguments than $\tilde{x}^i_{fj}$.
4. non-manipulability; every agent, once rationed, cannot increase the level of his/her transactions as well as others' transactions by increasing his/her demand or supply.
5. real and nominal voluntary exchange, where
   (a) real voluntary exchange:
   $\tilde{x}^i_{fj} F^i_j(\tilde{x}^1_{fj}, \ldots, \tilde{x}^I_{fj}) > 0$ and $|\tilde{x}^i_{fj}| \geq |F^i_j(\tilde{x}^1_{fj}, \ldots, \tilde{x}^I_{fj})|\ i = 1, \ldots, I,\ j = 1, \ldots, m$
   (b) nominal voluntary exchange:
   $(\bar{\boldsymbol{p}} \cdot \tilde{\boldsymbol{x}}^i_f)(\bar{\boldsymbol{p}} \cdot F^i(\tilde{\boldsymbol{x}}^1_f, \ldots, \tilde{\boldsymbol{x}}^I_f) > 0$ and $|\bar{\boldsymbol{p}} \cdot \tilde{\boldsymbol{x}}^i_f| \geq |\bar{\boldsymbol{p}} \cdot F^i(\tilde{\boldsymbol{x}}^1_f, \ldots, \tilde{\boldsymbol{x}}^I_f)|\ i = 1, \ldots, I$

These conditions above mentioned are more or less normal in the literature.

Under the given assumptions, we have a substantial proposition.

THEOREM 1. *In an economy $\mathcal{E}(\boldsymbol{p}, \{u^i, \boldsymbol{\omega}^i, F^i\}_i)$ there exist an equilibrium continuum that is unbounded, where 'unbounded' means that the continuum set of equilibrium price vectors $\boldsymbol{p}^c$ cannot be covered by any ball with a finite radius in $\boldsymbol{R}^n$. Each equilibrium in the continuum yields a different resource allocation.*

Proof. See Appendix.

This theorem contains both good and bad news. Good news is that whatever the ratio of n to m, flexible price markets compensate the rigidity of fix price markets, leading to equilibria. In this sense, the competitive price mechanism turns out to have a robust property in regard to the existence of equilibrium. On the other hand, bad news is the indeterminacy of equilibria. We are not able to locate any equilibrium. In order to ascertain the statement of the theorem, we provide an illustrative numerical example below.

Numerical example

We consider a pure exchange economy with two consumers ($i = A, B$) and three goods ($j = 1, 2, 3$). Let the initial endowments($(\bar{x}^i_j)_{j=1,2,3}$) of consumers ($i = A, B$) be (3, 4, 5) and (5, 4, 3) respectively. We assume that the price of the third good is fixed and set at 1. The price of other goods is denoted by $p_j (j = 1, 2)$. A utility function of each consumer is assumed to be expressed as follows.

$$u^A = x_1 \cdot x_2 \cdot x_3, \quad u^B = x_1 + 2x_2^{\frac{1}{2}} + \frac{1}{3}x_3^2 .$$

According to Step 1, we first calculate the demand for f-goods (good 3) by each consumer, obtaining the following.

$$x_3^A = p_1 + \frac{4}{3}p_2 + \frac{5}{3}, \quad x_3^B = \frac{3}{2p_1} .$$

We have two possible cases; (i) $x_3^A - 5 \geq 0$ and $x_3^B - 3 \leq 0$, (ii) $x_3^A - 5 < 0$ and $x_3^B - 3 > 0$. It suffices to only consider case (i). The other can be treated in the same way as described below. Since we have only two consumers, we may consider the short side rule to be the only possible rationing scheme. Then, the consequence of the rationing depends on the relative size of $|x_3^A - 5|$ and $|x_3^B - 3|$. Again, it suffices to consider one of two possible cases. We pick up the case in which $|x_3^A - 5| \geq |x_3^B - 3|$. In this case, consumer B can meet his/her desire (supply) whereas consumer A cannot. Obviously, we have that $F^A(p_1, p_2) = -(x_3^B - 3) = 3 - \frac{3}{2p_1}$ and $F^B(p_1, p_2) = x_3^B - 3 = \frac{3}{2p_1} - 3$.

We proceed to Step 2. Each consumer, then, confronts the following optimization problem.

$$\max\ u^A = x_1 \cdot x_2 \cdot \Big(8 - \frac{3}{2p_1}\Big) \quad s.t. \quad p_1x_1 + p_2x_2 = 3p_1 + 4p_2 - 3 + \frac{3}{2p_1}$$

$$\max\ u^B = x_1 + 2x_2^{\frac{1}{2}} + \frac{1}{3}\Big(\frac{3}{2p_1}\Big)^2 \quad s.t. \quad p_1x_1 + p_2x_2 = 5p_1 + 4p_2 + 3 - \frac{3}{2p_1}$$

It is worth noting that summing up their budget constraints leads to Walras' law; that is, $p_1(x_1^A + x_1^B - 8) + p_2(x_2^A + x_2^B - 8) = 0$. Thus, we have only to focus on one of the two goods and consider its equilibrium condition. Pick good 2, then, through calculation we have the following equilibrium condition for it.

$$\frac{1}{2p_2}\Big(3p_1 + 4p_2 - 3 + \frac{3}{2p_1}\Big) + \Big(\frac{p_1}{p_2}\Big)^2 = 8 .$$

This equation cannot be solved for $\frac{p_1}{p_2}$. Any pair (p_1, p_2) satisfying the above equation could be an equilibrium price system. More precisely, by particular case conditions (namely, $|x_3^A - 5| \geq |x_3^B - 3|$ as well as $x_3^A - 5 \geq 0$ and $x_3^B - 3 \leq 0$), there exists some $\underline{p_1}$ ($3 < \underline{p_1} < 4$) such that for $p_1 \geq \underline{p_1}$ any (p_1, p_2) satisfying the above equation consists of an equilibrium price system. Note that if $p_1 \geq \underline{p_1}$, then a strictly positive p_2 is always obtainable.

It is easily seen that each equilibrium price system yields a different resource allocation by the fact that for consumer B we always have that $x_3^B = \frac{3}{2p_1}$.

3. ROLE OF MONEY IN A MIXED-MARKET ECONOMY

3.1. Introduction of money: monetary equilibrium

Let us recall the basic reason why we have indeterminacy of equilibria. We need to stress two observations. Observation (1) Since both c-goods and f-goods are included in the unique budget constraint of each agent, his/her demand function for c-goods cannot be homogeneous of degree 0 in $\boldsymbol{p}^c$ unless he/she has no initial endowments of f-goods. Observation (2) Since we always have the balance of supply and demand for f-goods through the rationing schemes, the sum of the budget constraints over all agents leads to Walras' law for c-goods. Through these observations, we necessarily have discrepancy between the number of equations and the number of variables (=the number of flexible prices) in the equation system expressing an equilibrium state of the economy. More specifically, the number of equations is just one less than the number of variables in the system, which results in the indeterminacy of equilibria.

In order to cope with this problem, we consider an implication given by observation (1). Since the demand for c-goods is not homogeneous of degree 0 in $\boldsymbol{p}^c$, we cannot take $\boldsymbol{p}^c$ as relative prices. In other words, we must consider them to be absolute prices. Moreover, a serious problem about these absolute prices is that each one can take any value between 0 and $+\infty$. Thus, to avoid this incovenience, we are naturally required to have a certain amount of entity which plays a role of a unit of account. Nothing can be conceived but money as such an entity. So, let us see what will happen to the morass of indeterminacy if a certain amount of money is introduced into our model.

Money, put in our model, turns out to be a very special good. Because money can be seen as a fix price good (as a numéraire) while it is never rationed. Thus, the introduction of money adds one clearance condition without any effect on the number of variables, which quite likely serves to solve the above problem. Before that,however, we need a more or less reasonable way of incorporating money into our model.

As is well known, money has three fundamental workings; namely the unit of account, the medium of exchange and the store of value. Among them, the last one, i.e., the store of value, should be out of consideration in our model since our model is intrinsically static like a standard general competitive model; namely we do not take time passage into account in the model-building. In addition, our model, as well as a standard competitive model, assumes that an actual trade can only take place once for all after an equilibrium is established. Hence, all agents in our model have no incentive to hold money after trade, which implies that money should not be included in a budget constraint as well as a utility function of each agent since otherwise some one must hold money after trade. Actually, money is necessitated only once for the trade. In these circumstances, we are forced to artificially model the introduction of money as follows. That is, there exists the monetary authority that pumps a certain amount of money into an economy just before the trade and then withdraws them all after the trade. This is actually a popular convention in dealing with money in a static model (see, e.g., Gale (1982), esp., chap. 6).

The next problem is to consider how money is distributed among agents before trade. If we follow Clower (Clower (1967)) and adopt the view of 'cash in advance', it is appropriate to provide each agent with enough amount of money to meet his/her net demand. Along this line, the introduction of money would be formally expressed as follows.

$$\bar{M} = \sum_{i=1}^{I} m_i \,,$$
$$m_i = \boldsymbol{p}^c max\{\boldsymbol{x}^i_c(\boldsymbol{p}^c) - \boldsymbol{\omega}^i_c, \boldsymbol{0}\} + \bar{\boldsymbol{p}}\; max\{\boldsymbol{x}^i_f(\boldsymbol{p}^c) - \boldsymbol{\omega}^i_f, \boldsymbol{0}\} \quad i = 1, \ldots, I \,.$$

where $\bar{M}$ designates a total amount of money that is the so called money supply while m_i denotes a specific amount of money distributed to agent i ($i = 1, \ldots, I$). The sign max should be interpreted elementwisely for the relevant two vectors. It is worth noting that through trade the sum of money received by agent i will be $\boldsymbol{p}^c|min\{\boldsymbol{x}^i_c(\boldsymbol{p}^c - \boldsymbol{\omega}^i_c, \boldsymbol{0}\}| + \bar{\boldsymbol{p}}\;|min\{\boldsymbol{x}^i_f(\boldsymbol{p}^c) - \boldsymbol{\omega}^i_f, \boldsymbol{0}\}|$ which is equal to m_i by his/her budget constraint ($i = 1, \ldots, I$). Thus, the same amount of money as before will be withdrawn from every agent after trade.

This formulation, however, raises some behavioral problem. An agent, given some amount of money, would have an incentive to spend all and not to hold it again; namely he/she would be motivated to buy without selling. In order to get an agent to return the specific amount of money, we need some legal force or other (for instance, Gale needs a tax system for this purpose, see Gale (1982), *op.cit.*). Indeed, we can proceed along this line to obtain desirable consequences, but there is another way of controlling money which is feasible without resort to governmental compulsion, and moreover, facilitates our analysis. The way is to distribute money *not* to agents *but* to markets, i.e. auctioneers. Specifically, the authority provides each market with the amount of money equivalent to the aggregate supply in the market. Then, an agent sells his/her net supply to each market. Since every agent necessarily spends all his/her money to meet his/her demand, after trade each market always has the same amount of money as before, which is returned to the authority. This procedure is described by the following way of money distribution.

$$\bar{M} = \sum_{k=1}^{n} m^c_k + \sum_{j=1}^{m} m^f_j$$
$$m^c_k = \sum_{i=1}^{I} p^c_k |min\{x^i_{ck}(\boldsymbol{p}^c) - \omega^i_{ck}, 0\}| \quad k = 1, \ldots, n$$
$$m^f_j = \sum_{i=1}^{I} \bar{p}_j |min\{x^i_{fj}(\boldsymbol{p}^c) - \omega^i_{fj}, 0\}| \quad j = 1, \ldots, m \,.$$

It is worth noting that $\sum_{k=1}^{n} m_k^c + \sum_{j=1}^{m} m_j^f$ is always equal to $\sum_{i=1}^{I} m_i$. We adopt this method in the following. To be emphasized here, the more orthodox method described before can also be applicable in our argument but needs complicated treatment compared to our procedure. Since both lead to the same concequences, it is advisable to use this specific way for efficiency of our analysis.

We make an assumption about agents' supply to markets that is just for simplification of later analysis.

ASSUMPTION 5. Each agent supplies all his/her initial endowments to markets to get money.

It easily follows from this assumption that the relevant money distribution leads to the following condition.

$$\boldsymbol{p}^c \sum_i^I \boldsymbol{\omega}_c^i + \bar{\boldsymbol{p}} \sum_i^I \boldsymbol{\omega}_f^i = \bar{M}$$

Considering this condition, we are allowed to define the equilibrium for a mixed-market economy with money.

DEFINITION 3. An equilibrium for a mixed-market economy with money $\mathcal{E}(\bar{\boldsymbol{p}}, \{u^i, \boldsymbol{\omega}^i, \bar{M}, F^i\}_i)$ is characterized by a price vector $\boldsymbol{p}^c$ which satisfies
(1) $\sum_i^I \hat{\boldsymbol{x}}_c^i(\boldsymbol{p}^c) = \sum_i^I \boldsymbol{\omega}_c^i$
(2) $\boldsymbol{p}^c \sum_i^I \boldsymbol{\omega}_c^i + \bar{\boldsymbol{p}} \sum_i^I \boldsymbol{\omega}_f^i = \bar{M}$

3.2. Determinacy of equilibrium

Let us consider how the set of equilibria is in the model including money. First, we immediately obtain the following existence result.

PROPOSITION 1. *If $\bar{M} > \bar{\boldsymbol{p}} \sum_i^I \boldsymbol{\omega}_f^i$, then in an economy $\mathcal{E}(\bar{\boldsymbol{p}}, \{u^i, \boldsymbol{\omega}^i, \bar{M}, F^i\}_i)$ equilibrium price systems are bounded.*

Proof. As the proof of theorem 1 has shown, without condition (2) *of definition 3 we would have the unbounded set of equilibrium price systems. However, by this condition $\boldsymbol{p}^c \sum_i^I \boldsymbol{\omega}_c^i = \bar{M} - \bar{\boldsymbol{p}} \sum_i^I \boldsymbol{\omega}_f^i > 0$, which implies that $\boldsymbol{p}^c$ must be bounded. Thus, the claim is immediate.*

Remark [1]: In the process of equilibrium, we should note that the amount of money delivered to each market is determined simultaneously with the price of each c-good. More specifically, we should think of the following tâtonnement process. First, call a tentative price for each c-good market. Then, every agent offers its demand or supply for all goods, which brings about a specific amount of money to be delivered to each market (see m_k^c and m_j^f provided above) that can be seen as money demand. If all goods markets and money market (supply of money is given as $\bar{M}$ that should satisfy the condition of proposition 1) are not cleared by the nominated prices, then all prices are cancelled and some other prices are called. This process is repeated until both goods and money markets are cleared although we do not address the stability of equilibrium.

Remark [2]: The condition of the proposition is crucial. Otherwise, nobody could trade a competitive good through money.

To our regret, this result is not good enough. Admittedly, the proposition shows boundedness of equilibrium, but it does not say anything about the determinacy of equilibrium that is our main issue. Thus, we should proceed to investigate its determinacy. To this end, we should stop to think of what the determinacy is. Needless to say, a uniqueness of equilibrium is a perfect concept for the determinacy. It is, however, well known that very severe conditions are required for it even in the standard general competitive model. On the other hand, the indeterminacy of equilibrium means that arbitrarily close to every equilibrium we always find another equilibrium. Thus, it is legitimate for us to abandon a uniqueness of equilibrium, instead adopt a local uniqueness of equilibrium for the determinacy of equilibrium, where a local uniqueness of equilibrium allows a multiplicity of equilibria but excludes the existence of other equilibria in a neighborhood of each equilibrium.

It is effective in examining a local uniqueness of equilibrium to make use of differentiability of relevant functions. Thus, we assume the following.

ASSUMPTION 6. u^i and F^i are all differentiable (C^r class, $r \geq 2$) for all i in addition to the conditions provided in assumptions 1 and 4.

This assumption leads to differentiability of $\hat{\boldsymbol{x}}^i_c(\boldsymbol{p}^c)$.

PROPOSITION 2. *Under assumption 6, $\hat{\boldsymbol{x}}^i_c(\boldsymbol{p}^c)$ is differentiable in $\boldsymbol{p}^c$ for all i.*

Proof. A function $\boldsymbol{x}^i_f(\boldsymbol{p}^c)$ in the step 1 of deriving process of $\hat{\boldsymbol{x}}^i_c(\boldsymbol{p}^c)$ is differentiable since strict quasiconcavity and differentiability of u^i enables us to effectively apply the implicit function theorem to the first order conditions of the relevant optimization problem. Since F^i is also assumed to be differentiable, $\hat{\boldsymbol{x}}^i_f(\boldsymbol{p}^c)$ is differentiable. Then, in step 2, it is worth noting that the objective function of the optimization is strictly quasiconcave in $\boldsymbol{x}^i_c$ because of strict quasiconcavity of $u^i(\boldsymbol{x}^i_c, \boldsymbol{x}^i_f)$. Given this fact and differentiability of $\hat{\boldsymbol{x}}^i_f(\boldsymbol{p}^c)$, the implicit function theorem is again applicable to the first order conditions of this optimization problem to obtain a differentiable function $\hat{\boldsymbol{x}}^i_c(\boldsymbol{p}^c)$.

Then, we consider a matrix consisting of the price effect on the demand for c-goods as follows.

$$A(\boldsymbol{p}^c) = \begin{pmatrix} \sum_{i=1}^{I} \frac{\partial \hat{x}^i_{c1}(\boldsymbol{p}^c)}{\partial p^c_1} & \sum_{i=1}^{I} \frac{\partial \hat{x}^i_{c1}(\boldsymbol{p}^c)}{\partial p^c_2} & \cdots & \sum_{i=1}^{I} \frac{\partial \hat{x}^i_{c1}(\boldsymbol{p}^c)}{\partial p^c_n} \\ \sum_{i=1}^{I} \frac{\partial \hat{x}^i_{c2}(\boldsymbol{p}^c)}{\partial p^c_1} & \sum_{i=1}^{I} \frac{\partial \hat{x}^i_{c2}(\boldsymbol{p}^c)}{\partial p^c_2} & \cdots & \sum_{i=1}^{I} \frac{\partial \hat{x}^i_{c2}(\boldsymbol{p}^c)}{\partial p^c_n} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{i=1}^{I} \frac{\partial \hat{x}^i_{cn}(\boldsymbol{p}^c)}{\partial p^c_1} & \sum_{i=1}^{I} \frac{\partial \hat{x}^i_{cn}(\boldsymbol{p}^c)}{\partial p^c_2} & \cdots & \sum_{i=1}^{I} \frac{\partial \hat{x}^i_{cn}(\boldsymbol{p}^c)}{\partial p^c_n} \end{pmatrix}$$

where $\hat{x}^i_{cj}(\boldsymbol{p}^c)$ $j = 1, \ldots, n$ is an element function of $\hat{\boldsymbol{x}}^i_c(\boldsymbol{p}^c)$. Through this matrix, we obtain a sufficient condition for an equilibrium to be locally determinate.

THEOREM 2. *An equilibrium $\boldsymbol{p}^c$ for the monetary economy $\mathcal{E}(\bar{\boldsymbol{p}}, \{u^i, \boldsymbol{\omega}^i, \bar{M}, F^i\}_i)$ is locally unique if there is not any nonzero $\boldsymbol{y}$ $(\in \boldsymbol{R}^n)$ such that*

$$A(\boldsymbol{p}^c) \cdot \boldsymbol{y} = \boldsymbol{0} \quad and \quad \sum_i^I \boldsymbol{\omega}_c^i \cdot \boldsymbol{y} = 0$$

Proof. *See Appendix [2].*

The condition of the theorem specifies a geometric relation between an aggregate initial endowment vector of c-goods and a hyperplane spanned by those vectors which correspond to price effects on the demand for each good. This theorem shows that the introduction of money assures determinacy of equilibrium, but it is conditional. Since it is difficult to see how weak or strong the condition is for monetary economies, we cannot say how likely money yields the local uniqueness of equilibrium. So we consider the effectiveness of money from a different angle.

3.3. Determinacy of equilibrium as a generic property

We use the method of regular economies to investigate the issue from the generic viewpoint. To this end, we first take any strictly positive value of $\bar{M}$ and consider the set $\{(\bar{p}, \omega) \in \boldsymbol{R}^m_{++} \times \boldsymbol{R}^{(n+m)I}_{++} | \bar{M} > \bar{\boldsymbol{p}} \sum_i^I \omega_f^i\}$, which is denoted by $\mathcal{E}_{\bar{M}}$. It is worth noting that this set is open and that ω_{ci}^i is free from this set. In the following, we use $\mathcal{E}_{\bar{M}}$ as a parameter space and apply the technique of regular economies to it. More specifically, our strategy is to identify a particular subset of $\mathcal{E}_{\bar{M}}$ each element of which yields the local uniqueness of equilibrium. To proceed our argument along this line, we need to express the dependence of the demand function $\hat{\boldsymbol{x}}_c^i(\boldsymbol{p}^c)$ on the parameters. It is obvious from its construction that a final demand $\hat{\boldsymbol{x}}_c^i$ is dependent on $(\bar{\boldsymbol{p}}, \omega)$ as well as $\boldsymbol{p}^c$. Thus, we are allowed to express $\hat{\boldsymbol{x}}_c^i(\boldsymbol{p}^c; \bar{\boldsymbol{p}}, \omega)$ instead of $\hat{\boldsymbol{x}}_c^i(\boldsymbol{p}^c)$. It turns out that this function has a following property.

PROPOSITION 3. *$\hat{\boldsymbol{x}}_c^i(\boldsymbol{p}^c; \bar{\boldsymbol{p}}, \omega)$ is differentiable in both $\boldsymbol{p}^c$ and $(\bar{\boldsymbol{p}}, \omega)$ for all i.*

Proof. We may basically follow the same way as in the proof of proposition 2. Given assumption 6, we can obtain a differentiable function $\boldsymbol{x}_f^i(\boldsymbol{p}^c; \bar{\boldsymbol{p}}, \omega^i)$ in the step 1 of the deriving process of $\boldsymbol{x}_c^i$. Then, each agent i is rationed with $F^i(\boldsymbol{x}_f^1(\boldsymbol{p}^c; \bar{\boldsymbol{p}}, \omega^1), \ldots, \boldsymbol{x}_f^I(\boldsymbol{p}^c; \bar{\boldsymbol{p}}, \omega^I))$, so it has $\hat{\boldsymbol{x}}_f^i(\boldsymbol{p}^c; \bar{\boldsymbol{p}}, \omega)$ which is also differentiable through assumption 6. Thus, in the step 2 we are allowed to apply the implicit function theorem, obtaining the desired consequence.

Now we show a claim which gives a generic answer to the problem of indeterminacy of equilibrium.

THEOREM 3. *If there exists at least one agent who is rationed at all fix price markets, then for any $\bar{M}(> 0)$ the set of equilibrium price vectors is discrete for almost all $(\bar{\boldsymbol{p}}, \omega)$ in $\mathcal{E}_{\bar{M}}$.*

Proof. *See Appendix [3].*

It is worth noting that the condition of this theorem is totally independent of the one provided in Theorem 2. As long as we have at least one agent who is restricted for all f-goods by the rationing scheme, the introduction of money assures local uniqueness of equilibrium for almost all economies. We should recall that the resource allocation does not become definite until equilibrium is determined. Thus, it is money supply that realizes a specific resource allocation among agents. In this sense, money actually affects a real economy.

We have another interesting fact concerning the equilibrium set that is obtained through the technique of regular economies.

THEOREM 4. *For almost all* $(\bar{\boldsymbol{p}}, \omega)$ *in* $\mathcal{E}_{\bar{M}}$ *for any* $\bar{M}(> 0)$ *there exists an odd number of equilibrium price vectors.*

Proof. See Appendix [4].

It follows from these theorems that money truly plays a crucial role in a mixed type economy.

Lastly, we should refer to the efficiency of the equilibria. The equilibria obtained above, however, cannot be expected to have Pareto efficiency because of the rationing schemes, which is due to the following fact. Although we only required the most fundamental property of rationing schemes in definition 1, in order for the schemes to make sense some other properties should be added to. Above all, the condition of voluntary exchange is indispensable for any rationing scheme. However, Silvestre showed that Pareto efficiency and voluntary exchange go together *only* at the Walrasian allocation in which all the markets are competitive (Silvestre (1985)).

4. CONCLUDING REMARKS

We should refer to the literature dealing with a similar issue to ours. Drèze (1997) and Citanna et al. (2001) have also considered the combination of fixed/flexible prices. The latter, in particular, has obtained the similar result to one of ours on the basis of an economy with production; namely the set of equilibria constitutes a continuum (see Theorem 1 of section 2). However, as for a pure exchange economy they have only considered fix price markets. In addition, they have assumed that in every market only excess supply occurs. They have also considered an effect of expectation of supply opportunities on the decision making. These are structural differences from ours. What is more important, their goal is completely distinct from ours. They aim at showing that the relevant equilibrium set (called underemployment equilibria) constitutes a continuum, which necessarily leads to the indeterminacy of equilibrium, whereas our goal is to improve that situation.

We solved our problem with recourse to money which is not unfoundedly introduced but is naturally required by the structure of the model. A point to note is that money introduced in our model is not allowed to have the function of the store of value by our model being intrinsically static. In addition, the actual trade takes place only once after the equilibrium is established, which is the same presupposition as the one in the standard general competitive model. Thus, an agent does not have any incentive to hold money. Nevertheless, money supply proves to substantially affect a real economy, that is, it determines a specific resource allocation. In macroeconomics, indeed money plays a decisive role in determining real quantities, but it is crucially based on the assumption that agents do want to hold money through three types of motives. Thus, the real effect of money in our model stems from a totally different ground from that of macroeconomics. It seems that this working of money from a viewpoint of general competitive equilibrium has been overlooked. Since in reality an economy is likely to have a competitive market system with partial malfunction, this crucial role of money should be stressed.

If we take account of the distinction between the present and the future, then we need to think of the store of value as the function of money. In this case, we have to consider the behavior of money-holding of an agent. In this connection, Magill and Quinzii have shown that a similar difficulty, i.e., indeterminacy of equilibrium, can occur in the incomplete market system (Magill and Quinzii (1996), § 33). If our mixed-market model is interpreted as a temporary equilibrium model, we can also show that through an appropriate formalization of the money-holding behavior of an agent money plays the same effective role in determining the equilibrium (Nagata (2008)).

APPENDIX

[1] Proof of Theorem 1.

In order to prove Theorem 1, we need need two lemmas.

LEMMA 1. *$\boldsymbol{x}^i{}_f(\boldsymbol{p}^c)$ is continuous in $\boldsymbol{p}^c$, $i = 1, \dots, I$.*

Proof. We first consider the set $\Gamma(\boldsymbol{p}^c)$ defined by

$$\{(\boldsymbol{x}^i_c, \boldsymbol{x}^i{}_f) \in \boldsymbol{R}^{n+m}_+ \mid \boldsymbol{p}^c\boldsymbol{x}^i_c + \bar{\boldsymbol{p}}\boldsymbol{x}^i{}_f = \boldsymbol{p}^c\boldsymbol{\omega}^i_c + \bar{\boldsymbol{p}}\boldsymbol{\omega}^i{}_f\}\,.$$

$\Gamma(\boldsymbol{p}^c)$ can be seen as a correspondence, which particularly satisfies following conditions. (1) *For any $\boldsymbol{p}^c \in \boldsymbol{R}^n_{++}$, $\Gamma(\boldsymbol{p}^c)$ is compact. In fact, through assumptions 2 and 3, $\Gamma(\boldsymbol{p}^c)$ is obviously bounded. Thus, any sequence of $\Gamma(\boldsymbol{p}^c)$ has a convergent subsequence whose limit point, say $(\boldsymbol{x}^i_{c0}, \boldsymbol{x}^i{}_{f0})$, is easily shown to satisfy that $\boldsymbol{p}^c\boldsymbol{x}^i{}_{c0} + \bar{\boldsymbol{p}}\boldsymbol{x}^i{}_{f0} = \boldsymbol{p}^c\boldsymbol{\omega}^i_c + \bar{\boldsymbol{p}}\boldsymbol{\omega}^i{}_f$, which implies that $(\boldsymbol{x}^i_{c0}, \boldsymbol{x}^i{}_{f0}) \in \Gamma(\boldsymbol{p}^c)$.* (2) *For any $\boldsymbol{p}^c \in \boldsymbol{R}^n_{++}$, $\Gamma(\boldsymbol{p}^c)$ is convex. This is easily shown by the simple calculation.* (3) *$\Gamma(\boldsymbol{p}^c)$ is continuous in $\boldsymbol{p}^c$. First we show the upper hemi-continuity of Γ. For any given $\bar{\boldsymbol{p}}^c \in \boldsymbol{R}^n_{++}$, pick any sequence $\{\boldsymbol{p}^c_r\} \to \bar{\boldsymbol{p}}^c$ where $\{\boldsymbol{p}^c_r\} \subset \boldsymbol{R}^n_{++}$. Then we consider a sequence $(\boldsymbol{x}^i_{cr}, \boldsymbol{x}^i{}_{fr}) \in \Gamma(\boldsymbol{p}^c_r)$, $r = 1, 2, \dots$ which has a limit point, say $(\bar{\boldsymbol{x}}^i_c, \bar{\boldsymbol{x}}^i{}_f)$. We have only to show that $(\bar{\boldsymbol{x}}^i_c, \bar{\boldsymbol{x}}^i{}_f) \in \Gamma(\bar{\boldsymbol{p}}^c)$. Suppose not, then there exists a number r' such that $\bar{\boldsymbol{p}}^c\boldsymbol{x}^i_{cr'} + \bar{\boldsymbol{p}}\boldsymbol{x}^i{}_{fr'} > \bar{\boldsymbol{p}}^c\boldsymbol{\omega}^i_c + w$ where $w = \bar{\boldsymbol{p}}\boldsymbol{\omega}^i{}_f$. Hence, there exists another number r'' such that $\boldsymbol{p}^c_{r''}\boldsymbol{x}^i_{cr'} + \bar{\boldsymbol{p}}\boldsymbol{x}^i{}_{fr'} > \boldsymbol{p}^c_{r''}\boldsymbol{\omega}^i_c + w$. Set $r* = max\{r', r''\}$. Then, $\boldsymbol{p}^c_{r*}\boldsymbol{x}^i{}_{cr*} + \bar{\boldsymbol{p}}\boldsymbol{x}^i{}_{fr*} > \boldsymbol{p}^c_{r*}\boldsymbol{\omega}^i_c + w$, which is a contradiction. Next, we demonstrate the lower hemi-continuity of Γ. As before, first, for any given $\bar{\boldsymbol{p}}^c \in \boldsymbol{R}^n_{++}$, pick any sequence $\{\boldsymbol{p}^c_r\} \to \bar{\boldsymbol{p}}^c$ where $\{\boldsymbol{p}^c_r\} \subset \boldsymbol{R}^n_{++}$. We need to show that for any $(\bar{\boldsymbol{x}}^i_c, \bar{\boldsymbol{x}}^i{}_f) \in \Gamma(\bar{\boldsymbol{p}}^c)$ there*

exists a sequence $(\boldsymbol{x}^i_{cr}, \boldsymbol{x}^i_{fr}) \in \Gamma(\boldsymbol{p}^c_r),\ r = 1, 2, \ldots$ *which converges to* $(\bar{\boldsymbol{x}}^i_c, \bar{\boldsymbol{x}}^i_f)$. *Suppose not, then there exists a neighborhood* $N(\bar{\boldsymbol{x}}^i_c, \bar{\boldsymbol{x}}^i_f)$ *such that* $\Gamma(\boldsymbol{p}^c_r) \cap N(\bar{\boldsymbol{x}}^i_c, \bar{\boldsymbol{x}}^i_f) = \emptyset$ *for all* r. *But this is impossible because for any* $\boldsymbol{\epsilon} (= (\boldsymbol{\epsilon}_c, \boldsymbol{\epsilon}_f) \neq 0) \in \boldsymbol{R}^{n+m}$ *there always exists a* $\boldsymbol{\delta} \in \boldsymbol{R}^n$ *such that* $\boldsymbol{\delta}(\bar{\boldsymbol{x}}^i_c + \boldsymbol{\epsilon}_c - \boldsymbol{\omega}^i_c) = -\bar{\boldsymbol{p}}^c \boldsymbol{\epsilon}_c - \bar{\boldsymbol{p}} \boldsymbol{\epsilon}_f$, *which leads to the following equation.*

$$(\bar{\boldsymbol{p}}^c + \boldsymbol{\delta})(\bar{\boldsymbol{x}}^i_c + \boldsymbol{\epsilon}_c) + \bar{\boldsymbol{p}}(\bar{\boldsymbol{x}}^i_f + \boldsymbol{\epsilon}_f) = (\bar{\boldsymbol{p}}^c + \boldsymbol{\delta})\boldsymbol{\omega}^i_c + w\,.$$

Thus, we are allowed to use Berge's maximum theorem (Berge (1963), ch.6) to obtain a continuous demand function $\boldsymbol{x}^i_f(\boldsymbol{p}^c)$. *Note that the strict quasi-concavity of* u^i *makes* $\boldsymbol{x}^i_f(\boldsymbol{p}^c)$ *a function (not a correspondence).*

It is worth noting that through this proposition we have a continuous net demand function $\tilde{\boldsymbol{x}}^i_f(\boldsymbol{p}^c)$ $(= \boldsymbol{x}^i_f(\boldsymbol{p}^c) - \boldsymbol{\omega}^i_f)$ for all i, which in turn leads to a continuous $w^i(\boldsymbol{p}^c)$ $(= \bar{\boldsymbol{p}} F^i(\tilde{\boldsymbol{x}}^1_f(\boldsymbol{p}^c), \ldots, \tilde{\boldsymbol{x}}^I_f(\boldsymbol{p}^c))$ because of the continuity of F^i(assumption 4,(1)).

LEMMA 2. $\hat{\boldsymbol{x}}^i_c(\boldsymbol{p}^c)$ *is continuous in* $\boldsymbol{p}^c,\ i = 1, \ldots, I$.

Proof. Step 1
Since the set $\{\boldsymbol{x}^i_c \in \boldsymbol{R}^n_+ \mid \boldsymbol{p}^c \boldsymbol{x}^i_c = \boldsymbol{p}^c \boldsymbol{\omega}^i_c - w^i(\boldsymbol{p}^c)\},\ i = 1, \ldots, I$ *is well-defined owing to assumption 4,(5), let* $\Lambda(\boldsymbol{p}^c)$ *denote it (for simplicity of notation, we omit the superscript i for* Λ*). It is easily shown that* $\Lambda(\boldsymbol{p}^c)$ *is compact and convex for all* $\boldsymbol{p}^c \in \boldsymbol{R}^n_{++}$. *We demonstrate that* $\Lambda(\boldsymbol{p}^c)$ *as a correspondence is continuous in* $\boldsymbol{p}^c$. *Since the upper-hemicontinuity is shown in the same way as the proof of the upper hemi-continuity of* $\Gamma(\boldsymbol{p}^c)$ *in Proposition 3, we only show the lower-hemicontinuity. For any given* $\bar{\boldsymbol{p}}^c \in \boldsymbol{R}^n_{++}$, *pick any* $\bar{\boldsymbol{x}}^i_c \in \Lambda(\bar{\boldsymbol{p}}^c)$ *and fix it. We have to show that for a sequence* $\{\boldsymbol{p}^c_k\} \to \bar{\boldsymbol{p}}^c$ *there exists a sequence* $\{\boldsymbol{x}^i_{ck}\}$ *such that* $\boldsymbol{x}^i_{ck} \in \Lambda(\boldsymbol{p}^c_k),\ k = 1, 2, \ldots$ *and* $\{\boldsymbol{x}^i_{ck}\} \to \bar{\boldsymbol{x}}^i_c$. *To this end, we consider for any given* $\epsilon \in \boldsymbol{R}^n$ *a specific* $\boldsymbol{x}^i_c$ *that satisfies*

$$(\bar{\boldsymbol{p}}^c + \epsilon)(\bar{\boldsymbol{x}}^i_c - \boldsymbol{x}^i_c) = \epsilon(\bar{\boldsymbol{x}}^i_c - \boldsymbol{\omega}^i_c) + w^i(\bar{\boldsymbol{p}}^c + \epsilon) - w^i(\bar{\boldsymbol{p}}^c)\,.$$

Note that such a $\boldsymbol{x}^i_c$ *is always obtainable. Moreover, it is worth noting that such a* $\boldsymbol{x}^i_c$ *fulfills*

$$(\bar{\boldsymbol{p}}^c + \epsilon)\boldsymbol{x}^i_c = (\bar{\boldsymbol{p}}^c + \epsilon)\boldsymbol{\omega}^i_c - w^i(\bar{\boldsymbol{p}}^c + \epsilon)$$

and converges to $\bar{\boldsymbol{x}}^i_c$ *as* $\epsilon \to 0$, *which shows the lower hemi-continuity of* $\Lambda(\boldsymbol{p}^c)$ *at* $\bar{\boldsymbol{p}}^c$. *Since* $\bar{\boldsymbol{p}}^c$ *is arbitrarily chosen, the desired outcome follows.*

Step 2
Recall that $\hat{\boldsymbol{x}}^i_c(\boldsymbol{p}^c)$ *is derived through the following optimization problem.*

$$\max \quad u^i(\boldsymbol{x}^i_c, \hat{\boldsymbol{x}}^i_f(\boldsymbol{p}^c) + \boldsymbol{\omega}^i_f)$$
$$s.t.\ \boldsymbol{x}^i_c \in \Lambda(\boldsymbol{p}^c)$$

where we should recall that $\hat{\boldsymbol{x}}^i_f(\boldsymbol{p}^c) = F^i(\tilde{\boldsymbol{x}}^1_f(\boldsymbol{p}^c), \ldots, \tilde{\boldsymbol{x}}^I_f(\boldsymbol{p}^c))$. *Since* $\Lambda(\boldsymbol{p}^c)$ *is compact and convex and* u^i *is strictly quasi-concave, the optimal* $\boldsymbol{x}^i_c$ *is uniquely determined in the above problem. Thus,* $\boldsymbol{x}^i_c(\boldsymbol{p}^c)$ *is a well-defined function. We show that this function is continuous in* $\boldsymbol{p}^c$. *To this end, pick any* $\bar{\boldsymbol{p}}^c \in \boldsymbol{R}^n_{++}$ *and fix it. Then, it suffices to*

show that
$\lim_{\{\boldsymbol{p}_k^c\}\to \bar{\boldsymbol{p}}^c} \|argmax_{\boldsymbol{x}_c^i \in \Lambda(\bar{\boldsymbol{p}}^c)} u^i(\boldsymbol{x}_c^i, \hat{\boldsymbol{x}}_f^i(\bar{\boldsymbol{p}}^c) + \boldsymbol{\omega}_f^i) - argmax_{\boldsymbol{x}_c^i \in \Lambda(\boldsymbol{p}_k^c)} u^i(\boldsymbol{x}_c^i, \hat{\boldsymbol{x}}_f^i(\boldsymbol{p}_k^c) + \boldsymbol{\omega}_f^i)\| = 0.$
Note that
$\|argmax_{\boldsymbol{x}_c^i \in \Lambda(\bar{\boldsymbol{p}}^c)} u^i(\boldsymbol{x}_c^i, \hat{\boldsymbol{x}}_f^i(\bar{\boldsymbol{p}}^c) + \boldsymbol{\omega}_f^i) - argmax_{\boldsymbol{x}_c^i \in \Lambda(\boldsymbol{p}_k^c)} u^i(\boldsymbol{x}_c^i, \hat{\boldsymbol{x}}_f^i(\boldsymbol{p}_k^c) + \boldsymbol{\omega}_f^i)\|$
$= \|argmax_{\boldsymbol{x}_c^i \in \Lambda(\bar{\boldsymbol{p}}^c)} u^i(\boldsymbol{x}_c^i, \hat{\boldsymbol{x}}_f^i(\bar{\boldsymbol{p}}^c) + \boldsymbol{\omega}_f^i) - argmax_{\boldsymbol{x}_c^i \in \Lambda(\boldsymbol{p}_k^c)} u^i(\boldsymbol{x}_c^i, \hat{\boldsymbol{x}}_f^i(\bar{\boldsymbol{p}}^c) + \boldsymbol{\omega}_f^i)$
$+ argmax_{\boldsymbol{x}_c^i \in \Lambda(\boldsymbol{p}_k^c)} u^i(\boldsymbol{x}_c^i, \hat{\boldsymbol{x}}_f^i(\bar{\boldsymbol{p}}^c) + \boldsymbol{\omega}_f^i) - argmax_{\boldsymbol{x}_c^i \in \Lambda(\boldsymbol{p}_k^c)} u^i(\boldsymbol{x}_c^i, \hat{\boldsymbol{x}}_f^i(\boldsymbol{p}_k^c) + \boldsymbol{\omega}_f^i)\|$
$\leq \|argmax_{\boldsymbol{x}_c^i \in \Lambda(\bar{\boldsymbol{p}}^c)} u^i(\boldsymbol{x}_c^i, \hat{\boldsymbol{x}}_f^i(\bar{\boldsymbol{p}}^c) + \boldsymbol{\omega}_f^i) - argmax_{\boldsymbol{x}_c^i \in \Lambda(\boldsymbol{p}_k^c)} u^i(\boldsymbol{x}_c^i, \hat{\boldsymbol{x}}_f^i(\bar{\boldsymbol{p}}^c)\|$
$+ \|argmax_{\boldsymbol{x}_c^i \in \Lambda(\boldsymbol{p}_k^c)} u^i(\boldsymbol{x}_c^i, \hat{\boldsymbol{x}}_f^i(\bar{\boldsymbol{p}}^c) + \boldsymbol{\omega}_f^i) - argmax_{\boldsymbol{x}_c^i \in \Lambda(\boldsymbol{p}_k^c)} u^i(\boldsymbol{x}_c^i, \hat{\boldsymbol{x}}_f^i(\boldsymbol{p}_k^c) + \boldsymbol{\omega}_f^i)\|.$
Then, through Berge's maximum theorem, we have

$$\lim_{\{\boldsymbol{p}_k^c\}\to \bar{\boldsymbol{p}}^c} \|argmax_{\boldsymbol{x}_c^i \in \Lambda(\bar{\boldsymbol{p}}^c)} u^i(\boldsymbol{x}_c^i, \hat{\boldsymbol{x}}_f^i(\bar{\boldsymbol{p}}^c) + \boldsymbol{\omega}_f^i) - argmax_{\boldsymbol{x}_c^i \in \Lambda(\boldsymbol{p}_k^c)} u^i(\boldsymbol{x}_c^i, \hat{\boldsymbol{x}}_f^i(\bar{\boldsymbol{p}}^c)\| = 0$$

and from the fact that u^i is continuous in $\hat{\boldsymbol{x}}_f^i$ which is in turn continuous in $\boldsymbol{p}^c$, it follows that
$\lim_{\{\boldsymbol{p}_k^c\}\to \bar{\boldsymbol{p}}^c} \|argmax_{\boldsymbol{x}_c^i \in \Lambda(\boldsymbol{p}_k^c)} u^i(\boldsymbol{x}_c^i, \hat{\boldsymbol{x}}_f^i(\bar{\boldsymbol{p}}^c) + \boldsymbol{\omega}_f^i) - argmax_{\boldsymbol{x}_c^i \in \Lambda(\boldsymbol{p}_k^c)} u^i(\boldsymbol{x}_c^i, \hat{\boldsymbol{x}}_f^i(\boldsymbol{p}_k^c) + \boldsymbol{\omega}_f^i)\| = 0.$
These observations lead to the equation to be shown.

These lemmas lead to the proof of Theorem 1, which is as follows.

Proof. We consider the aggregate excess demand function for c-goods which is denoted by $g : \boldsymbol{R}^n_{++} \to \boldsymbol{R}^n$. Obviously, the price vector $\boldsymbol{p}^c$ satisfying that $g(\boldsymbol{p}^c) = 0$ characterizes an equilibrium. Since g is defined as

$$g(\boldsymbol{p}^c) = \sum_i^I \hat{\boldsymbol{x}}_c^i(\boldsymbol{p}^c) - \sum_i^I \boldsymbol{\omega}_c^i$$

, we can establish the following properties concerning g.

(1) g is continuous in $\boldsymbol{p}^c$ (see lemma 2).

(2) For any given $\boldsymbol{p}^c \in \boldsymbol{R}^n_{++}$, $\boldsymbol{p}^c g(\boldsymbol{p}^c) = 0$. In fact, for each i, we have that $\boldsymbol{p}^c \hat{\boldsymbol{x}}_c^i(\boldsymbol{p}^c) = \boldsymbol{p}^c \boldsymbol{\omega}_c^i - w^i(\boldsymbol{p}^c)$. According to the property of rationing schemes, $\sum_i^I F^i(\boldsymbol{p}^c) = 0$, which implies that $\sum_i^I w^i(\boldsymbol{p}^c) = \bar{\boldsymbol{p}} \sum_i^I F^i(\boldsymbol{p}^c) = 0$. Thus, we have that $\sum_i^I \boldsymbol{p}^c \hat{\boldsymbol{x}}_c^i(\boldsymbol{p}^c) = \sum_i^I \boldsymbol{p}^c \boldsymbol{\omega}_c^i$.

(3) Pick a $\bar{\boldsymbol{p}}^c \in \partial \boldsymbol{R}^n_{++}$ such that $\bar{\boldsymbol{p}}^c \neq 0$ and $\bar{p}_j^c = 0$. Then, through assumption 1 (esp. $1-2.$), we have that $\lim_{\boldsymbol{p}^c \to \bar{\boldsymbol{p}}^c} g_j(\boldsymbol{p}^c) > 0$.

Now we take any $\epsilon(> 0)$ and consider a ϵ-sphere S^ϵ in $\boldsymbol{R}^n$. Let the set $\boldsymbol{R}^n_{++} \cap S^\epsilon$ be S^ϵ_{++}. Then, we truncate S^ϵ_{++} to make $\bar{S}$ that includes the boundary close enough to $\partial \boldsymbol{R}^n_{++}$. Restricting g on $\bar{S}$, g is interpreted as a continuous vector field on $\bar{S}$ (see (1) and (2) stated above) that points inward at the boundary (see (3) mentioned above). Therefore, the vector field g has zeros; namely there exists a vector $\boldsymbol{p}^c \in \bar{S}$ such that $g(\boldsymbol{p}^c) = 0$. Thus, the existence of an equilibrium is proved.

(4) From the argument of (3), we see that each sphere with ϵ as the radius has its own

equilibrium competitive price vectors on it. We are, however, not allowed to specify any ϵ since every agent's demand function $\hat{\boldsymbol{x}}_c^i$ $(i = 1, \ldots, I)$ is not homogeneous of degree 0 in $\boldsymbol{p}^c$. It follows that the set of equilibrium competitive price vectors contains a continuum that expands infinitely. Needless to say, the demand function is not a constant function. Thus, inhomogeneity of the demand function also guarantees that resource allocations each corresponding to an equilibrium price vector are different one another.

[2] Proof of Theorem 2.

Step 1.

Let E_1 denote the set $\{\boldsymbol{p}^c \in \boldsymbol{R}^n_{++} | \sum_i^I \hat{\boldsymbol{x}}_c^i(\boldsymbol{p}^c) = \sum_i^I \boldsymbol{\omega}_c^i\}$ and E_2 denote $\{\boldsymbol{p}^c \in \boldsymbol{R}^n_{++} | \boldsymbol{p}^c \sum_i^I \boldsymbol{\omega}_c^i + \bar{\boldsymbol{p}} \sum_i^I \boldsymbol{\omega}_f^i = \bar{M}\}$. Obviously, the set of equilibrium price vectors is equal to $E_1 \cap E_2$. Take an arbitrary equilibrium price vector $\boldsymbol{p}^c$ and fix it. Since E_1 is an unbounded continuum, we are allowed to consider a pass through $\boldsymbol{p}^c$ in E_1, which can be expressed by a function $\boldsymbol{p}_1^c : [0, +\infty) \to E_1$. We may assume that for some $\bar{t} \in (0, +\infty)$, $\boldsymbol{p}^c = \boldsymbol{p}_1^c(\bar{t})$. Through proposition 2, we may consider $\boldsymbol{p}_1^c$ to be differentiable or at least approximated by a differentiable function. As for E_2, we are also allowed to think of a pass expressed by $\boldsymbol{p}_2^c : [0, +\infty) \to E_2$ for which we may assume the differentiability. In addition we may assume that $\boldsymbol{p}^c = \boldsymbol{p}_2^c(\bar{t})$.

Step 2.

For $\boldsymbol{p}_1^c(t)$, we obviously have that $\sum_i^I \boldsymbol{x}_c^i(\boldsymbol{p}_1^c(t)) = \sum_i^I \boldsymbol{\omega}_c^i$. Thus, differentiating both sides by t, we obtain the following.

$$A(\boldsymbol{p}_1^c(t)) \cdot \frac{d\boldsymbol{p}_1^c(t)}{dt} = 0$$

In particular, we have

$$A(\boldsymbol{p}_1^c(\bar{t})) \cdot \frac{d\boldsymbol{p}_1^c(\bar{t})}{dt} = 0 \tag{1}$$

Since the above equation holds for any pass through $\boldsymbol{p}^c$, $|A(\boldsymbol{p}_1^c(\bar{t}))| \neq 0$.

For $\boldsymbol{p}_2^c(t)$, we follow the similar procedure to have that

$$\sum_i^I \boldsymbol{\omega}_c^i \cdot \frac{d\boldsymbol{p}_2^c(\bar{t})}{dt} = 0 \tag{2}$$

Step 3.

The equilibrium price vector $\boldsymbol{p}^c$ is locally unique if and only if in a neighborhood of $\boldsymbol{p}^c$, $\boldsymbol{p}_1^c(t)$ and $\boldsymbol{p}_2^c(t)$ never coincide. Alternatively put, $\frac{d\boldsymbol{p}_1^c(\bar{t})}{dt} \neq \frac{d\boldsymbol{p}_2^c(\bar{t})}{dt}$ for any $\boldsymbol{p}_i^c(t)$ through $\boldsymbol{p}^c$ $(i = 1, 2)$, which is, through the above equations (1) and (2), assured by the condition provided in theorem 2.

[3] Proof of Theorem 3.

First we take an arbitrary $\bar{M}(> 0)$ and fix it. Then we denote $(\bar{\boldsymbol{p}}, \omega)$ of $\mathcal{E}_{\bar{M}}$ by e and call it an economy for simplicity in the following.

Step 1.

We consider aggregate excess demand functions for c-goods. For j-th of c-goods an aggregate excess demand function $g_j : \boldsymbol{R}^n_{++} \times \mathcal{E}_{\bar{M}} \to \boldsymbol{R}$ is defined by

$$g_j(\boldsymbol{p}^c, e) = \sum_i^I (\hat{x}^i_{cj}(\boldsymbol{p}^c, e) - \omega^i_{cj}) .$$

For money we consider the following function $m : \boldsymbol{R}^n_{++} \times \mathcal{E}_{\bar{M}} \to \boldsymbol{R}$ given by

$$m(\boldsymbol{p}^c, e) = \boldsymbol{p}^c \sum_i^I \boldsymbol{\omega}^i_c + \bar{\boldsymbol{p}} \sum_i^I \boldsymbol{\omega}^i_f - \bar{M} .$$

Note that g_j ($j = 1, \ldots, n$) and m are all differentiable through proposition 3. By putting these functions together, we finally construct the following function $f : \boldsymbol{R}^n_{++} \times \mathcal{E}_{\bar{M}} \to \boldsymbol{R}^n$ defined by

$$f(\boldsymbol{p}^c, e) = (g_1(\boldsymbol{p}^c, e), \ldots, g_{n-1}(\boldsymbol{p}^c, e), m(\boldsymbol{p}^c, e)) .$$

We often write $f_e(\boldsymbol{p}^c)$ instead of $f(\boldsymbol{p}^c, e)$ to distinguish intrinsic variables and complementary ones. Note that f only contains $n - 1$ of c-goods. Since Walras' law holds for c-goods, we immediately have that $f_e^{-1}(0)$ forms the set of equilibria for the economy e. Thus, our job is to investigate the structure of $f_e^{-1}(0)$.

Step 2.

Since we explicitly consider an economy e, we need to emphasize that the net demand $\tilde{x}^i_f$ of f-goods for each agent i is dependent on $\bar{\boldsymbol{p}}$, $\boldsymbol{\omega}^i_c$, $\boldsymbol{\omega}^i_f$ as well as $\boldsymbol{p}^c$. According to this observation, the rationing for agent i should be written as $F^i(\boldsymbol{p}^c, e)$, which in turn converts $w^i(\boldsymbol{p}^c)$ into $w^i(\boldsymbol{p}^c, e)$ since $w^i = \bar{\boldsymbol{p}} F^i$. Thus, the budget constraint for c-goods for agent i should be expressed as follows.

$$\boldsymbol{p}^c \boldsymbol{x}^i_c = \boldsymbol{p}^c \boldsymbol{\omega}^i_c - w^i(\boldsymbol{p}^c, e) .$$

Let the right side of the equation be $z^i(\boldsymbol{p}^c, e)$ which is obviously interpreted as his/her income. Then, the demand function for c-goods can be written as $\hat{\boldsymbol{x}}^i_c(\boldsymbol{p}^c, z^i(\boldsymbol{p}^c, e))$ instead of $\hat{\boldsymbol{x}}^i_c(\boldsymbol{p}^c, e)$.

Step 3.

Now, we will show that $f : \boldsymbol{R}^n_{++} \times \mathcal{E}_{\bar{M}} \to \boldsymbol{R}^n$ is a submersion. To this end, for any given $(\boldsymbol{p}^c, e)(\in \boldsymbol{R}^n_{++} \times \mathcal{E}_{\bar{M}})$, we demonstrate that the derivative $df_{\boldsymbol{p}^c, e} : \boldsymbol{R}^n \times \boldsymbol{R}^{(n+m)I+m+1} \to \boldsymbol{R}^n$ is surjective. Since there exists an agent who is rationed at all f-goods markets, we may assume without loss of generality that the first agent 1 is such an agent. We focus on an effect of ω^1_c since it is a free variable in $\mathcal{E}_{\bar{M}}$. Then, through assumptions 4,(4) and 6 we have that $\partial F^i / \partial \omega^1_{cj} = 0$, $i = 1, \ldots, I$, $j = 1, \ldots, n$. Thus, the part of $df_{\boldsymbol{p}^c, e}$ that corresponds to $\boldsymbol{\omega}^1_c$ turns out to be as follows.

$$
\begin{array}{c}
\\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n
\end{array}
\begin{array}{c}
\begin{array}{ccccc} \omega^1_{c1} & \omega^1_{c2} & \cdots & \omega^1_{cn-1} & \omega^1_{cn} \end{array} \\
\begin{pmatrix}
\frac{\partial \hat{x}^1_{c1}}{\partial z^1} p^c_1 - 1 & \frac{\partial \hat{x}^1_{c1}}{\partial z^1} p^c_2 & \dots & \frac{\partial \hat{x}^1_{c1}}{\partial z^1} p^c_{n-1} & \frac{\partial \hat{x}^1_{c1}}{\partial z^1} p^c_n \\
\frac{\partial \hat{x}^1_{c2}}{\partial z^1} p^c_1 & \frac{\partial \hat{x}^1_{c2}}{\partial z^1} p^c_2 - 1 & \dots & \frac{\partial \hat{x}^1_{c2}}{\partial z^1} p^c_{n-1} & \frac{\partial \hat{x}^1_{c2}}{\partial z^1} p^c_n \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial \hat{x}^1_{cn-1}}{\partial z^1} p^c_1 & \frac{\partial \hat{x}^1_{cn-1}}{\partial z^1} p^c_2 & \dots & \frac{\partial \hat{x}^1_{cn-1}}{\partial z^1} p^c_{n-1} - 1 & \frac{\partial \hat{x}^1_{cn-1}}{\partial z^1} p^c_n \\
p^c_1 & p^c_2 & \dots & p^c_{n-1} & p^c_n
\end{pmatrix}
\end{array}
$$

The linear independence of this submatrix can be proved as follows. First multiply each j-th column by λ_j $(j = 1, \dots, n)$ and set their sum equal to 0, resulting in the system of n linear equations. We particularly take the last equation of the system that is

$$\sum_j^n \lambda_j p^c_j = 0\,. \qquad (*)$$

On the other hand, the k-th component equation of the system $(k = 1, \dots, n-1)$ is written as

$$\frac{\partial \hat{x}^1_{ck}}{\partial z^1}\left(\sum_j^n \lambda_j p^c_j\right) - \lambda_k = 0\,.$$

It follows from $(*)$ that $\lambda_k = 0$ $(k = 1, \dots, n-1)$, which in turn implies that $\lambda_n = 0$ since $p^c_n > 0$.
Step 4.

Now that we have shown the linear independence of the above submatrix, the surjectivity of $df_{\boldsymbol{p}^c,e}$ is immediate because the tangent space on which the above submatrix operates is obviously $\boldsymbol{R}^n$.

Since f proved to be a submersion, any point of the range of f is a regular value for f. In particular, 0 of $\boldsymbol{R}^n$ is a regular value for f. Thus, through the preimage theorem (Guillemin and Pollack (1974), p.21), $f^{-1}(0)$ constitutes a $(n+m)I+m+1$-dimensional submanifold of $\boldsymbol{R}^n_{++} \times \mathcal{E}_{\bar{M}}$. Put $T = f^{-1}(0)$ for simplicity of notation. Then, we consider the projection $\pi : \boldsymbol{R}^n_{++} \times \mathcal{E}_{\bar{M}} \to \mathcal{E}_{\bar{M}}$ and restrict it on T, which can be expressed as $\pi|_T$. Note that for any e of $\mathcal{E}_{\bar{M}}$, $\pi|_T^{-1}(e)$ expresses the set $\{(equilibrium\ price\ vectors\ of\ e, e)\}$ which is nonempty through theorem 1. Let e be a regular value for $\pi|_T$. Then, $\pi|_T^{-1}(e)$ forms a 0-dimensional submanifold since $\mathcal{E}_{\bar{M}}$ can be regarded as a manifold the dimension of which is equal to that of T. Hence, as long as e is a regular value for $\pi|_T$, its equilibrium price vectors are locally unique, forming a discrete set. Through Sard's theorem, the set of regular value is dense in $\mathcal{E}_{\bar{M}}$, which completes the proof.

[4] Proof of Theorem 4.

In this proof, we follow the convention of a symbolic usage provided in the proof of theorem 3.

Before stating the proof, we present a lemma.

LEMMA 3. *For any economy e of* $\mathcal{E}_{\bar{M}}$,

$$\lim_{\boldsymbol{p}^c \to \partial \boldsymbol{R}^n_{++}} \|f(\boldsymbol{p}^c, e)\| = +\infty$$

Proof. *We first examine* $w^i(\boldsymbol{p}^c, e)$ *as* $\boldsymbol{p}^c \to \partial \boldsymbol{R}^n_{++}$. *Let* j *be the good whose price approaches to* 0. *In order for* $\boldsymbol{x}^i_f$ *to satisfy FOC for the optimization problem underlying* $w^i(\boldsymbol{p}^c, e)$, *the marginal rate of substitution of* x^i_{fk} *for* x^i_{cj} *must converge to* 0 *as* $p^c_j \to 0$, $k = 1, \ldots, m$. *Considering assumptions 2, 3 and the budget constraint, this convergency requires* x^i_{fk} *to converge to* ω^i_{fk} *from below* $(k = 1, \ldots, m)$ *while* x^i_{cj} *is required to indefinitely increase. Hence,* $\tilde{x}^i_{fk}(\boldsymbol{p}^c, e) \to 0$ *from below* $(k = 1, \ldots, m)$ *as* $p^c_j \to 0$. *Note that this holds for all* i. *Thus, through the operation of rationing schemes,* $F^i(\boldsymbol{p}^c, e) = 0$ *when* p^c_j *is close enough to* 0, *which implies that* $w^i(\boldsymbol{p}^c, e) = 0$ *as long as* $\boldsymbol{p}^c$ *remains close enough to* $\partial \boldsymbol{R}^n_{++}$.

As long as $\boldsymbol{p}^c$ *remains in a small neighborhood of* $\partial \boldsymbol{R}^n_{++}$, *we may set* $w^i(\boldsymbol{p}^c, e) = 0$. *Then, through the argument analogous to the preceding paragraph, we have that* $\hat{x}^i_{cj}(\boldsymbol{p}^c, e) \to \infty$ *where* $j \in \{j \in \{1, \ldots, n\} | \ p^c_j \to 0\}$. *In the light of the construction of* $f(\boldsymbol{p}^c, e)$, *the desired outcome follows.*

Then, we proceed to the proof of Theorem 4.

Proof. *We apply the modulo* 2 *degree theory to this issue (for this theory, see Guillemin and Pollack (1974)).*

Step 1.

We will show that $f_e^{-1}(0)$ *is a compact set for almost all economy e obtained in Theorem 3. We have shown that* $f_e^{-1}(0)$ *is bounded. Thus, any sequence* $(\boldsymbol{p}^c_r)_r$ *in it has a convergent subsequence whose limit point, say* $\boldsymbol{p}^c_0$, *belongs to* $\boldsymbol{R}^n_{++}$ *through lemma 1. Since* f_e *is continuous,* $f_e(\boldsymbol{p}^c_0) = 0$, *thus* $\boldsymbol{p}^c_0 \in f_e^{-1}(0)$, *which implies that* $f_e^{-1}(0)$ *is compact.*

Step 2.

Consider the map $h : \boldsymbol{R}^n_{++} \to \boldsymbol{R}^n$ *given by*

$$h(p^c_1, \ldots, p^c_n) = (p^c_1, \ldots, p^c_n) - (1, \ldots, 1) .$$

Obviously, h is smooth and $h^{-1}(0)$ *is a singleton, thus a compact set. Then, we construct a smooth homotopy* $H : \boldsymbol{R}^n_{++} \times [0, 1] \to \boldsymbol{R}^n$ *between* f_e *and h as follows.*

$$H(\boldsymbol{p}^c, t) = tf_e(\boldsymbol{p}^c) + (1 - t)h(\boldsymbol{p}^c) .$$

We conventionally use $H_t(\boldsymbol{p}^c)$ *instead of* $H(\boldsymbol{p}^c, t)$. *Then, through lemma 1,* $H_t^{-1}(0)$ *is proved to be compact for any* $t \in [0, 1]$, *which implies that* $H^{-1}(0)$ *is compact.*

Step 3.

From step 2, we conclude that $\sharp f_e^{-1}(0) = \sharp h^{-1}(0)(mod\ 2)$, *i.e., the modulo* 2 *residue class of* $f_e^{-1}(0)$ *is equal to that of* $h^{-1}(0)$. *Obviously, the latter is* 1, *thus* $f_e^{-1}(0)$ *has an odd number of elements.*

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