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[^0]
# A FORMULA FOR CALCULATING THE SLUTSKY MATRIX 

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Abstract: This article aims to provide a formula to calculate the Slutsky matrix. Our formula uses only the first- or second-order derivatives of the utility function and does not require any computation process on optimization problem.

Key words: Slutsky matrix, bordered Hessian, substitute good, complementary good.
JEL Classification Number: D11.

## 1. INTRODUCTION

In economics, a commodity is a substitute (resp. complementary) good of another commodity if and only if the corresponding element of the Slutsky matrix is positive (resp. negative). These notions are widely used and have been applied in many works. Hence, the signs of the elements of the Slutsky matrix are very important.

In this article, we aim to show that the sign of the $(i, j)$-th element of the Slutsky matrix is the same as that of $(-1)^{n} \times \tilde{B}_{i j}$, where $\tilde{B}_{i j}$ denotes the $(i, j)$-th cofactor of the bordered Hessian matrix. This result follows from a formula to calculate the Slutsky matrix using only the first- or second-order derivatives of the utility function. Using this formula, we can calculate the Slutsky matrix without any computation to solve optimization problems.

In Section 2, we introduce the formal statement of our result and prove it. We discuss our conclusion in Section 3.

## 2. THE RESULT

Let $\Omega=\mathbb{R}_{++}^{n}$ denote the consumption space and $u: \Omega \rightarrow \mathbb{R}$ denote the utility function. We assume that $u$ is $C^{2}$-class, $D u(x) \gg 0$ for any $x \in \Omega$, and

[^1]\[

(-1)^{i}\left|$$
\begin{array}{cccc}
u_{11}(x) & \ldots & u_{1 i}(x) & u_{1}(x)  \tag{1}\\
\vdots & \ddots & \vdots & \vdots \\
u_{i 1}(x) & \ldots & u_{i i}(x) & u_{i}(x) \\
u_{1}(x) & \ldots & u_{i}(x) & 0
\end{array}
$$\right|>0
\]

for any $i=2, \ldots, n$ and any $x \in \Omega$, where $u_{k}=\frac{\partial u}{\partial x_{k}}$ and $u_{j k}=\frac{\partial^{2} u}{\partial x_{k} \partial x_{j}}$. Note that if $i=n$, the matrix in (1) is called the bordered Hessian matrix. Under these conditions, $u$ must be strictly quasi-concave. Define $f(p, m)=\arg \max \{u(x) \mid p \cdot x \leq m\}$. It is well known that under these assumptions, $f$ is single-valued and $C^{1}$-class on $\operatorname{dom}(f)=\left\{(p, m) \in \mathbb{R}_{++}^{n} \times \mathbb{R}_{++} \mid f(p, m) \neq \emptyset\right\}$. Thus, we can define the Slutsky matrix $S(p, m)=\left(s_{i j}(p, m)\right)_{i, j=1}^{n}=D_{p} f(p, m)+D_{m} f(p, m)(f(p, m))^{T} .{ }^{1}$ It is also well known that $S(p, m)$ is negative semi-definite and symmetric, the rank of $S(p, m)$ is $n-1, p^{T} S(p, m)=0^{T}, S(p, m) p=0$, and the diagonal elements of $S(p, m)$ must be negative. Usually, we say that good $i$ is substitute (resp. complementary) to good $j$ if and only if $s_{i j}(p, m)>0\left(\right.$ resp. $\left.s_{i j}(p, m)<0\right)$.

We now introduce our main result.
Theorem. ${ }^{2}$ Suppose $n \geq 3$ and $(p, m) \in \operatorname{dom}(f)$, and let $x=f(p, m)$. For any $i, j \in\{1, \ldots, n\}$,

$$
\operatorname{sgn}\left(s_{i j}(p, m)\right)=(-1)^{n} \operatorname{sgn}\left(\tilde{B}_{i j}(x)\right),
$$

where $\tilde{B}_{i j}(x)$ denotes the $(i, j)$-th cofactor of the bordered Hessian matrix.
Using this theorem, one can decide whether good $i$ is a substitute or complementary good of good $j$ by calculating a very simple determinant consisting of the first- or second-order derivatives of the utility function. We think that this result is useful for many applied researches.

In fact, our theorem arises from the following lemma.
Lemma 1. Suppose $n \geq 3$ and choose any $i, j \in\{1, \ldots, n\}$. Fix any $k \in\{1, \ldots, n\}$ such that $i \neq k \neq j$ and for any $\ell \in\{1, \ldots, n-1\}$, define $\bar{\ell}=\ell$ if $\ell<k$ and $\bar{\ell}=\ell+1$ otherwise. Let

$$
a_{\ell m}=\frac{-1}{\left(u_{k}\right)^{3}}\left|\begin{array}{ccc}
u_{\bar{l} \bar{m}} & u_{\bar{\ell} k} & u_{\bar{\ell}} \\
u_{k \bar{m}} & u_{k k} & u_{k} \\
u_{\bar{m}} & u_{k} & 0
\end{array}\right|,
$$

for any $\ell, m \in\{1, \ldots, n-1\}$ and $A=\left(a_{\ell m}\right)_{\ell, m=1}^{n-1}$. Let $\tilde{B}_{i j}(x)$ denote the $(i, j)-t h$ cofactor of the bordered Hessian matrix. Then,

$$
s_{i j}(p, m)=\frac{-1}{|A(x)|\left(u_{k}(x)\right)^{n}} \tilde{B}_{i j}(x) .
$$

1 See Debreu (1972).
${ }^{2}$ It can easily be shown that this theorem is correct even if $n=2$. In fact, since $S(p, m) p=0$, we have $p_{1} s_{11}+p_{2} s_{12}=0$. Since $s_{11}<0$, we have $s_{12}>0$ and thus good 1 must be substitute to good 2 . Meanwhile,

$$
\tilde{B}_{12}=(-1)^{1+2}\left|\begin{array}{cc}
u_{12} & u_{1} \\
u_{2} & 0
\end{array}\right|=u_{1} u_{2}>0
$$

and thus we have $\operatorname{sgn}\left(s_{12}\right)=\operatorname{sgn}\left(\tilde{B}_{12}\right)$.

Suppose that Lemma 1 is correct. We can then check that the matrix A is negative definite and symmetric. ${ }^{3}$ Hence, the sign of $|A|$ is the same as $(-1)^{n-1}$ and our theorem holds.

Proof of Lemma 1. In this proof, we abbreviate $(p, m)$ and $x$ for notational simplicity. Without loss of generality, we assume that $i \neq n \neq j$ and choose $k=n$. Define $g=\frac{1}{u_{n}} D u$. If $x=f(p, m)$, then by the Lagrange multiplier rule, there exists $\lambda>0$ such that $D u(x)=\lambda p$. Therefore, there exists $\mu>0$ such that $g(x)=\mu p$. By Walras' law of $f$, we have

$$
g(x) \cdot x=\mu p \cdot x=\mu p \cdot f(p, m)=\mu m .
$$

Therefore, by homogeneity of degree zero of $f$,

$$
x=f(p, m)=f\left(\frac{1}{\mu} p, \frac{1}{\mu} m\right)=f(g(x), g(x) \cdot x) .
$$

Such a function, $g$, is sometimes called an inverse demand function.
Now, the matrix $\left(\frac{\partial g_{\ell}}{\partial x_{m}}-\frac{\partial g_{\ell}}{\partial x_{n}} g_{m}\right)_{\ell, m=1}^{n-1}$ is called the Antonelli matrix of the inverse demand function $g .{ }^{4}$ To compute this,

$$
\begin{aligned}
\frac{\partial g_{\ell}}{\partial x_{m}}-\frac{\partial g_{\ell}}{\partial x_{n}} g_{m} & =\frac{u_{\ell m} u_{n}-u_{\ell} u_{n m}}{\left(u_{n}\right)^{2}}-\frac{u_{\ell n} u_{n}-u_{\ell} u_{n n}}{\left(u_{n}\right)^{2}} \frac{u_{m}}{u_{n}} \\
& =\frac{-1}{\left(u_{n}\right)^{3}}\left[-u_{\ell m}\left(u_{n}\right)^{2}+u_{\ell} u_{n} u_{n m}+u_{\ell n} u_{m} u_{n}-u_{\ell} u_{m} u_{n n}\right] \\
& =\frac{-1}{\left(u_{n}\right)^{3}}\left|\begin{array}{ccc}
u_{\ell m} & u_{\ell n} & u_{\ell} \\
u_{n m} & u_{n n} & u_{n} \\
u_{m} & u_{n} & 0
\end{array}\right| \\
& =a_{\ell m} .
\end{aligned}
$$

Hence, we have that $A$ is the Antonelli matrix of $g$.
Samuelson (1950) showed that, ${ }^{5}$

$$
\left(\begin{array}{ccc}
s_{11} & \ldots & s_{1, n-1} \\
\vdots & \ddots & \vdots \\
s_{n-1,1} & \ldots & s_{n-1, n-1}
\end{array}\right)=A^{-1}
$$

Hence, $A$ is negative definite and symmetric. Let $\tilde{A}_{i j}$ denote the $(i, j)$-th cofactor of $A$. Then,

$$
s_{i j}=s_{j i}=\frac{1}{|A|} \tilde{A}_{i j}
$$

It suffices to show that,

$$
\tilde{A}_{i j}=\frac{-1}{\left(u_{n}\right)^{n}} \tilde{B}_{i j}
$$

Now, we introduce a lemma.

[^2]LEMMA 2. For any $\ell \in\{1, \ldots, n-1\}$, let $\hat{\ell}=\ell$ if $\ell<j$ and $\hat{\ell}=\ell+1$ otherwise. Define

$$
h:\left(\begin{array}{ccc}
c_{11} & \ldots & c_{1 n} \\
\vdots & \ddots & \vdots \\
c_{n-2,1} & \ldots & c_{n-2, n}
\end{array}\right) \mapsto\left|\begin{array}{ccc}
b_{11} & \ldots & b_{1, n-2} \\
\vdots & \ddots & \vdots \\
b_{n-2,1} & \ldots & b_{n-2, n-2}
\end{array}\right|
$$

where,

$$
b_{\ell m}=\frac{-1}{\left(u_{n}\right)^{3}}\left|\begin{array}{ccc}
c_{\ell m} & c_{\ell, n-1} & c_{\ell n} \\
u_{n \hat{m}} & u_{n n} & u_{n} \\
u_{\hat{m}} & u_{n} & 0
\end{array}\right|
$$

for any $\ell, m \in\{1, \ldots, n-2\}$. Then,

$$
h\left(c_{1}, \ldots, c_{n-2}\right)=\frac{-1}{\left(u_{n}\right)^{n}}\left|\begin{array}{ccccccc}
c_{11} & \ldots & c_{1, j-1} & c_{1 j} & \ldots & c_{1, n-1} & c_{1 n} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots \\
c_{n-2,1} & \ldots & c_{n-2 . j-1} & c_{n-2, j} & \ldots & c_{n-2, n-1} & c_{n-2, n} \\
u_{n 1} & & u_{n, j-1} & u_{n, j+1} & \ldots & u_{n n} & u_{n} \\
u_{1} & \ldots & u_{j-1} & u_{j+1} & \ldots & u_{n} & 0
\end{array}\right|
$$

PROOF OF LEMMA 2. In general, a real-valued function $L$ defined on $(V)^{\alpha}$, where $V$ is any linear space, is called an $\alpha$-tensor on $V$ if for any $\beta \in\{1, \ldots, \alpha\}$ and fixed $v_{1}, \ldots, v_{\beta-1}, v_{\beta+1}, \ldots, v_{\alpha} \in V$, the function

$$
v \mapsto L\left(v_{1}, \ldots, v_{\beta-1}, v, v_{\beta+1}, \ldots, v_{\alpha}\right)
$$

is linear. A tensor $L$ is called antisymmetric if for any $\beta, \gamma \in\{1, \ldots, \alpha\}$ with $\beta \neq \gamma$,

$$
L\left(v_{1}, \ldots, v_{\beta}, \ldots, v_{\gamma}, \ldots, v_{\alpha}\right)=-L\left(v_{1}, \ldots, v_{\gamma}, \ldots, v_{\beta}, \ldots, v_{\alpha}\right)
$$

It is known that if $V$ is an $n$-dimensional space and $\alpha \leq n$, then the space of all $\alpha$ antisymmetric tensors is an $\frac{n!}{\alpha!(n-\alpha)!}$-dimensional linear space. ${ }^{6}$

Let $e_{k}$ denote the $k$-th unit vector. Then, the functions $L_{\ell, m}:\left(c_{1}, \ldots, c_{n-2}\right) \mapsto$ $\operatorname{det}\left(c_{1}, \ldots, c_{n-2}, e_{\ell}, e_{m}\right)$ with $1 \leq m<\ell \leq n$ are $(n-2)$-antisymmetric tensors on $\mathbb{R}^{n}$. Moreover, the family $\left(L_{\ell, m}\right)$ consists of a basis of the space of all $(n-2)$-antisymmetric tensors on $\mathbb{R}^{n}$. ${ }^{7}$ Now, it can easily be shown that $h$ is also an $(n-2)$-antisymmetric tensor on $\mathbb{R}^{n}$. Therefore, there exists $x_{2,1}, x_{3,1}, \ldots, x_{n, n-1} \in \mathbb{R}$ such that

$$
h=\sum_{1 \leq m<\ell \leq n} x_{\ell, m} L_{\ell, m}
$$

We shall detect the numbers $x_{\ell, m}$. At first, to set $c_{s}=e_{s}$ for all $s \in\{1, \ldots, n-2\}$, we have $L_{q, r}\left(c_{1}, \ldots, c_{n-2}\right)=0$ if $q \neq n$ or $r \neq n-1$, and $L_{n, n-1}\left(c_{1}, \ldots, c_{n-2}\right)=-1$. Meanwhile, $b_{q r}=\frac{1}{u_{n}}$ if $q=r$ and $b_{q r}=0$ otherwise. Hence, $h\left(c_{1}, \ldots, c_{n-2}\right)=$ $\frac{1}{\left(u_{n}\right)^{n-2}}$. Therefore, we obtain

[^3]$$
x_{n, n-1}=\frac{-1}{\left(u_{n}\right)^{n}}\left(u_{n}\right)^{2} .
$$

Secondly, let $m \in\{1, \ldots, n-2\}$ and set $c_{s}=e_{s}$ if $s<m$ and $c_{s}=e_{s+1}$ otherwise. Then, $L_{q, r}\left(c_{1}, \ldots, c_{n-2}\right)=0$ if $q \neq n$ or $r \neq m$, and $L_{n, m}\left(c_{1}, \ldots, c_{n-2}\right)=(-1)^{n-m}$. Meanwhile, $b_{q r}=\frac{1}{u_{n}}$ if $q=r \leq m-1$ or $m \leq q=r-1 \leq n-3, b_{n-2, r}=\frac{-u_{\hat{r}}}{\left(u_{n}\right)^{2}}$, and $b_{q r}=0$ otherwise. Therefore, $h\left(c_{1}, \ldots, c_{n-2}\right)=\frac{(-1)^{n-m-1}}{\left(u_{n}\right)^{n-1}} u_{\hat{m}}$, and thus we obtain

$$
x_{n, m}=\frac{-1}{\left(u_{n}\right)^{n}} u_{\hat{m}} u_{n} .
$$

Thirdly, let $m \in\{1, \ldots, n-2\}$ and set $c_{s}=e_{s}$ if $s<m, c_{s}=e_{s+1}$ if $m \leq s<$ $n-2$ and $c_{n-2}=e_{n}$. Then, $L_{q, r}\left(c_{1}, \ldots, c_{n-2}\right)=0$ if $q \neq n-1$ or $r \neq m$, and $L_{n-1, m}\left(c_{1}, \ldots, c_{n-2}\right)=(-1)^{n-m-1}$. Meanwhile, $b_{q r}=\frac{1}{u_{n}}$ if $q=r \leq m-1$ or $m \leq q=r-1 \leq n-3, b_{n-2, r}=\frac{-1}{\left(u_{n}\right)^{3}}\left(u_{n \hat{r}} u_{n}-u_{n n} u_{\hat{r}}\right)$ and $b_{q r}=0$ otherwise. Therefore, $h\left(c_{1}, \ldots, c_{n-2}\right)=\frac{(-1)^{n-m}}{\left(u_{n}\right)^{n}}\left(u_{n n} u_{\hat{m}}-u_{n \hat{m}} u_{n}\right)$, and thus we obtain

$$
x_{n-1, m}=\frac{-1}{\left(u_{n}\right)^{n}}\left(u_{n n} u_{\hat{m}}-u_{n \hat{m}} u_{n}\right) .
$$

Finally, let $\ell, m \in\{1, \ldots, n-2\}$ and $m<\ell$, and set $c_{s}=e_{s}$ if $s<m, c_{s}=e_{s+1}$ if $m \leq s<\ell-1$ and $c_{s}=e_{s+2}$ otherwise. Then, $L_{q, r}\left(c_{1}, \ldots, c_{n-2}\right)=0$ if $q \neq \ell$ or $r \neq m$, and $L_{\ell, m}\left(c_{1}, \ldots, c_{n-2}\right)=(-1)^{n-\ell-m}$. Meanwhile, $b_{q r}=\frac{1}{u_{n}}$ if $q=r \leq m-1$, $m \leq q=r-1 \leq \ell-2$, or $\ell-1 \leq q=r-2 \leq n-4, b_{n-3, r}=\frac{-u_{\hat{r}}}{\left(u_{n}\right)^{2}}, b_{n-2, r}=$ $\frac{-1}{\left(u_{n}\right)^{3}}\left(u_{n \hat{r}} u_{n}-u_{n n} u_{\hat{r}}\right)$, and $b_{q r}=0$ otherwise. Therefore,

$$
\begin{aligned}
h\left(c_{1}, \ldots, c_{n-2}\right) & =\frac{(-1)^{n-\ell-m-1}}{\left(u_{n}\right)^{n+1}}\left(u_{\hat{m}}\left(u_{n \hat{\ell}} u_{n}-u_{n n} u_{\hat{\ell}}\right)-u_{\hat{\ell}}\left(u_{n \hat{m}} u_{n}-u_{n n} u_{\hat{m}}\right)\right) \\
& =\frac{(-1)^{n-\ell-m-1}}{\left(u_{n}\right)^{n}}\left(u_{n \hat{\ell}} u_{\hat{m}}-u_{n \hat{m}} u_{\hat{\ell}}\right),
\end{aligned}
$$

and thus, we obtain

$$
x_{\ell, m}=\frac{-1}{\left(u_{n}\right)^{n}}\left(u_{n \hat{\ell}} u_{\hat{m}}-u_{n \hat{m}} u_{\hat{\ell}}\right) .
$$

Then, the claim of Lemma 2 can be easily verified by the multilinearity of the determinant.

Now, set $c_{\ell m}=u_{\ell m}$ if $\ell<i$ and $m<j, c_{\ell m}=u_{\ell, m+1}$ if $\ell<i$ and $j \leq m<n$, $c_{\ell n}=u_{\ell}$ if $\ell<i, c_{\ell m}=u_{\ell+1, m}$ if $i \leq \ell$ and $m<j, c_{\ell m}=u_{\ell+1, m+1}$ if $i \leq \ell$ and $j \leq m<n$, and $c_{\ell n}=u_{\ell+1}$ if $i \leq \ell$. Then,

$$
\tilde{A}_{i j}=(-1)^{i+j} h\left(c_{1}, \ldots, c_{n-2}\right)
$$

$$
\begin{aligned}
& =\frac{(-1)^{i+j+1}}{\left(u_{n}\right)^{n}}\left|\begin{array}{ccccccc}
u_{11} & \ldots & u_{1, j-1} & u_{1, j+1} & \ldots & u_{1 n} & u_{1} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots \\
u_{i-1,1} & \ldots & u_{i-1, j-1} & u_{i-1, j+1} & \ldots & u_{i-1, n} & u_{i-1} \\
u_{i+1,1} & \ldots & u_{i+1, j-1} & u_{i+1, j+1} & \ldots & u_{i+1, n} & u_{i+1} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots \\
u_{n 1} & \ldots & u_{n, j-1} & u_{n, j+1} & \ldots & u_{n n} & u_{n} \\
u_{1} & \ldots & u_{j-1} & u_{j+1} & \ldots & u_{n} & 0
\end{array}\right| \\
& =\frac{-1}{\left(u_{n}\right)^{n}} \tilde{B}_{i j},
\end{aligned}
$$

This completes the proof of Lemma 1.

## 3. CONCLUSION

We have provided a formula calculating the sign of the $(i, j)$-th element of the Slutsky matrix. This sign can be used to determine whether good $i$ is substitute or complementary to good $j$. Hence, our formula is useful for determining whether some good is a substitute good of another good or not.

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[^1]:    Acknowledgments. We are grateful for Masataka Eguchi for his comments and suggestions.
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[^2]:    ${ }^{3}$ We will verify this result later.
    ${ }^{4}$ See Katzner (1970) or Hurwicz and Richter (1979) for more detailed arguments.
    ${ }^{5}$ See also Hosoya (2010), or Hosoya and Yu (2012).

[^3]:    ${ }^{6}$ See Ch. 4 of Guillemin and Pollack (1974) for more detailed arguments.
    7 We can determine that this space is an $\frac{n(n-1)}{2}$ dimensional linear space using the formula of dimension on the space of antisymmetric tensors. Because ( $L_{\ell, m}$ ) is an $\frac{n(n-1)}{2}$ linearly independent family of antisymmetric tensors, it is a basis of this space.

