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ENDOGENOUS GROWTH WITH ENVIRONMENTAL POLLUTION AND DEPRECIATION OF PUBLIC CAPITAL: A THEORETICAL NOTE

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Abstract: In this note, we develop a model of endogenous growth in which public infrastructure maintenance expenditure and abatement expenditure play special role to explain economic growth. We consider industrial production as the source of environmental pollution; and treat public capital depreciation as endogenous, being dependent upon environmental quality and maintenance expenditure. We find out the properties of the steady-state equilibrium growth path; and then analyze the properties of growth rate maximizing fiscal policy along this path.

Key words: Endogenous growth, Public capital, Depreciation, Maintenance expenditure, Environmental pollution, Abatement expenditure.

JEL Classification Number: H2, H3, H4, O4, Q2, Q5.

1. INTRODUCTION

The literature on endogenous growth theory has received substantial contribution from models dealing with the role of public capital accumulation. Models developed by Barro (1990), Futagami, Morita, Shibata (hereafter called FMS) (1993), Dasgupta (1999) etc.[1], belong to this set. However, these models do not deal with the problem of depreciation of public capital and the role of maintenance expenditure.

Rioja (2003) first introduces these problems in a FMS (1993) type of model where domestic income tax revenues finance maintenance expenditure and foreign aid finances

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[1] See, for example, Irmen and Kuehnel (2009).

new public investment. Kalaitzidakis and Kalyvitis (hereafter called KK) (2004) introduce learning by doing effect and an adjustment cost of private investment in a FMS (1993) type of model; and show that the steady-state equilibrium growth rate maximizing tax rate exceeds the competitive output share of public capital. Dioikitopoulos and Kalyvitis (hereafter called DK) (2008) introduce a negative congestion effect of public capital but ignore the role of learning by doing effect of private capital accumulation and derive results similar to those in KK (2004). In Agenor (2009) the maintenance expenditure plays a dual role to increase the durability as well as the efficiency of public capital. However, none of these works considers the role of environmental pollution on economic growth and on the depreciation of public capital. On the contrary, Greiner (2005) and Economides and Philippopoulos (hereafter called EP) (2008), who deal with the interaction between economic growth and environmental pollution using the Barro-FMS framework do not analyze the problem of depreciation of public capital.

In this paper, we develop a model of endogenous growth where the problem of depreciation of public capital is worsened by environmental pollution. In our model, the level of net public capital investment not only varies inversely with the size of private capital and directly with the level of maintenance expenditure but also varies directly with the environmental quality. Environmental quality accumulates over time through abatement activities and degrades through pollution.

Environmental quality enters as an additional argument in the depreciation function in the present model. We now turn to explain the motivation of this assumption. For example, public irrigation programme uses canal and river water to irrigate fields of crops. With pollutants in the water, government has to bear the cost to treat and cleanse it before it can release the water to the fields. Industrial pollutants, emitted as smoke react with air forming oxides, which precipitate in the form of acid rain. This causes severe damage to heritage buildings as well as other public properties increasing their maintenance cost. Industrial effluents also contaminate water posing serious health hazards to workers. In turn such loss of health takes a heavy toll on public health insurance payment; and thus government has to spend more to maintain proper health among the population. Global warming leads to natural disasters like floods, earthquakes, cyclones, etc.; and these, in turn, cause severe damages to infrastructural capitals like roads, electric lines, power plants, buildings, industrial plants, etc.

We derive following results analyzing our model. The growth rate maximizing income tax rate and the abatement expenditure rate in the steady state equilibrium depend on the pollution-output coefficient. However, the share of maintenance expenditure in the budget appears to be independent of the pollution-output coefficient. In DK (2008), KK (2004) and also in Agenor (2009), there is no environmental pollution; and hence the growth rate maximizing income tax rate and the ratio of public investment to national income in the steady state equilibrium do not depend on pollution-output coefficient. Secondly, the optimum ratio of combined expenditure on net public investment and maintenance to national income is not unambiguously greater than the competitive output share of the public capital in this steady-state equilibrium. Moreover, this optimum ratio is dependent on the pollution-output coefficient. Both DK (2008) and KK

(2004) show that the optimum ratio of combined expenditure on net public investment and maintenance to national income is always greater than the competitive output share of public capital, while it is shown to be equal to the latter by Agenor (2009).

The rest of the paper is organized as follows. Section 2 presents the basic model. Properties of the steady-state equilibrium and optimal fiscal policies are analyzed in section 3. Concluding remarks are made in section 4.

2. THE MODEL

The economy produces only one final good with private capital and public capital as inputs. Labour endowment is normalized to unity. All markets are competitive; and every producer maximizes profit. The government imposes a proportional income tax on the representative household who consumes a part of the post-tax income and invests the other part. The environmental quality is an accumulable input. It deteriorates with pollution caused by the production of the final good; and is improved by abatement activities of the government. Every household maximizes her lifetime utility defined as the infinite integral of the discounted present value of instantaneous utility where instantaneous utility is a positive and concave function of the level of consumption; and the rate of discount is assumed to be a constant.

Following equations describe the model.

$$Y = K^{\alpha} G^{1-\alpha} \quad \text{with} \quad 0 < \alpha < 1\,; \tag{1}$$

$$\dot{K} = (1 - \tau)\, Y - C\,; \tag{2}$$

$$I = \mu\, (\tau - T)\, Y\,; \tag{3}$$

$$M = (1 - \mu)\, (\tau - T)\, Y\,; \tag{4}$$

$$\dot{G} = \frac{I}{m}\,; \tag{5}$$

$$m = K^{\psi+\eta} E^{-\psi} M^{-\eta} \quad \text{with} \quad 0 < \psi\,, \quad \eta < 1\,; \tag{6}$$

$$\dot{E} = TY - \delta Y \quad \text{with} \quad 0 < \delta < 1\,; \tag{7}$$

and

$$u\,(C) = \frac{C^{1-\sigma}}{1-\sigma} \quad \text{with} \quad \sigma > 0\,. \tag{8}$$

Equation (1) describes the CRS Cobb-Douglas production function of the final good Here Y, K and G represent level of output produced, stocks of private capital and public capital respectively.

The budget constraint of the representative household is given by equation (2). Here τ is the proportional income tax rate.

Equation (3) shows the fraction of non-abatement expenditure used to finance public investment. T is the ratio of abatement expenditure to national income; and μ is the fraction of non-abatement expenditure used to finance public investment. Equation (4) shows the fraction of non-abatement expenditure going to the maintenance of public capital.

Accumulation of public capital takes place according to equation (5). Here $\dot{G}$ is the net investment to public capital formation. $I - \dot{G}$ is the level of depreciation of public capital. Thus using equation (5) we have

$$I - \dot{G} = I\left(1 - \frac{1}{m}\right).$$

Hence this equation combined with equation (6) shows that the level of depreciation of public capital varies positively with the stock of private capital and inversely with the level of maintenance expenditure and the stock of environmental quality[2]. Here E stands for the stock of environmental quality and M stands for the level of maintenance expenditure.

The increased usage of public infrastructure made by private firms lowers the durability of public capital; and thus the depreciation of public capital varies positively with the scale of expansion of the private economy. Maintenance of public investment goods and protection of environment raise its durability and thus lowers the depreciation rate[3].

Equation (7) shows how environmental quality changes over time depending upon the magnitudes of pollution and abatement activity. That abatement activities bring improvements in environmental quality is supported by empirical works[4]. Here environmental pollution is proportional to the level of production with $\delta > 0$ being the constant pollution-output coefficient. Many models of environmental pollution assume pollution to be a positive function of the level of production[5] of the final good.

Equation (8) describes the instantaneous utility function of the household which is familiar in the literature.

3. STEADY-STATE EQUILIBRIUM

The representative household's problem is to maximize $\int_0^\infty u(C)e^{-\rho t}dt$ subject to equations (1), (2) and (8).

The demand rate of growth of consumption is derived from this maximizing problem as follows.

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}\left[\alpha(1-\tau)\left(\frac{E}{K}\right)^{1-\alpha}\left(\frac{G}{E}\right)^{1-\alpha} - \rho\right]. \tag{9}$$

The growth rates of the three state variables, K, G and E, can be expressed as follows.

$$\frac{\dot{K}}{K} = (1-\tau)\left(\frac{E}{K}\right)^{1-\alpha}\left(\frac{G}{E}\right)^{1-\alpha} - \frac{C}{K}; \tag{10}$$

2 Total depreciation of public capital may not be positive always. Here, depending upon the ratios of maintenance expenditure to private capital and environmental quality to private capital, total depreciation can take a negative value. This can be interpreted as an efficiency gain or a virtual expansion of the existing public capital stock brought about by maintenance expenditure as well as the stock of environmental quality.

3 Many authors like Fisher and Turnovsky (1998) only consider scale of the economy congesting public capital services available to private firms. Its effects on durability of public capital are not considered, nor does environment play a role in its durability in their model.

4 See the works of Liddle (2001), Managi (2006), Dinda (2005), Di Vita (2008), Smulders and Gradus (1996), Byrne (1997), etc.

5 For example, see the works of Liddle (2001), Oueslati (2002), Hartwick (1991), Smulders and Gradus (1996), Byrne (1997), Gruver (1976), Dinda (2005), etc.

$$\frac{\dot{G}}{G} = \mu(1-\mu)^{\eta}(\tau - T)^{1+\eta}\left(\frac{E}{K}\right)^{(1+\eta)(1-\alpha)-1+\psi}\left(\frac{G}{E}\right)^{(1+\eta)(1-\alpha)-1} ; \tag{11}$$

and

$$\frac{\dot{E}}{E} = (T-\delta)\left(\frac{E}{K}\right)^{-\alpha}\left(\frac{G}{E}\right)^{1-\alpha} . \tag{12}$$

We consider a steady-state growth equilibrium where all macroeconomic variables grow at the same rate, g. It can be easily shown that this steady-state equilibrium satisfies saddle-point stability. Hence, we have

$$\frac{\dot{C}}{C} = \frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{\dot{E}}{E} = \frac{\dot{G}}{G} = g . \tag{13}$$

Using equations (9) to (13) we arrive at the following equation[6] to solve for g.

$$g^{(1+\psi)(1-\alpha)}(\sigma g + \rho)^{\alpha-(\psi+\eta)(1-\alpha)} = \alpha^{\alpha-(\psi+\eta)(1-\alpha)}\mu^{1-\alpha}(1-\mu)^{\eta(1-\alpha)}$$
$$(T-\delta)^{\psi(1-\alpha)}(1-\tau)^{\alpha-(\psi+\eta)(1-\alpha)}(\tau - T)^{(1+\eta)(1-\alpha)} . \tag{14}$$

Here $\alpha - (\psi + \eta)(1-\alpha)$ represents the social elasticity of output with respect to private capital in the steady-state equilibrium because this is obtained when the negative external effect of private capital accumulation on the depreciation of public capital is internalized. This internalization can be done only by a social planner and not by a competitive private firm. We assume that $\alpha - (\psi + \eta)(1-\alpha) > 0$. This implies that the positive marginal technological contribution of private capital on output exceeds its negative marginal external effect that takes place through depreciation of public capital. It can be easily shown that the L. H. S. of equation (14) is always a monotonically increasing function of g; and this is also shown in appendix (B). The R. H. S. of equation (14) is independent of g. Hence the existence of unique steady-state equilibrium growth rate is guaranteed when $0 < \mu < 1$ and $\delta < T < \tau < 1$. We summarize the result analyzed above in the following proposition.

PROPOSITION 1. *There exists a unique steady state equilibrium growth rate in the decentralized economy given the interior values of the income tax rate, the abatement expenditure rate and the public investment allocation share when $\delta < T < \tau < 1$.*

4. GROWTH MAXIMIZING POLICY

Here, we assume that the government maximizes the steady-state equilibrium growth rate with respect to fiscal instruments. The L. H. S. of equation (14) is a monotonically increasing function of g. Thus maximization of the growth rate with respect to policy variables subject to equation (14) is synonymous to maximizing the R. H. S. of equation (14) with respect to those policy variables.

Maximizing the R. H. S. of equation (14) with respect to τ, T and μ, we obtain the following equations.

$$\tau^{\star} = 1 - (1-\delta)\{\alpha - (\eta + \psi)(1-\alpha)\} ; \tag{15}$$

[6] Equation (14) is derived in appendix (B).

$$T^\star = \delta + (1-\delta)\,\psi\,(1-\alpha)\ ; \tag{16}$$

and

$$\mu^\star = \frac{1}{1+\eta}\,. \tag{17}$$

Using equations (15) and (16), we have

$$\tau^\star - T^\star = (1-\delta)\,(1-\alpha)\,(1+\eta) = (1-\alpha) + (1-\alpha)\,(\eta - \delta - \eta\delta)\ . \tag{18}$$

Here $\tau^\star - T^\star$ is the growth rate maximizing combined share of public investment expenditure and maintenance expenditure in the total output; and it is greater (less) than the competitive output share of the public capital when $\eta - \delta - \eta\delta > (<)\ 0$. Here, $1 - \mu^\star = \frac{\eta}{1+\eta}$; and hence $\eta - \delta - \eta\delta < 0$ implies that the pollution-output coefficient, δ, is larger than the share of maintenance expenditure, $(1 - \mu^\star)$. Hence, if δ takes a high value, then it will cause the growth rate maximizing combined share of public investment and maintenance expenditure in national income to scale down sufficiently so that it falls short of the competitive output share of the public capital. This result is different from that found in KK (2004) and DK (2008) where this optimum combined share is unambiguously greater than the competitive output share of the public capital. This is so because $\delta = 0$ in these models. This growth rate maximizing share is equal to the competitive output share of public capital in Agenor (2009) due to absence of pollution and the dual role of maintenance expenditure. However, the growth rate maximizing income tax rate is equal to the competitive output share of the public capital in Greiner (2005) due to absence of depreciation of public capital. If $\eta = \delta = 0$, then $\tau^\star - T^\star = (1-\alpha)$; and thus, in this model we get back the result of Barro (1990) and of Futagami *et al.* (1993).

Since δ is the pollution-output coefficient, $(1-\delta)$ can be interpreted as the fraction of unpolluted output produced in the system. Hence$(1-\delta)\,\{\alpha - (\eta + \psi)\,(1-\alpha)\}$ is the social elasticity of unpolluted output with respect to private capital; and equation (15) implies that the optimum income tax rate is equal to one minus this social elasticity. In Futagami *et al.* (1993), $\eta + \psi = \delta = 0$, because there is neither any external effect of private capital accumulation nor any environmental pollution. So the social elasticity of unpolluted output equals to the private elasticity of total output; and hence the growth rate maximizing income tax rate is equal to the competitive output share of public capital. The same is true in Greiner (2005). In the present model, private elasticity of total output with respect to private capital always exceeds the corresponding social elasticity of unpolluted output. Hence the optimum income tax rate exceeds the competitive output share of public capital in this model. In KK (2004), the optimal income tax rate exceeds the social elasticity of output with respect to public capital due to the positive external learning-by-doing effect of private capital accumulation.

We can state the following proposition.

PROPOSITION 2. (i) *The growth rate maximizing income tax rate, abatement expenditure rate and public investment allocation share in the steady state growth equilibrium are given by*

$$\tau^{\star} = 1 - (1-\delta)\left\{\alpha - (\eta + \psi)(1-\alpha)\right\} ,$$
$$T^{\star} = \delta + (1-\delta)\,\psi\,(1-\alpha) ,$$

and

$$\mu^{\star} = \frac{1}{1+\eta} .$$

The income tax rate is less than unity if the social elasticity of unpolluted output with respect to private capital is less than unity.

(ii) *The growth rate maximizing ratio of combined expenditure on net public investment and on maintenance of public capital to national income in the steady-state growth equilibrium varies inversely with the magnitude of the pollution-output coefficient. It is greater (less) than the competitive output share of public capital if the pollution-output coefficient is smaller (greater) than the share of maintenance expenditure.*

5. CONCLUSION

In this note, we develop an endogenous growth model with special reference to the analysis of the depreciation of public capital and its interaction with maintenance expenditure and environmental pollution. The model is more general than the existing models like KK (2004), DK (2008), Agenor (2009), Greiner (2005), EP (2008), etc.

We derive some interesting results from our model. First, the optimal income tax rate exceeds the competitive output share of the public capital and this result is different from the FMS (1993) result that establishes equality between these two. The optimal share of the combined public expenditure on investment and maintenance of public capital in national income is not necessarily greater than the competitive output share of public capital in the final good sector. This result is different from the corresponding one found in DK (2008), KK (2004) and Agenor (2009). Secondly, the optimum income tax rate and the optimum share of the combined public expenditure to national income are dependent on the pollution-output coefficient.

Our model is abstract and fails to consider many important features. For example we do not consider unemployment problem. The possibilities of technological change and human capital accumulation are also ruled out here. Technological change and human capital accumulation may help to reduce the depreciation of public capital; and the increase in unemployment may aggravate this problem. We also ignore the external effect of public capital and environmental quality on the utility function of the consumer. We encourage future researchers to take care of these problems.

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APPENDIX (A)

DERIVATION OF EQUATION (14) IN SECTION 3.1

Using equations (1) to (7), (9) and (13) we have the following equations.

$$g = \frac{\dot{C}}{C} = \frac{1}{\sigma}\left[\alpha\,(1-\tau)\left(\frac{E}{K}\right)^{1-\alpha}\left(\frac{G}{E}\right)^{1-\alpha} - \rho\right]; \tag{A1}$$

$$g = \frac{\dot{K}}{K} = (1-\tau)\left(\frac{E}{K}\right)^{1-\alpha}\left(\frac{G}{E}\right)^{1-\alpha} - \frac{C}{K}; \tag{A2}$$

$$g = \frac{\dot{E}}{E} = (T-\delta)\left(\frac{E}{K}\right)^{-\alpha}\left(\frac{G}{E}\right)^{1-\alpha}; \tag{A3}$$

and

$$g = \frac{\dot{G}}{G} = \mu\,(1-\mu)^{\eta}\,(\tau-T)^{1+\eta}\left(\frac{E}{K}\right)^{(1+\eta)(1-\alpha)-1+\psi}\left(\frac{G}{E}\right)^{(1+\eta)(1-\alpha)-1}; \tag{A4}$$

From equation (A1) we have,

$$\frac{E}{K} = \left\{ \frac{(\sigma g + \rho)}{\alpha (1 - \tau)} \right\}^{\frac{1}{1-\alpha}} \left(\frac{G}{E} \right)^{-1} . \tag{A5}$$

Again, from equation (A3) we have,

$$\frac{E}{K} = \left(\frac{g}{T - \delta} \right)^{-\frac{1}{\alpha}} \left(\frac{G}{E} \right)^{\frac{1-\alpha}{\alpha}} . \tag{A6}$$

Using equations (A5) and (A6) we derive the following equation.

$$\frac{G}{E} = \left(\frac{g}{T - \delta} \right) \left\{ \frac{(\sigma g + \rho)}{\alpha (1 - \tau)} \right\}^{\frac{\alpha}{1-\alpha}} . \tag{A7}$$

Using equations (A6) and (A7) we obtain the following equation.

$$\frac{E}{K} = \frac{(\sigma g + \rho)}{g} \frac{(T - \delta)}{\alpha (1 - \tau)} . \tag{A8}$$

Now, using equations (A4), (A7) and (A8) we derive the following equation.

$$g = \mu (1 - \mu)^{\eta} (\tau - T)^{1+\eta} \left[\frac{(\sigma g + \rho)}{g} \frac{(T - \delta)}{\alpha (1 - \tau)} \right]^{(1+\eta)(1-\alpha)-1+\psi}$$
$$\times \left[\left(\frac{g}{T - \delta} \right) \left\{ \frac{(\sigma g + \rho)}{\alpha (1 - \tau)} \right\}^{\frac{\alpha}{1-\alpha}} \right]^{(1+\eta)(1-\alpha)-1} ,$$

or,

$$g^{(1+\psi)(1-\alpha)} (\sigma g + \rho)^{\alpha-(\psi+\eta)(1-\alpha)} = \alpha^{\alpha-(\psi+\eta)(1-\alpha)} \mu^{1-\alpha} (1 - \mu)^{\eta(1-\alpha)}$$
$$(T - \delta)^{\psi(1-\alpha)} (1 - \tau)^{\alpha-(\psi+\eta)(1-\alpha)} (\tau - T)^{(1+\eta)(1-\alpha)} . \tag{A9}$$

This is same as equation (14) in the body of the paper.

We denote the L. H. S. of equation (A9) by $H (g)$ and find that

$$\begin{aligned} H' (g) &= g^{(1+\psi)(1-\alpha)-1} (\sigma g + \rho)^{\alpha-(\psi+\eta)(1-\alpha)-1} \big[(1 + \psi) (1 - \alpha) (\sigma g + \rho) \\ &\quad + \{\alpha - (\psi + \eta) (1 - \alpha)\} \sigma g \big] \\ &= g^{(1+\psi)(1-\alpha)-1} (\sigma g + \rho)^{\alpha-(\psi+\eta)(1-\alpha)-1} \big[(1 + \psi) (1 - \alpha) \rho \\ &\quad + \{1 - \eta (1 - \alpha)\} \sigma g \big] . \end{aligned}$$

Here, $H' (g) > 0$ because $0 < \alpha, \eta < 1$. So the L. H. S. of equation (A9) is a monotonically increasing function of g.

APPENDIX (B)

DERIVATION OF EQUATIONS (15), (16) AND (17) AND THE SECOND ORDER CONDITIONS IN SECTION 3.2

Maximizing the R. H. S. of equation (14) with respect to τT and μ respectively we have

$$(1+\eta)(1-\alpha)(\tau-T)^{-1} - \left\{\alpha - (\eta+\psi)(1-\alpha)\right\}(1-\tau)^{-1} = 0\,, \tag{B1}$$

$$\left[\psi(1-\alpha)(T-\delta)^{-1} - (1+\eta)(1-\alpha)(\tau-T)^{-1}\right] = 0\,, \tag{B2}$$

and

$$\left[(1-\alpha)\mu^{-1} - \eta(1-\alpha)(1-\mu)^{-1}\right] = 0\,. \tag{B3}$$

Using equations (B1), (B2) and (B3) we obtain the following expressions.

$$\tau^\star = 1 - (1-\delta)\{\alpha - (1-\alpha)(\eta+\psi)\}\ ;$$

$$T^\star = \delta + (1-\delta)\psi(1-\alpha)\ ;$$

and

$$\mu^\star = \frac{1}{1+\eta}\,.$$

These are same as equations (15), (16) and (17) in the body of the paper.

To check the second order conditions to be satisfied, we obtain the followings.

$$\begin{aligned}
&-\left[(1+\psi)(1-\alpha)g^{-2} + \sigma^2\{\alpha-(1-\alpha)(\eta+\psi)\}(\sigma g+\rho)^{-2}\right]\left(\frac{\partial g}{\partial \tau}\right)^2 \\
&+\left[(1+\psi)(1-\alpha)g^{-1} + \sigma\{\alpha-(1-\alpha)(\eta+\psi)\}(\sigma g+\rho)^{-1}\right]\frac{\partial^2 g}{\partial \tau^2} \\
&= -\left[(1+\eta)(1-\alpha)(\tau-\mathrm{T})^{-2} + \{\alpha-(1-\alpha)(\eta+\psi)\}(1-\tau)^{-2}\right];
\end{aligned} \tag{B4}$$

$$\begin{aligned}
&-\left[(1+\psi)(1-\alpha)g^{-2} + \sigma^2\{\alpha-(1-\alpha)(\eta+\psi)\}(\sigma g+\rho)^{-2}\right]\left(\frac{\partial g}{\partial T}\right)^2 \\
&+\left[(1+\psi)(1-\alpha)g^{-1} + \sigma\{\alpha-(1-\alpha)(\eta+\psi)\}(\sigma g+\rho)^{-1}\right]\frac{\partial^2 g}{\partial T^2} \\
&= -\left[\psi(1-\alpha)(T-\delta)^{-2} + (1+\eta)(1-\alpha)(\tau-\mathrm{T})^{-2}\right];
\end{aligned} \tag{B5}$$

and

$$\begin{aligned}
&-\left[(1+\psi)(1-\alpha)g^{-2} + \sigma^2\{\alpha-(1-\alpha)(\eta+\psi)\}(\sigma g+\rho)^{-2}\right]\left(\frac{\partial g}{\partial \mu}\right)^2 \\
&+\left[(1+\psi)(1-\alpha)g^{-1} + \sigma\{\alpha-(1-\alpha)(\eta+\psi)\}(\sigma g+\rho)^{-1}\right]\frac{\partial^2 g}{\partial \mu^2} \\
&= -\left[(1-\alpha)\mu^{-2} + \eta(1-\alpha)(1-\mu)^{-2}\right];
\end{aligned} \tag{B6}$$

Using equations (B1), (B2) and (B3) we obtain the followings.

$$\frac{\partial^2 g}{\partial \tau^2} = -\frac{(1+\eta)(1-\alpha)(\tau^\star - T^\star)^{-2} + \{\alpha-(1-\alpha)(\eta+\psi)\}(1-\tau^\star)^{-2}}{\left[(1+\psi)(1-\alpha)g^{-1} + \sigma\{\alpha-(1-\alpha)(\eta+\psi)\}(\sigma g+\rho)^{-1}\right]} < 0\,;$$

$$\frac{\partial^2 g}{\partial T^2} = -\frac{(1+\eta)(1-\alpha)(\tau^\star - T^\star)^{-2} + \psi(1-\alpha)(T^\star - \delta)^{-2}}{\left[(1+\psi)(1-\alpha)g^{-1} + \sigma\{\alpha - (1-\alpha)(\eta+\psi)\}(\sigma g + \rho)^{-1}\right]} < 0\,;$$

and

$$\frac{\partial^2 g}{\partial \mu^2} = -\frac{\left[(1-\alpha)\mu^{\star^{-2}} + \eta(1-\alpha)(1-\mu^\star)^{-2}\right]}{\left[(1+\psi)(1-\alpha)g^{-1} + \sigma\{\alpha - (1-\alpha)(\eta+\psi)\}(\sigma g + \rho)^{-1}\right]} < 0\,.$$

Thus the second order conditions are also satisfied.