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DYNAMIC DUOPOLY WITH VERTICAL DIFFERENTIATION

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Abstract: I analyse a differential duopoly game where firms, through capital accumulation over time, supply vertically differentiated goods. I show that (i) the instantaneous R&D effort of the high quality firm is larger than the low quality firm's, and therefore (ii) there are quality ranges such that, in proximity of the steady state, the low quality firm's profits are larger than the high quality firm's; (iii) the optimal quality ratio is 4/7, as in the static model by Choi and Shin (1992).

Keywords: differential games, capital accumulation, R&D, product quality
JEL Classification Number: C73, L13, O31

1. INTRODUCTION

I propose a dynamic approach to the strategic use of non-price tools in a differential game model of vertical differentiation. Non-price variables typically include product and/or process R&D, product differentiation and advertising, that firms may use in isolation or together, so as to increase the profitability of their price or quantity strategies.

Ever since the pioneering work of Spence (1975) and Mussa and Rosen (1978) on the provision of product quality by a monopolist, vertical differentiation has received wide attention within the theory of industrial organization. Several issues have been investigated in oligopoly models where firms supply goods of different quality. In Gabszewicz and Thisse (1979, 1980) and Shaked and Sutton (1982, 1983), the so-called *finiteness property* is established, according to which the number of firms that can survive in a vertically is finite. This result holds if unit costs of quality are flat enough, and the overall cost associated with the improvement of quality is an R&D cost unrelated with the scale of production. In their approach, the only costs explicitly modelled is a fixed cost which is assumed to be exogenous and arbitrarily small. Therefore, the finiteness property essentially depends on demand rather than technological conditions. The influence of the shape of the cost function on prices, market shares and profits is the topic

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of several contributions, where the cost of quality is alternatively related or unrelated with the output scale.[1]

More recent contributions deal several aspects of the technology associated with product innovation in vertically differentiated markets, through either independent ventures (Beath *et al.*, 1987; Dutta *et al.*, 1995; Rosenkranz, 1997) or joint ventures (Motta, 1992; Rosenkranz, 1995; Lambertini, 2000; Lambertini *et al.*, 2002). A result common to all these contributions is that the highest quality good is more profitable than all inferior varieties, irrespective of the specification of the cost function and, in particular, notwithstanding the assumption, common to all this literature, that the higher the quality of a good, the higher its cost.

With the exception of Beath *et al.* (1987) and Dutta *et al.* (1995), where quality improvement is modelled as the outcome of an uncertain innovation race, the above literature adopts a static approach where firms set qualities and prices (or outputs) in two stages. To the best of my knowledge, the problem of quality supply has been investigated in optimal control and differential game models only in relation with advertising strategies designed to increase goodwill and/or to compete for market shares (Kotowitz and Mathewson, 1979; Conrad, 1985; and Ringbeck, 1985).

I investigate a differential duopoly game where firms supply goods of different quality, which is the result of capital accumulation over time. The degree of vertical (quality) differentiation interacts with prices in determining market shares at every instant. The setup of market demand is borrowed from well known static models, while qualities increase over time as a result of firms' R&D investments. Instantaneous R&D costs are linear in the quality level, while the dynamics of quality is characterised by decreasing returns to R&D efforts. It is shown that the dynamic model produces situations where the low quality firm may earn, in proximity of the steady state, higher profits than the high quality firm. This conclusion holds irrespectively of whether the technologically feasible quality set is exogenously given or instead changes endogenously as a consequence of firms' optimal decisions.[2] The dynamic results also reinforce some of the wisdom we are accustomed with from the static literature. One such result is that the high quality firm always invests more than the low-quality firm. The second is that the optimal quality ratio is 4/7, i.e., the same as in the static model with zero production costs, as in Choi and Shin (1992). When dealing with the static literature on product differentiation, one may consider the case of costless quality improvements as a theoretical *curiosum* or a simplifying assumption towards the analysis of equilibrium market structure as described by the finiteness property. The dynamic analysis presented here

[1] For models where the development of quality bears upon variable costs, see Moorthy (1988); Champsaur and Rochet (1989); Cremer and Thisse (1994); Lambertini (1996). For those where quality represents a fixed cost, see Aoki and Prusa (1997); Lehmann-Grube (1997) and Lambertini (1999). A comparative evaluation is in Motta (1993).

[2] The time dimension can play a decisive role in reversing the usual profit ranking between high- and low-quality suppliers also for other reasons, as shown in a differential game by Colombo and Lambertini (2003). Provided the low-quality firm is more efficient than the high-quality firm in terms of advertising activity or investment in productive capacity, then offering a superior quality does not necessarily entail higher profits than the rival's.

seems to point out that the so-called *4/7 rule* is more than that, since the static two-stage model of Choi and Shin summarises the essential steady-state properties of a dynamic approach to the issue of vertical differentiation. When comparing static and differential games, this appears to be the exception rather than the rule.[3]

The remainder of the paper is organised as follows. Section 2 contains a brief review of the static model. The setup of the differential game is described in section 3. Section 4 deals with the closed-loop solution. In section 5 possible extensions are discussed. Section 6 contains some concluding remarks.

2. PRELIMINARIES: THE STATIC TWO-STAGE GAME

Here I briefly summarise the static two-stage model analysed in several contributions (Choi and Shin, 1992; Dutta *et al.*, 1995; Lambertini *et al.*, 2002, *inter alia*). Two single-product firms, labelled as H and L, supply goods of qualities $Q > q_H \geq q_L \geq 0$. Consumers are uniformly distributed with density equal to one over the interval $[\Theta - 1, \Theta]$, with $\Theta > 1$. Therefore, the total population of consumers is represented by a unit square. Each consumer is indexed by a marginal willingness to pay for quality $\theta \in [\Theta - 1, \Theta]$, and his net utility from consumption is:

$$U = \begin{cases} \theta q_i - p_i \geq 0 & \text{if he buys} \\ 0 & \text{if he doesn't buy} \end{cases} \tag{1}$$

where p_i is the price of the good supplied by firm i.

All variable costs are assumed to be nil, and quality improvements involve an exogenous R&D effort k_i. The unit cost of capital is ρ, and therefore the R&D cost is ρk_i. As this cost is not explicitly defined as a function of quality, it does not affect first order conditions. It is sensible to assume that a higher quality requires a higher R&D effort, and therefore $k_H > k_L$.

As in Choi and Shin (1992), partial coverage is assumed, so that market demands for the two goods are:

$$x_H = \Theta - \theta_H \; ; \; x_L = \theta_H - \theta_L , \tag{2}$$

where θ_H is the marginal willingness to pay for quality characterising the consumer who is indifferent between q_H and q_L at the price vector $\{p_H , p_L\}$, i.e., it is the solution to:

$$\theta_H q_H - p_H = \theta_H q_L - p_L \Leftrightarrow \theta_H = \frac{p_H - p_L}{q_H - q_L} , \tag{3}$$

while θ_L is the marginal evaluation of quality associated with the consumer who is indifferent between buying the low quality good and not buying at all, $\theta_L = p_L/q_L$. Firms' profits, which coincide with revenues, are:

$$\pi_H = p_H \left(\Theta - \frac{p_H - p_L}{q_H - q_L} \right) - \rho k_H \; ; \; \pi_L = p_L \left(\frac{p_H - p_L}{q_H - q_L} - \frac{p_L}{q_L} \right) - \rho k_L . \tag{4}$$

[3] For an overview, see Dockner *et al.* (2000). Another instance of the same kind is in Cellini and Lambertini (1998), where Cournot and Bertrand equilibria with differentiated products *à la* Singh and Vives (1984) are derived using the Ramsey capital accumulation model.

Firms play simultaneously a non-cooperative two-stage game, where they set qualities in the first stage and price in the second. As usual, the solution concept is subgame perfection by backward induction. The outcome is summarised in the following Proposition, the complete proof of which can be found in Choi and Shin (1992) and Wauthy (1996) (superscript *sp* stands for *subgame perfect*):

PROPOSITION 1. *At the subgame perfect equilibrium,*

- *qualities are* $q_H^{sp} = Q$ *and* $q_L^{sp} = 4Q/7$;
- *output levels are* $x_H^{sp} = 7\Theta/12$ *and* $x_L^{sp} = 7\Theta/24 = x_H^{sp}/2$;
- *prices are* $p_H^{sp} = \Theta Q/4$ *and* $p_L^{sp} = \Theta Q/14$;
- *profits are* $\pi_H^{sp} = \dfrac{7\Theta^2 q_H}{48} - \rho k_H$; $\pi_L^{sp} = \dfrac{\Theta^2 q_H}{48} - \rho k_L$.

The result that $q_L^{sp} = 4Q/7$ tells that the high quality is exactly equal to the highest (and exogenously given) feasible quality Q, while the low quality locates slightly above the middle of the quality spectrum. In the remainder of the paper, I will refer to this result as the *4/7 rule*. The above Proposition also allows to establish that $x_H^{sp} + x_L^{sp} < 1$ for all $\Theta < 8/7$. Hence, for all $\Theta \geq 8/7$, the demand functions (2) are not valid and the model must be re-specified to allow for full market coverage, with $x_L = \theta_H - (\Theta - 1)$.[4]

Now observe that

COROLLARY 1. *The duopoly equilibrium is sustainable iff*:

1. $\pi_i^{sp} > 0$ *for all* i, *which entails* $q_H > \max\left\{\dfrac{48\rho k_L}{\Theta^2}, \dfrac{48\rho k_H}{7\Theta^2}\right\}$, *where*

$$\max\left\{\frac{48\rho k_L}{\Theta^2}, \frac{48\rho k_H}{7\Theta^2}\right\} = \begin{cases} \dfrac{48\rho k_H}{7\Theta^2} & \text{for all } k_H > 7k_L \\ \dfrac{48\rho k_L}{\Theta^2} & \text{for all } k_H \in (k_L, 7k_L) \end{cases}$$

and

2. $\pi_H^{sp} > \pi_L^{sp}$, *which requires* $k_H - k_L < \dfrac{\Theta^2 q_H}{8\rho}$.

In particular, if the second condition in Corollary 1 is not met, there is no incentive for either firm to enter first and offer the high-quality good. Consequently the market remains inactive because producing the superior variety is not convenient.

Moreover, $x_H^{sp} = 7\Theta/12$ implies that $x_L^{sp} > 0$ iff $\Theta < 12/7$. Therefore, Proposition 1 also produces the following Corollary:

COROLLARY 2. *For all* $\Theta \geq 12/7$, *the market is monopolised by the high quality firm.*

Corollary 1 is an instance of the so-called *finiteness property* (Shaked and Sutton, 1983), which establishes that the demand structure of a vertically differentiated market allows for a finite number of firms operating with positive demand and profits at the

[4] See Tirole (1988, appendix to ch. 7), Rosenkranz (1995) and Wauthy (1996).

subgame perfect equilibrium. In particular, the above case is what Shaked and Sutton label as a *natural monopoly*. They use consumer income, while here I use the marginal evaluation for quality, as in Mussa and Rosen (1978) and Gabszewicz and Thisse (1979, 1980). It can be easily shown that the two approaches are equivalent, provided that consumer's utility function is concave in income.[5]

The finiteness property can be shown to hold also in models where quality affects fixed costs (see, e.g., Motta, 1993; Lehmann-Grube, 1997; and Lambertini, 1999). Of course, considering any endogenous cost function defined in terms of quality entails that the *4/7 rule* does not hold in slightly more sophisticated reformulations of the static model.

For future reference, it is worth noting that the (exogenously imposed) upper bound of the quality spectrum, Q (the highest technologically feasible quality), generates the corner solution $q_H^{sp} = Q$, as the revenues of firm H are everywhere increasing in q_H. The level of Q may be determined by the existing technology. Think, e.g., about the car industry: the maximum feasible quality of a car is largely determined by the fact that vehicles use engines burning a liquid fuel like gasoline. As soon as the new environmental-friendly engines burning hydrogen come into mass production, the level of Q will change.

3. THE DYNAMIC MODEL

The market exists over $t \in [0, \infty)$. At any t, as in the static version, a constant population of consumers is uniformly distributed with density equal to one over the interval $[\Theta - 1, \Theta]$, $\Theta > 1$. Again, the total mass of consumer is 1. The consumers who are able to buy do so at each instant $t \in [0, \infty)$. To this regard, it is worth stressing that, as purchases repeat over time, this model suits the case of non-durable goods. Each consumer is characterised by a marginal willingness to pay for quality $\theta \in [\Theta - 1, \Theta]$, and his net instantaneous utility from consumption is now defined as:

$$U = \begin{cases} \theta q_i(t) - p_i(t) \geq 0 & \text{if he buys} \\ 0 & \text{if he doesn't buy} \end{cases} \tag{5}$$

where $q_i(t)$ is the quality and $p_i(t)$ is the price of the good supplied by firm i at time t. Two single-product firms, labelled as H and L, supply goods of qualities $Q > q_H(t) \geq q_L(t) \geq 0$. The discount rate $\rho > 0$ is common to both firms.

The quality of firm i's product increases over time according to the following dynamics:

$$\frac{dq_i(t)}{dt} = a\sqrt{k_i(t)}\,, \tag{6}$$

where $k_i(t)$ is the instantaneous investment of firm i in an R&D process aimed at improving product quality, and a is a positive parameter. The initial condition for

[5] Under this condition, $\theta = \beta/u_y$, where $u_y \equiv \partial u(y)/\partial y$ is the marginal utility of income and β is a positive parameter. If $u_{yy} \equiv \partial^2 u(y)/\partial y^2 \leq 0$, the marginal willingness to pay for quality increases as income increases (see Tirole, 1988, ch. 2).

firm i is $q_i(0) = q_{i0} \geq 0$. The instantaneous cost associated to the R&D activity is $C_i\,(k_i(t)) = \rho k_i(t)$, i.e., I assume that the rental price of the capital input be equal to the discount rate ρ.[6] The foregoing assumption concerning instantaneous costs and the dynamics of quality amount to saying that decreasing returns in the production of quality take place through (6) rather than instantaneous R&D costs. Each firm bears no costs other than $C_i\,(k_i(t))$. That is, operative production costs are assumed to be nil, and therefore instantaneous profits are given by the difference between revenues and the cost of investment.

The definition of market demands is analogous to the static setup, i.e., I will focus on partial market coverage only. At any t, market demands for the two varieties are defined as follows:

$$x_H(t) = \Theta - \theta_H(t)\,;\; x_L(t) = \theta_H(t) - \theta_L(t)\,, \tag{7}$$

where $\theta_H(t)$ is the marginal willingness to pay for quality characterising the consumer who is indifferent between $q_H(t)$ and $q_L(t)$ at the price vector $\{p_H(t)\,,\ p_L(t)\}$: $\theta_H(t) = [p_H(t) - p_L(t)]/[q_H(t) - q_L(t)]$, while $\theta_L(t) = p_L(t)/q_L(t)$. Accordingly, instantaneous profits are:

$$\pi_H(t) = p_H(t)\left(\Theta - \frac{p_H(t) - p_L(t)}{q_H(t) - q_L(t)}\right) - \rho k_H(t)\,; \tag{8}$$

$$\pi_L(t) = p_L(t)\left(\frac{p_H(t) - p_L(t)}{q_H(t) - q_L(t)} - \frac{p_L(t)}{q_L(t)}\right) - \rho k_L(t)\,, \tag{9}$$

provided that $x_H(t) + x_L(t) \leq 1$.

Control variables are the price $p_i(t)$ and the R&D effort $k_i(t)$, while quality $q_i(t)$ is the state variable. Firms play simultaneously and non-cooperatively. Given the dynamic constraints and the instantaneous profit functions, the resulting Hamiltonian of firm i is not written in a linear-quadratic form, and consequently the feedback solution cannot be obtained analytically through Bellman's equation.[7] Therefore, in characterising strongly time consistent equilibria, I will focus on the memoryless closed-loop solution.

4. THE CLOSED-LOOP SOLUTION

Firm i's Hamiltonian is:

$$\mathcal{H}_i(t) = e^{-\rho t} \cdot \left\{\pi_i(t) + \lambda_{ii}(t)a\sqrt{k_i(t)} + \lambda_{ij}(t)a\sqrt{k_j(t)}\right\}, \tag{10}$$

where $\lambda_{ij}(t) = \mu_{ij}(t)e^{\rho t}$, and $\mu_{ij}(t)$ is the co-state variable associated to $q_j(t)$. The FOCs are (for brevity, in the remainder I will drop the indication of time as well as exponential discounting $e^{-\rho t}$):

[6] This is acceptable if financial markets are efficient.

[7] On the difference between the closed-loop memoryless solution and the feedback solution, see Başar and Olsder (1982, 1995^2, ch. 6; in particular, Proposition 6.1).

$$\frac{\partial \mathcal{H}_H}{\partial p_H} = \frac{p_L - 2p_H + \Theta\left(q_H - q_L\right)}{q_H - q_L} = 0\,; \tag{11}$$

$$\frac{\partial \mathcal{H}_L}{\partial p_L} = \frac{p_H q_L - 2p_L q_H}{q_L\left(q_H - q_L\right)} = 0\,; \tag{12}$$

$$\frac{\partial \mathcal{H}_i}{\partial k_i} = -\rho + \frac{a\lambda_{ii}}{2\sqrt{k_i}} = 0\,, \quad i = H, L\,; \tag{13}$$

$$-\frac{\partial \mathcal{H}_H}{\partial q_H} - \frac{\partial \mathcal{H}_H}{\partial p_L}\cdot\frac{\partial p_L^*}{\partial q_H} - \frac{\partial \mathcal{H}_H}{\partial k_L}\cdot\frac{\partial k_L^*}{\partial q_H} = \frac{\partial \mu_{HH}}{\partial t} \Rightarrow$$

$$\frac{\partial \lambda_{HH}}{\partial t} = \rho\lambda_{HH} - \frac{p_H\left(p_H - p_L\right)}{\left(q_H - q_L\right)^2} - \frac{3\Theta p_H q_L^2}{\left(q_H - q_L\right)\left(4q_H - q_L\right)^2}\,; \tag{14}$$

$$-\frac{\partial \mathcal{H}_H}{\partial q_L} - \frac{\partial \mathcal{H}_H}{\partial p_L}\cdot\frac{\partial p_L^*}{\partial q_L} - \frac{\partial \mathcal{H}_H}{\partial k_L}\cdot\frac{\partial k_L^*}{\partial q_L} = \frac{\partial \mu_{HL}}{\partial t} \tag{15}$$

$$-\frac{\partial \mathcal{H}_L}{\partial q_L} - \frac{\partial \mathcal{H}_L}{\partial p_H}\cdot\frac{\partial p_H^*}{\partial q_L} - \frac{\partial \mathcal{H}_L}{\partial k_H}\cdot\frac{\partial k_H^*}{\partial q_L} = \frac{\partial \mu_{LL}}{\partial t} \Rightarrow$$

$$\frac{\partial \lambda_{LL}}{\partial t} = \rho\lambda_{LL} - \frac{p_L\left(p_H q_L^2 - 2p_L q_H q_L + p_L q_H^2\right)}{\left[q_L\left(q_H - q_L\right)\right]^2} + \frac{6\Theta p_L q_H^2}{\left(q_H - q_L\right)\left(4q_H - q_L\right)^2}\,; \tag{16}$$

$$-\frac{\partial \mathcal{H}_L}{\partial q_H} - \frac{\partial \mathcal{H}_L}{\partial p_H}\cdot\frac{\partial p_H^*}{\partial q_H} - \frac{\partial \mathcal{H}_L}{\partial k_H}\cdot\frac{\partial k_H^*}{\partial q_H} = \frac{\partial \mu_{LH}}{\partial t} \tag{17}$$

$$\lim_{t\to\infty} \mu_{ij}(t)\cdot q_i(t) = 0\,, \quad i, j = H, L\,. \tag{18}$$

First, note that the feedback effects $\frac{\partial u_i^*}{\partial q_j}$ must be calculated on the basis of the optimal levels of $u_i = \{p_i\,,\ k_i\}$ (hence the star) obtaining from the relevant first order conditions. This immediately reveals that

$$\frac{\partial k_i^*}{\partial q_j} = 0 \quad \text{for all } i, j \tag{19}$$

and therefore feedback rules are affected only by the effect of state variables on optimal prices, which, solving (11–12) yields the equilibrium prices:

$$p_H^* = \frac{2\Theta q_H\left(q_H - q_L\right)}{\left(4q_H - q_L\right)}\,; \quad p_L^* = \frac{\Theta q_L\left(q_H - q_L\right)}{\left(4q_H - q_L\right)}\,, \tag{20}$$

defined for a generic quality pair. Second, observe that (13) depends upon λ_{ii} only. Consequently, the kinematic equation of k_i is a function of $\partial\lambda_{ii}/\partial t$ but not of $\partial\lambda_{ij}/\partial t$. Therefore, co-state equation (15) and (17) are redundant and the problem admits the solution $\lambda_{HL} = \lambda_{LH} = 0$. From (13) I obtain:

$$\lambda_{ii} = \frac{2\rho\sqrt{k_i}}{a}\,; \quad \frac{\partial k_i}{\partial t} = \frac{a^2\lambda_{ii}}{2\rho^2}\cdot\frac{\partial \lambda_{ii}}{\partial t}\,. \tag{21}$$

Then, using (), () and (21), together with optimal prices $\{p_i^*\}$, I can verify that:

$$\frac{\partial k_H}{\partial t} = 0 \quad \text{at} \quad k_H^* = \frac{4a^2\Theta^4 q_H^2[q_H(4q_H - 3q_L) + 2q_L^2]^2}{\rho^4\,(4q_H - q_L)^6}\,; \tag{22}$$

$$\frac{\partial k_L}{\partial t} = 0 \quad \text{at} \quad k_L^* = \frac{a^2\Theta^4 q_H^4[4q_H - 7q_L]^2}{4\rho^4\,(4q_H - q_L)^6}\,, \tag{23}$$

where it is easily shown that $k_H^* > k_L^*$ always.[8] Moreover, $k_H^* > 0$ always, which entails that $q_H^* = Q$ in steady state, and $k_L^* = 0$ at $q_L^* = 4q_H^*/7$, which is the dynamic counterpart of the so-called 4/7 rule derived by Choi and Shin (1992) at the two-stage subgame perfect equilibrium of the static game. Using $q_L^* = 4q_H^*/7 = 4Q/7$, k_H^* simplifies as $k_H^* = 49a^2\Theta^4/\left(2304\rho^4\right)$, but of course $k_H^* = 0$ when $q_H^* = Q$.

At $\left\{q_H^* = Q\,,\, q_L^* = 4Q/7\,,\, k_H^* = k_L^* = 0\right\}$, steady state quantities, prices and profits coincide with those obtained from the static game:

$$x_H^* = \frac{7\Theta}{12}\,; \quad x_L^* = \frac{7\Theta}{24}\,; \tag{24}$$

$$p_H^* = \frac{\Theta Q}{4}\,; \quad p_L^* = \frac{\Theta Q}{14}\,; \tag{25}$$

$$\pi_H^* = \frac{7\Theta^2 q_H}{48}\,; \quad \pi_L^* = \frac{\Theta^2 Q}{48}\,, \tag{26}$$

with $\pi_H^* > \pi_L^*$. Partial market coverage holds for all $\Theta \in [1, 8/7)$. The foregoing discussion produces the following:

PROPOSITION 2. *Suppose $\Theta \in [1\,,\, 8/7)$. The closed-loop solution of the game entails partial market coverage. Equilibrium qualities are $q_H^* = Q$ and $q_L^* = 4Q/7$, where $\partial k_L/\partial t = 0$ and $\partial k_H/\partial t > 0$.*

The properties of the dynamic system can be evaluated to show that:

PROPOSITION 3. *The closed-loop equilibrium*

$$q_H^* = Q\,, \quad q_L^* = 4Q/7\,, \quad k_H^* = k_L^* = 0$$

is stable in the saddle point sense.

Proof. See the Appendix. □

Notice, however, that k_L^* smoothly approaches zero as q_L approaches $4q_H/7$ from below, while k_H^* sharply drops to zero as soon as q_H reaches the top feasible quality level Q. This entails that, in the left neighbourhood of $\left\{q_H^* = Q\,,\, q_L^* = 4Q/7\right\}$, firm H is still investing in R&D a strictly positive amount of resources, while firm L is investing almost nil. Therefore, it is interesting to investigate a little further what happens in such neighbourhood, in particular in terms of profits. If we take the values:

[8] There exists another solution where $k_H = k_L = 0$, which is economically irrelevant and unstable.

$$q_H^* = Q\,,\; q_L^* = 4Q/7\,,\quad k_H^* = 49a^2\Theta^4/2304\rho^4\,,\quad k_L^* = 0\,,$$

and measure the corresponding profits, we find:

$$\pi_H^* = \frac{7\Theta^2\left(48Q\rho^3 - 7a^2\Theta^2\right)}{2304\rho^4}\,;\quad \pi_L^* = \frac{\Theta^2 Q}{48} \tag{27}$$

with $\pi_H^* > 0$ for all $Q > \widehat{Q} = 7a^2\Theta^2/\left(48\rho^3\right)$ and $\pi_H^* > \pi_L^*$ for all $Q > \overline{Q} = 49a^2\Theta^2/\left(288\rho^3\right)$; $\overline{Q} > \widehat{Q}$ always. This yields:

PROPOSITION 4. *In the left neighbourhood of the steady state qualities* $q_H^* = Q$; $q_L^* = 4Q/7$,

1] $\pi_L^* > 0 > \pi_H^*$ *for all* $Q < \widehat{Q}$;

2] $\pi_L^* > \pi_H^* > 0$ *for all* $Q \in (\widehat{Q}\,,\,\overline{Q})$;

3] $\pi_H^* > \pi_L^* > 0$ *for all* $Q > \overline{Q}$.

I.e., there are quality ranges where (i) the high-quality firm's profits can be negative, so that firm H bears *ad interim* losses, with a view to the dominant position she will enjoy in steady state; (ii) firm H's profits are positive *but* lower than firm L's. However, if Q is larger than the critical threshold $\overline{Q}$, we have the familiar outcome from the static literature, where the profits attached to the high quality are higher than those attached to the low quality.

5. EXTENSIONS

The foregoing analysis can be extended along several directions, allowing, e.g., for Cournot competition, full market coverage and endogenous improvements of the highest technologically feasible quality Q. Here I will not discuss the first two perspectives, for the following reasons. Considering competition in output levels rather than in prices is indeed a natural alternative, and promises new interesting results. However, this would require reworking the entire model anew, starting from the instantaneous demand functions. As to the possibility of full market coverage, we know from Cremer and Thisse (1991) that nesting this assumption in a vertically differentiated duopoly makes the model pretty similar to a Hotelling model with convex transportation costs, and produces a qualitatively analogous prediction as far as optimal product differentiation is concerned, i.e., maximum differentiation.[9] Given the type of quality dynamics adopted in the present model, the incentive to maximise differentiation usually associated with full coverage would be in conflict with the assumption whereby the low-quality firm invests to increase quality according to (6). Hence it appears that full coverage calls for a separate analysis where R&D for quality improvement is treated in a completely different way.

Instead, the third perspective can be easily nested into the present model, as follows. The basic structure, summarised by (5–9) is unchanged. Let

[9] See Tirole (1988, pp. 296–98) and Wauthy (1996) for the analysis of the static game with Bertrand competition under full market coverage.

$$\frac{dQ(t)}{dt} = b\sqrt{q_H(t) + q_L(t)} - \eta Q(t) \tag{28}$$

describe the dynamics of the highest feasible quality level, with parameters $b, \eta > 0$. The above equations entails that $Q(t)$ would exhibit a constant decay rate (whereby $Q(t)$ would eventually drop to zero) if firms were not there to supply their respective qualities.

Accordingly, firm i's Hamiltonian is:

$$\mathcal{H}_i(t) = e^{-\rho t}\{\pi_i(t) + \lambda_{ii}(t)a\sqrt{k_i(t)} + \lambda_{ij}(t)a\sqrt{k_j(t)} + \lambda_{iQ}(t)[b\sqrt{q_H(t) + q_L(t)} - \eta Q(t)]\} \tag{29}$$

where $\lambda_{iQ}(t) = \mu_{iQ}(t)e^{\rho t}$ is the co-state attached to $Q(t)$ by firm i. Clearly, this leaves unaffected the set of FOCs concerning firms' controls, eqs. (11–13), while the corresponding co-state equations (again written for the memoryless closed-loop solution) change considerably due to the presence of (28):

$$\frac{\partial\lambda_{HH}}{\partial t} = \rho\lambda_{HH} - \frac{p_H(p_H - p_L)}{(q_H - q_L)^2} - \frac{3\Theta p_H q_L^2}{(q_H - q_L)(4q_H - q_L)^2} - \frac{b\lambda_{HQ}}{2\sqrt{q_H(t) + q_L(t)}}; \tag{30}$$

$$\frac{\partial\lambda_{LL}}{\partial t} = \rho\lambda_{LL} - \frac{p_L(p_H q_L^2 - 2p_L q_H q_L + p_L q_H^2)}{[q_L(q_H - q_L)]^2} + \frac{6\Theta p_L q_H^2}{(q_H - q_L)(4q_H - q_L)^2} - \frac{b\lambda_{LQ}}{2\sqrt{q_H(t) + q_L(t)}}; \tag{31}$$

$$\frac{\partial\lambda_{iQ}}{\partial t} = \lambda_{iQ}(\rho - \eta), \quad i = H, L. \tag{32}$$

The transversality conditions are now:

$$\lim_{t\to\infty} \mu_{ij}q_i = 0, \quad i, j = H, L \tag{33}$$

$$\lim_{t\to\infty} \mu_{iQ}Q = 0, \quad i = H, L. \tag{34}$$

As in the previous case, it can be easily verified that $\partial\lambda_{ij}/\partial t$, $i \neq j$, are separable differential equations and therefore one can set $\lambda_{HL} = \lambda_{LH} = 0$ at all t. Additionally, this is also true for $\partial\lambda_{iQ}/\partial t$, whereby $\lambda_{HQ} = \lambda_{LQ} = 0$ at all t.

Then, using (20) and (21), optimal investments k_i^* coincide with (22–23). Therefore, in steady state we have $\{q_H^* = Q,\ q_L^* = 4Q/7,\ k_H^* = k_L^* = 0\}$ and

$$\frac{dQ}{dt} = b\sqrt{11Q/7} - \eta Q = 0 \text{ in } Q = \begin{cases} 0 \\ \dfrac{11b^2}{7\eta^2}, \end{cases} \tag{35}$$

with $Q = 0$ being clearly unacceptable. In correspondence of $Q^* = q_H^* = 11b^2/(7\eta^2)$, market shares coincide with (24), while prices and profits are:

$$p_H^* = \frac{11b^2\Theta}{28\eta^2}\,; \quad p_L^* = \frac{11b^2\Theta}{98\eta^2}\,; \tag{36}$$

$$\pi_H^* = \frac{11b^2\Theta^2}{48\eta^2}\,; \quad \pi_L^* = \frac{11b^2\Theta^2}{336\eta^2} \tag{37}$$

with $\pi_H^* > \pi_L^*$. Proceeding as for Proposition 3, the following result can be shown to hold:[10]

PROPOSITION 5. *The closed-loop equilibrium*

$$Q^* = q_H^* = 11b^2/(7\eta^2)\,, \quad q_L^* = 4Q/7\,, \quad k_H^* = k_L^* = 0$$

is stable in the saddle point sense.

However, in the left neighbourhood of the steady state equilibrium, where $k_L^* = 0$ but $k_H^* = 49a^2\Theta^4/2304\rho^4$, the profits of firm H are positive iff $\Theta \in (0, 4b\sqrt{33\rho^3}/(7a\eta))$, and again the sign of the inequality between firms' profits may change, since

$$\pi_H^* > \pi_L^* \Leftrightarrow \Theta \in \left(0, \frac{12b\sqrt{22\rho^3}}{7\sqrt{7}a\eta}\right), \tag{38}$$

so that I can state:

PROPOSITION 6. *In the left neighbourhood of the steady state qualities* $Q^* = q_H^* = 11b^2/(7\eta^2)$; $q_L^* = 4Q/7$,

1] $\pi_L^* > 0 > \pi_H^*$ *for all* $\Theta > \dfrac{4b\sqrt{33\rho^3}}{7a\eta}$;

2] $\pi_L^* > \pi_H^* > 0$ *for all* $\Theta \in \left(\dfrac{12b\sqrt{22\rho^3}}{7\sqrt{7}a\eta}, \dfrac{4b\sqrt{33\rho^3}}{7a\eta}\right)$;

3] $\pi_H^* > \pi_L^* > 0$ *for all* $\Theta \in \left(0, \dfrac{12b\sqrt{22\rho^3}}{7\sqrt{7}a\eta}\right)$.

At first sight, Proposition 5 is rather counterintuitive, as it claims that the position of the high-quality firm becomes progressively less (resp. more) favourable as Θ increases (resp., decreases). For sufficiently high values of the marginal willingness to pay for quality of the richest consumer in the market, π_H^* becomes first lower than π_L^* (but still positive) and then drops below zero. Yet, this feature of the present version of the model can be intuitively explained by observing that, while $k_L^* = 0$, in the left neighbourhood of the steady state k_H^* is positive and sharply increasing in Θ: indeed, while k_H^* is proportional to Θ^4, the revenues of firm H are proportional to Θ^2, since p_H^* and x_H^* are both linear in Θ. Therefore, higher values of Θ entail higher R&D costs which are not fully counterbalanced by the parallel increase in revenues.

Quite intuitively, a further extension would be now called for, to account for endogenous changes in consumers' marginal willingness to pay for quality. This, however, is

[10] In this case, the Jacobian matrix of the dynamic system is a 5x5 matrix with two negative eigenvalues and three positive ones, which ensures stability in the saddle point sense. The details of the proof are omitted for brevity.

a rather delicate aspect as it ultimately involves modifying the structure of preferences, e.g. through advertising campaigns.

6. CONCLUDING REMARKS

I have analysed a differential game where firms, through capital accumulation over time, supply vertically differentiated goods. The explicit treatment of R&D activity as a capital accumulation process proves that several results which are seemingly well established in the static approach are not robust. In particular, I have shown three main results: (i) the high quality firm always invests more than the low quality firm; as a consequence of this property, (ii) there exist quality ranges where the low quality firm's *ad interim* profits are larger than the high quality firm's; (iii) at the closed-loop equilibrium, the optimal quality ratio is always 4/7, as in the static model with costless (or exogenously given costs of) quality improvements (Choi and Shin, 1992). These results have been shown to be robust to the endogenisation of the highest technically feasible quality level of the industry, which has been usually considered as an exogenously given in most of the existing literature in this field.

APPENDIX

Proof of Proposition 3. Given the quasi-static solution for prices, the relevant differential equations for firm i are:

$$\frac{\partial q_i(t)}{\partial t} \equiv \dot{q}_i = a\sqrt{k_i(t)} \tag{a1}$$

$$\frac{\partial k_i(t)}{\partial t} \equiv \dot{k}_i = \frac{a^2\lambda_{ii}}{2\rho^2}\cdot\frac{\partial\lambda_{ii}}{\partial t}, \tag{a2}$$

where the r.h.s. of (a2) may be reformulated by using the appropriate expressions for λ_{ii} and $\partial\lambda_{ii}/\partial t$. The properties of the system must be established on the basis of the sign of the trace and determinant of the Jacobian matrix:

$$\Xi_i = \left.\begin{bmatrix} \dfrac{\partial\dot{q}_i}{\partial q_i} & \dfrac{\partial\dot{q}_i}{\partial k_i} \\ \dfrac{\partial\dot{k}_i}{\partial q_i} & \dfrac{\partial\dot{k}_i}{\partial k_i} \end{bmatrix}\right|_{q_L=4q_H/7, k_i^{CL}} \tag{a3}$$

for both firms. Consider first firm H :

$$\dot{k}_H = \frac{2\sqrt{k_H}[\rho^2\sqrt{k_H}(4q_H-q_L)^3 - 2a\Theta^2 q_H[q_H(4q_H-3q_L)+2q_L^2]]}{\rho(4q_H-q_L)^3} \tag{a4}$$

and the trace and determinant of Ξ_H , are:

$$Tr(\Xi_H) = 2\rho - \frac{2a\Theta^2 q_H(4q_H^2-3q_Hq_L+2q_L^2)}{\rho\sqrt{k_H}(4q_H-q_L)^3}\ ; \tag{a5}$$

$$\Delta(\Xi_H) = -\frac{4a^2\Theta^2 q_L^2(5q_H + q_L)}{\rho(4q_H - q_L)^4} . \tag{a6}$$

Now notice that $\Delta(\Xi_H) < 0$ always, while evaluating $Tr(\Xi_H)$ at $q_H^* = Q$, $q_L^* = 4Q/7$, one obtains:

$$Tr(\Xi_H) = 2\rho - \frac{7a\Theta^2}{48\rho\sqrt{k_H}} , \tag{a7}$$

which, using De L'Hôpital's rule, can be simplified in the limit, for $k_H \to 0$, as $Tr(\Xi_H) = 2\rho > 0$. In $k_H = 0$, the determinant is $\Delta(\Xi_H) = -91a^2\Theta^2/1728\rho q_H < 0$.

Examine now firm L. The equation of motion of k_L is:

$$\dot{k}_L = \frac{\sqrt{k_L}[2\rho^2\sqrt{k_L}\,(4q_H - q_L)^3 - a\Theta^2 q_H^2[4q_H - 7q_L]]}{\rho(4q_H - q_L)^3} \tag{a8}$$

and the trace and determinant of Ξ_L are:

$$Tr(\Xi_L) = 2\rho - \frac{a\Theta^2 q_H^2(4q_H - 7q_L)}{2\rho\sqrt{k_L}(4q_H - q_L)^3} ; \tag{a9}$$

$$\Delta(\Xi_L) = -\frac{4a^2\Theta^2 q_H^2(8q_H + 7q_L)}{\rho(4q_H - q_L)^4} , \tag{a10}$$

with $\Delta(\Xi_H) < 0$ always, while $Tr(\Xi_H)$ evaluated at $q_H^* = Q$, $q_L^* = 4Q/7$ yields $Tr(\Xi_L) = 2\rho$ for all k_L. Finally, in

$$\{q_H^* = Q\,,\ q_L^* = 4Q/7\,,\ k_H^* = k_L^* = 0\} , \tag{a11}$$

the determinant rewrites as $\Delta(\Xi_L) = -2401a^2\Theta^2/27648\rho q_H < 0$. Therefore the closed-loop equilibrium is a saddle point. □

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