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THE CHARACTERIZATION OF THE COMMON PRIOR ASSUMPTION WITHOUT COMMON KNOWLEDGE

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Abstract: The purpose of this paper is to examine the situation in which agents are not hyper-rational and knowledge is incomplete, and to explore how agents find out the structure of games which they play or analyze, and how they construct a coherent model to the true game. That is, we study how they form a subjective model and learn the situation from past information. We prove that agents can know the true game according to an information accumulating process. Moreover we characterize the common prior assumption using a learning approach without common knowledge.

Key words: Bayesian game, common prior assumption.
JEL Classification Number: C72, C73, D82, D83.

1. INTRODUCTION

Game theory has been widely employed as a useful tool for modeling a strategic interaction under certainty or uncertainty. In applying this method, however, we often need some assumptions. Each player in a game theoretical model and analyzers to use this model have to know who takes part in this game, which strategies players can use, what the outcomes are, and so forth. Moreover, the players are assumed to have the common prior probability under uncertainty. These suppositions are unrealistic in many complex situations, especially in modeling a social situation.

Exploring this incomplete situation where individuals do not have the above knowledge, Harsanyi (1967/68) introduces the new class of games which is called the Bayesian game. In Harsanyi's paper each individual is assumed to be able to access to the same information and to have the same prior probability. This philosophy is called the Common Prior Assumption (henceforth CPA). Any difference in posterior probability assessments must be the results of difference in information under the CPA. Namely, individuals assign the same probability if they have the same information.

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Harsanyi's approach is very useful and general in studying the incomplete information. Not surprisingly, the CPA has been criticized as an artificial assumption. Aumann (1987) said that the CPA had no rational basis for people who had always been fed precisely the same information.

To justify the CPA, various ideas have been examined: for example Aumann (1976), Morris (1995), Samet (1998) and Bonanno and Nehring (1999). Aumann (1976) shows that if the subjective probability of an event of each individual is common knowledge then these probabilities must be the same. In his agreement theorem, he gave a necessary condition for the existence of the common prior. Bonanno and Nehring (1999) extend his result to the game situation with incomplete information. Morris (1995) summarizes some approaches (logical justifications, frequentist justifications and so on) to justify the CPA. Samet (1998) proposes a necessary and sufficient condition for the existence of the common prior on a finite type space. His condition is the one in which it is common knowledge that the iterated expectation to converge to the same value among players. Most approaches have to assume something which is common knowledge.

One of the main concern of this paper is to propose a theoretical framework to justify the CPA without common knowledge. To do so, we use another word that agents forecast a situation in a subjective way. Moreover, we also consider that agents face constantly the similar or new situation in society and can learn from additional information and their experiences. Namely, the agents are assumed to accumulate their knowledge and information. Under these hypotheses, it turns out that their personal forecast may become the correct one. That is, knowledge of an agent is revised according to the information accumulating process.

Before discussing the formal model, we introduce related literatures. Kalai and Lehrer (1995), Matsushima (1997) and Kaneko and Matsui (1999) have the similar motivation and background with our paper.

Kalai and Lehrer (1995) investigate a situation in which no player possesses objective knowledge of the game and maximizes his subjective expected payoff based on his belief. They showed that subjective optimizers converged to a subjective equilibrium during the repeated play of the game. Matsushima (1997) explores the similar situation in which players know the set of actions but do not know their true payoff functions. Then players have to formulate their own game in a subjective manner. According to him, these games are characterized through inductive learning procedures, by a trivial game. In this game, there is a unique action which is efficient and strictly dominated. These two papers deal with the adapting process of each player's belief using a learning approach. On the other hand, Kaneko and Matsui (1999) establish the inductive game theory, using a differential approach with a similar motivation. In their paper, the player does not know the structure of the games, but may infer from his experiences, what has been occurring. Roughly speaking, they prove that each player can choose a Nash equilibrium strategy using accumulated active experiences.

Our paper discusses the information process in the mind of the agents and the structure of their knowledge instead of these previous approaches. So, our motivation is similar to Kaneko and Matsui. However, we mainly consider when the player can recognize the structure of games, while Kaneko and Matsui consider when the player can play a Nash equilibrium strategy. We prove that the agents can know the true game according to the information accumulating process. We consider that CPA is justified from this result even if there is no common knowledge.

The rest of this paper is organized as follows. In section 2, we introduce a new class of games, which is called an extended complete game, and define the structure of information and learning for an individual. In section 3, we prove that the complete games and the incomplete games are represented by the limit of an extended complete game under certain assumptions. In section 4, we show that assumption 1 is derived from another assumption which is more fundamental. We state the main theorem in this paper. The last section presents concluding remarks and extensions.

2. EXTENDED COMPLETE GAMES AND THE TRUE GAME

2.1. *Bayesian Games and Extended Complete Games*

In this subsection we define a new class of games, which is an extension of complete games. This generalized form enables us to characterize incomplete games and complete games in the same method. We begin with explaining a relationship between a Bayesian game and a new class of games.

DEFINITION 1. (Harsanyi 1967)

A *Bayesian game* is a system $(N, (S_i)_{i\in N}, (T_i)_{i\in N}, (V_i)_{i\in N}, R)$, where

(1) N is the (finite) set of players,

(2) for every $i \in N$, S_i is the set of actions of player i,

(3) for every $i \in N$, T_i is the finite set of possible types of player i,

(4) for every $i \in N$, $V_i : S \times T \to \mathbf{R}$ is the payoff function of player i, where $S := \prod_{i\in N} S_i$ and $T := \prod_{i\in N} T_i$, and

(5) R is the objective probability on T, which all players know in advance.

This objective probability R is called a common prior. To define an extended complete game we prepare a Game space.

DEFINITION 2. We define a *Game Space* as

$$\Gamma = \left\{ (N, (S_i)_{i\in N}, (u_i)_{i\in N}) \,\middle|\, N \text{ is a set. } S_i \text{ is a set. } u_i : \prod_{j\in N} S_j \to \mathbf{R} \right\},$$

where

(1) N is the set of agents to play a game,

(2) S_i is the strategy set of player i,

(3) $u_i : \prod_{i\in N} S_i \to \mathbf{R}$ is the payoff function of player i.

Intuitively this space consists of all complete games. We consider the situation where an agent is supposed to choose some games from this game space and assign probabilities to them. Formally we define this situation as a game called an extended complete game.

DEFINITION 3. An *Extended Complete Game* is a system $(\{G^k\}_{k=1,\dots,m}, P)$, where
(1) $G^k = (N^k, \{S_j^k\}_{j\in N^k}, \{u_j^k\}_{j\in N^k}) \in \Gamma$ for all $k = 1,\dots,m$,
(2) $u_i^k : \prod_{j\in N^k} S_j^k \to \mathbf{R}$ is the payoff function at the game G^k, and
(3) P is the probability on $\{G^k\}_{k=1,\dots,m}$.

Note that, in case $P(G^j) = 1$ or $m = 1$, an extended complete game becomes a complete game. In this sense $(\{G^k\}_{k=1,\dots,m}, P)$ is an extension of complete games. Figure 1 illustrates an example of extended complete games.

	A	B
A	a_1, b_1	a_3, b_3
B	a_2, b_2	a_4, b_4

G^1

	A	B
A	c_1, d_1	c_3, d_3
B	c_2, d_2	c_4, d_4

G^2

Figure 1. These matrices illustrate the extended complete game $(\{G^1, G^2\}, P)$. The left side matrix G^1 and the right side matrix G^2 denote 2×2 games. $P(G^1)$ and $P(G^2)$ are probability to occur the game G^1 and G^2 respectively.

A Bayesian game does not coincide with an extended complete game. In spite of this, if Bayesian games and extended complete games satisfy the following five conditions:
(1) $|T| = m$, that is, the number of types is equal to the number of games,
(2) for all $k = 1,\dots,m$, $N = N^k$, that is, the set of players is the same,
(3) for all $k = 1,\dots,m$, and for all $i \in N$, $S_i = S_i^k$, that is, the set of actions is equal to the set of strategy in the game G^k,
(4) for all $t \in T$, $V_i(t) = u_i^t$, and
(5) for all $t \in T$, $R(t) = P(G^t)$,

then we can state a relationship between them.

PROPOSITION 1. *Under the above five conditions,*
1. *A Bayesian game has a unique extended complete game.*
2. *An extended complete game defines some Bayesian games.*

Proposition 1-1 suggests that a Bayesian game is characterized by an extended complete game. At the same time we can also conclude from proposition 1-2 that an extended complete game is represented by some Bayesian games. In this sense we call the extended complete game, which satisfies the previous five conditions, the *quasi-Bayesian Game*.

The following result says the equivalence about quasi-Bayesian games.

LEMMA 1. *Let $(\{G^k\}_{k=1,\dots,m}, P)$ be the quasi-Bayesian game. The game, to which P-null sets are added, is also the same quasi-Bayesian game.*

Proof. If and only if G is the P-null set, $P(G) = 0$. Then, even if we add P-null sets, the structure of the quasi-Bayesian game does not change. □

2.2. *The Structure of Games for an Agent and the True Game*

Next we introduce the extended complete game for an agent and the true game. The former is a game that an agent forms or guesses in a subjective way, and the latter is the game that describes the true situation and that an agent does not always know in advance. We suppose that the situation is determined by the true game and that each agent forecasts it in a personal way.

DEFINITION 4. An *extended complete game for agent* i is a system $(\{G_i^j\}_{j=1,\dots,g(i)}, P_i)$, where

(1) $G_i^j \in \Gamma$ for all $j = 1, \dots, g(i)$, and

(2) P_i is the subjective probability of agent i on $\{G_i^j\}_{j=1,\dots,g(i)}$.

In this paper, we consider an agent as a player to join a game or an analyst to construct a game theoretical model. In any case, we assume that each agent takes some games, which is finite, from the game space Γ. Agent i has a subjective probability P_i over $\{G_i^j\}_{j=1,\dots,g(i)}$ because of the axioms of Savage (1954), Anscombe and Aumann (1963) and so on. In this sense, we may regard the extended complete game for an agent as a subjective concept. We consider the extended complete game for an agent as the game which he expects or believes in a personal way. For notational convenience we write simply G_i instead of $\{G_i^j\}_{j=1,\dots,g(i)}$.

The true game is supposed to be given by the extended complete game. The true situation is also assumed to be determined by the true game in our paper. This game is denoted by $(\{G^k\}_{k=1,\dots,m}, Q)$, where $G^k \in \Gamma$ for all $k = 1, \dots, m$, and Q is the probability over $\{G^k\}_{k=1,\dots,m}$. In the same reason we write TG instead of $\{G^k\}_{k=1,\dots,m}$.

In the similar way, the extended complete game for agent i at period t is defined as $(\{G_i^1, \dots, G_i^{g(i)}\}_t, P_{i,t})$, where $\{G_i^1, \dots, G_i^{g(i)}\}_t$ is the set of games for him to select from Γ at period t, and $P_{i,t}$ is his subjective probability on $\{G_i^1, \dots, G_i^{g(i)}\}_t$ at period t. For notational convenience $G_{i,t} = \{G_i^1, \dots, G_i^{g(i)}\}_t$. Moreover, the true game at period t is described by (TG_t, Q_t), where $TG_t := TG$ is the set of games and $Q_t := Q$ is the probability over TG_t. We assume TG_t and Q_t do not depend on times, because we consider the true game is always given uniquely and does not change through times.

2.3. *Information and Learning*

Let $(G_{i,t}, P_{i,t})$ be the extended complete game for agent i at period t. Since $P_{i,t}$ is the probability measure on $G_{i,t}$, agent i's information is expressed by a σ-algebra of the subsets of $G_{i,t}$. In addition to this notion we consider that agent i can observe the game, which occurred at the previous period, and learn from his past information. To make this idea concrete we introduce a dynamic model where agents are assumed to face the same situations for several periods, and in which they are supposed to be informed of the basic situations over time.

Let $\mathbb{G}_i := \prod_{t=1}^{\infty} G_{i,t}$ be the product set of $G_{i,t}$ and $\mathbb{P}_i$ be the product probability measure over $\mathbb{G}_i$. To characterize an agent's information, we adopt a partition of $\mathbb{G}_i$.[1]

[1] This partition approach is sometimes adopted. Monderer and Samet (1995) use the same approach.

We define a partition of $\mathbb{G}_i$ at period t as $\mathcal{P}_{i,t}$. $\mathcal{P}_{i,t}(G)$ denotes the element of $\mathcal{P}_{i,t}$ which contains $G \in \mathbb{G}_i$. Moreover, we assume that these partitions satisfy

$$\mathcal{P}_{i,t}(G) \supset \mathcal{P}_{i,t+1}(G)\,.$$

This supposition describes the situation where agent i's information is refined and accumulated by observing past information. According to this refinement processes, an agent knows the realized game and learns from his past experience. Then, a sequence of σ-algebras $\{\mathcal{F}_{i,t}\}$ generated by the partition $\{\mathcal{P}_{i,t}\}$ forms an information increasing class.[2] In other words $\{\mathcal{F}_{i,t}\}$ satisfies

$$\mathcal{F}_{i,t} \subset \mathcal{F}_{i,t+1}\,.$$

We summarize the previous notions.

DEFINITION 5. *The structure of information and learning for an agent i* is defined as the filtered probability space $(\mathbb{G}_i, \mathcal{F}, \{\mathcal{F}_{i,t}\}, \mathbb{P}_i)$, where

(1) $\mathbb{G}_i$ is the product set of $G_{i,t}$.
(2) $\mathcal{F}$ is a product σ-algebra of the subsets of $\mathbb{G}_i$.
(3) $\{\mathcal{F}_{i,t}\}$ is a filtration, and
(4) $\mathbb{P}_i$ is the product probability measure and is called subjective probability.

This specification shows essentially that each agent constructs a subjective probability of what games are going to happen, based on information about what games occurred in the past. Concretely $\mathbb{P}_i$ denotes his forecast for games at the initial period. Besides, the condition $\mathcal{F}_{i,t}$ of the conditional probability $\mathbb{P}_i(\cdot|\mathcal{F}_{i,t})$ indicates his information and observation until the period t. For this reason, the conditional probability $\mathbb{P}_i(\cdot|\mathcal{F}_{i,t})$ expresses his forecast for games at period $t+1$. Namely, an agent is assumed to revise his prediction $\mathbb{P}_i(\cdot|\mathcal{F}_{i,t})$ by Bayesian learning.

Finally, we state the product set of the true game in the same way. The product set of the true game is denoted by $(\mathbb{TG}, \mathbb{Q})$, where $\mathbb{TG} := \prod_{t=1}^{\infty} TG_t$ and $\mathbb{Q}$ is the product probability measure on $\mathbb{TG}$.

3. THEOREM

If an agent faces and observes the same situations repeatedly, he may have another forecast about games. Namely, he becomes familiar with the game, which he joins or analyzes, by using his experience, intuition and so on. As it turned out, it may be possible for him to find out the true game. In this section we state a positive massage to this prospect. We prove that the quasi-Bayesian games and complete games are represented by the limit of the extended complete game for an agent i under some assumptions.

[2] The filtration is also given by another way. For example when the game, which player i observes at period t, is expressed by the random variable f_i^t and his information $\mathcal{F}_{i,t}$ is given by $f_i^1, \ldots, f_i^n$, his information $\mathcal{F}_{i,t}$ becomes a filtration.

First of all, we adopt the next assumption 1 to prove the results simply. The reader may think this assumption is too strong to verify the theorem, however we show that it is derived from more fundamental and weaker assumption in section 4.

ASSUMPTION 1. $\{G^k\}_{k=1,\ldots,m,t} \subset \{G_i^j\}_{j=1,\ldots,g(i),t}$ for all i and $t = 1, 2, \ldots$.

This assumption suggests that for each t the extended complete game for agent i contains the true game. Using assumption 1, we can add the games $(G_{i,t} \backslash TG_t)$ to the true game as Q-null sets. We abuse the notation TG_t and regard this new games $((G_{i,t} \backslash TG_t) \cup TG_t)$ as TG_t.

For this reason we can assume the following condition through this technical operation.

ASSUMPTION 2. $Q_t \ll P_{i,t}$ for all i and $t = 1, 2, \ldots$.[3]

Assumption 2, which is called an *absolutely continuous condition*, means that $Q_t(G) > 0$ implies $P_{i,t}(G) > 0$ for all $G \subset TG_t$, so that the prediction by Q_t is also predicted by $P_{i,t}$. Moreover, even if there is a game not to happen in the true game, an agent can assign a positive probability to this game. Accordingly we permit that he can have a wrong expectation. This assumption is usually adopted in Bayesian learning literatures, (for example, see Feldman (1987), Kalai and Lehrer (1993, 1995), and Nyarko (1998)).

LEMMA 2. *If $Q_t \ll P_{i,t}$ for all $t = 1, 2, \ldots$ then $\mathbb{Q} \ll \mathbb{P}_i$.*

Proof. We will show that for all $G \in \mathcal{F}_\infty$, $\mathbb{Q}(G) > 0$ implies $\mathbb{P}_i(G) > 0$. Let $G = \prod_{l=1}^{\infty} G_l$ be the product set in $\mathbb{TG}$. For all $G \in \mathcal{F}_\infty$,

$$\begin{aligned} \mathbb{Q}(G) > 0 &\Rightarrow \forall t = 1, 2, \ldots\ Q(G_l) > 0 \text{ for all } l = 1, 2, \ldots \\ &\Rightarrow \forall t = 1, 2, \ldots\ P_{i,t}(G_l) > 0 \text{ for all } l = 1, 2, \ldots \\ &\Rightarrow \mathbb{P}_i(G) > 0 . \end{aligned}$$

□

Using lemma 2, we may interpret assumption 2 in another way. If assumption 2 follows for all agents, we can consider that they have a common accessible information, which is described by Q.

The following result is useful to prove theorem 1.

LEMMA 3. *For all $f \in L^1$ and for every monotone increasing sequence $\{\mathcal{B}_n\}$ of σ-algebras converging to a σ-algebra $\mathcal{B}$,*

$$\lim_{n\to\infty} \mathrm{E}[f|\mathcal{B}_n] = \mathrm{E}[f|\mathcal{B}] \ \textit{almost everywhere}.$$

Proof. The proof is basically the same as Theorem 35.6 in Billingsley (1995). □

[3] Let P and Q be the probability measures on a set Ω, and Σ be a σ-algebra of subsets of Ω. Now, $Q \ll P$ denotes that Q is absolutely continuous with respect to P, and means that $Q(A) > 0$ implies $P(A) > 0$ for all $A \in \Sigma$. In our setting, we regard Ω as $TG_t = G_{i,t}$ from assumption 1 and the technical operation.

The following theorem is a basic result in this paper.

THEOREM 1. *Under assumption* 1 *and* 2,

$$\mathbb{Q}(A|\mathcal{F}_{i,\infty}) = \lim_{t\to\infty} \mathbb{P}_i(A|\mathcal{F}_{i,t}) \quad (a.e.)$$

for all $A \in \mathcal{P}_{i,\infty} \subset \mathbb{TG}$.

Intuitively, theorem 1 tells us that an agent can find out the true game, which is characterized by the extended complete game, as t goes to infinity, since his updating process works well under assumption 1 and 2.

Proof. From lemma 2, $\mathbb{Q} \ll \mathbb{P}_i$ follows. We can see from Radon-Nikodym's theorem that

$$\exists f(G)\ :\mathcal{F}_{i,\infty}\text{-measurable s.t.} \int_{\mathcal{P}} f(G)\mathrm{d}\mathbb{P}_i = \mathbb{Q}(\mathcal{P}) \tag{1}$$

for all $\mathcal{P} \in \mathcal{F}_{i,\infty}$. Using the definition of conditional expectation and lemma 3, we examine the following two cases, for all $\mathcal{P}_{i,t-1} \in \mathcal{F}_{i,t}$.

Case 1. $\mathbb{P}_i(\mathcal{P}_{i,t-1}(G)) \neq 0$.

$$\begin{aligned}
\mathrm{E}_{\mathbb{P}_i}[f(G)|\mathcal{P}_{i,t-1}(G)] &= \frac{1}{\mathbb{P}_i(\mathcal{P}_{i,t-1}(G))} \int_{\mathcal{P}_{i,t-1}(G)} f(G)\mathrm{d}\mathbb{P}_i(G) \\
&= \frac{\mathbb{Q}(\mathcal{P}_{i,t-1}(G))}{\mathbb{P}_i(\mathcal{P}_{i,t-1}(G))} && \text{(from (1))} \\
&\to \frac{\mathbb{Q}(\mathcal{P}_{i,\infty}(G))}{\mathbb{P}_i(\mathcal{P}_{i,\infty}(G))} \quad (a.e.)\,. && \text{(Lemma 3)}
\end{aligned}$$

Case 2. $\mathbb{P}_i(\mathcal{P}_{i,t-1}(G)) = 0$.

In this case, theorem 1 is trivial by $\mathbb{Q} \ll \mathbb{P}_i$. Let us consider the case 1 as follows.

Since $\mathcal{P}_{i,\infty}(G) \subset \mathcal{P}_{i,t-1}(G)$,

$$\lim_{t\to\infty} \frac{\mathbb{P}_i(\mathcal{P}_{i,\infty}(G)|\mathcal{P}_{i,t-1}(G))}{\mathbb{Q}(\mathcal{P}_{i,\infty}(G)|\mathcal{P}_{i,t-1}(G))} = 1 \quad (a.e.)\,.$$

This equation indicates that there exists $\mathbb{P}_i$-null set F suth that

$$\forall\varepsilon > 0,\ \forall\, G \in \mathbb{TG},\ \exists\, t_0 \in \mathbf{N};$$

$$t \geq t_0 \Rightarrow \left| \frac{\mathbb{P}_i(\mathcal{P}_{i,\infty}(G)|\mathcal{P}_{i,t-1}(G))}{\mathbb{Q}(\mathcal{P}_{i,\infty}(G)|\mathcal{P}_{i,t-1}(G))} - 1 \right| < \varepsilon$$

on $\mathbb{TG}\backslash F$. Then,

$$t \geq t_0 \Rightarrow 1 - \varepsilon < \frac{\mathbb{P}_i(\mathcal{P}_{i,\infty}(G)|\mathcal{P}_{i,t-1}(G))}{\mathbb{Q}(\mathcal{P}_{i,\infty}(G)|\mathcal{P}_{i,t-1}(G))} < 1 + \varepsilon\,.$$

Therefore, for all $t \geq t_o$, we obtain

$$|\mathbb{P}_i(\mathcal{P}_{i,\infty}(G)|\mathcal{P}_{i,t-1}(G)) - \mathbb{Q}(\mathcal{P}_{i,\infty}(G)|\mathcal{P}_{i,t-1}(G))| < \varepsilon\mathbb{Q}(\mathcal{P}_{i,\infty}(G)|\mathcal{P}_{i,t-1}(G))\,.$$

Noting that $0 \leq \mathbb{Q}(\mathcal{P}_{i,\infty}(G)|\mathcal{P}_{i,n-1}(G)) \leq 1$, we can conclude there exists $\mathbb{P}_i$-null set F such that, for all $\varepsilon > 0$, $G \in \mathbb{TG}$ and for some $t_0 \in \mathbf{N}$,

$$t \geq t_0 \Rightarrow |\mathbb{P}_i(\mathcal{P}_{i,\infty}(G)|\mathcal{P}_{i,t-1}(G)) - \mathbb{Q}(\mathcal{P}_{i,\infty}(G)|\mathcal{P}_{i,t-1}(G))| < \varepsilon$$

on $\mathbb{TG}\backslash F$. Let A be an atom of $\mathcal{F}_{i,\infty}$. We can see

$$t \geq t_0 \Rightarrow |\mathbb{P}_i(A|\mathcal{F}_{i,t}) - \mathbb{Q}(A|\mathcal{F}_{i,t})| < \varepsilon \, .$$

In fact, since A is not contained in $\mathcal{P}_{i,t}(G)$, it follows that $\mathbb{P}_i(A|\mathcal{F}_{i,t}) = 0$ and $\mathbb{Q}(A|\mathcal{F}_{i,t}) = 0$.

Hence,

$$\lim_{t\to\infty} \mathbb{P}_i(A|\mathcal{F}_{i,t}) = \mathbb{Q}(A|\mathcal{F}_{i,\infty}) \quad (a.e.) \, .$$

The proof is completed. □

Based on this theorem each agent can find out the same information which is also true, if they have a common accessible information. Since the conditional subjective probability for agent i converges to the conditional true probability, we may interpret that the limit of his prediction coincides with the true probability. As a result the quasi-Bayesian game is represented by the limit of the extended complete game for an agent. Roughly speaking, a Bayesian game is characterized by the limit of extended complete game. Moreover this theorem tells us that when agents satisfy assumption 1 and 2, they can find out the same game and agree to this game theoretical situation. Namely, even if they do not have the common prior, they can know the quasi-Bayesian game under these assumptions. From this theorem, the common prior is characterized by the limit of a subjective probability. *the CPA is justified without common knowledge*. However, when the ture game is a Bayesian game, each player may know another Bayesian game and cannot learn the true game, according to Proposition 1 and Theorem 1.[4] An extended complete game and a quasi-Bayesian game consist of finite complete games $(G^1, \ldots, G^m)$. For this reason, the player can know the true game at the positive provability when the true game is a Bayesian game. Furthermore, we can prove a more powerful statement as follows, since complete games are considered as a special case of extended complete games,

PROPOSITION 2. *A complete game is represented by the limit of extended complete game for agent i under assumption* 1 *and* 2.

Proof. A complete game is the special case of extended complete game where the true game TG_t is a singleton and a probability measure Q_t over TG_t assigns one to this unique set. For this reason we can apply the same proof of theorem 1 to this proposition and our conclusion follows. □

Using this result, even if agents do not have the common prior and common knowledge of the structure of the true game, they can perceive the complete game, which describes the true states. For example let the true states be described by the famous complete games: the Prisoner's Dilemma, the coordination game, Matching Pennies and so on. Players and analysts can recognize these games as time goes to infinity. In this sense, our results are opposite to Matsushima (1997).

[4] This comment is suggested by referees.

4. THE POSSIBILITY TO ASSUMPTION 1

It is natural that an agent should experience and observe the game which he does not predict. Likewise it is not realistic for him to know the all elements in $\{G^k\}_{k=1,\dots,m}$ (the true game) at the initial period. For this reason the reader may feel that assumption 1 is strong. We can prove, however, from more fundamental and consistent assumption that assumption 1 holds with probability one.

This basic hypothesis is informally that if an agent faces the game which he does not expect, then he chooses games from the game space Γ including this game again. By this assumption, an agent's information increases and he becomes familiar with the situations whenever he experiences an unpredictable game. In short, an agent is assumed to learn the situations and to accumulate his knowledge from his past experiences and events. In this sense, we can consider that the following assumption is consistent with the situation where an agent learns from his experiences.

ASSUMPTION 3. Let $G^k \notin \{G_i^j\}_{j=1,\dots,g(i),t}$. If G^k occurs at period t, then $\{G_i^l\}_{l,t+m} \supset \{G^k\} \cup \{G_i^j\}_{j=1,\dots,g(i),t}$ until the finite period $t+m$.

PROPOSITION 3. *Under assumption* 3. $\{G^k\}_{k=1,\dots,m} \subset \{G_i^j\}_{j,\infty}$ *holds with* Q-*probability one.*

Proof. Let A_n be the event where the game $G^i \in \{G^k\}_{k=1,\dots,m}$ occurs at period n. Since the game occurs independently for every period, A_n and A_m are independent events. Moreover, noting that $Q(A_n) = q > 0$ for each $n \in \mathbf{N}$, it follows that

$$\sum_{n=1}^{\infty} Q(A_n) = \sum_{n=1}^{\infty} q = \infty .$$

Then we can conclude from Borel-Cantelli's theorem,[5]

$$Q(\limsup_{n\to\infty} A_n) = 1 .$$

Since the previous proof does not depend on $i = 1, \dots, m$, it follows from the definition of the upper limit of A_n that $G^1, \dots, G^m$ occur for infinitely many periods with Q-probability one. By assumption 3, we can conclude that $\{G^k\}_{k=1,\dots,m} \subset \{G_i^j\}_{j,\infty}$ holds with Q-probability one. □

COROLLARY 1. *Under assumption* 3, $\{G^k\}_{k=1,\dots,m} \subset \{G_i^j\}_{j=1,\dots,g(i),t}$ *holds with positive* Q-*probability.*

If the true game is a complete game, the following statement follows directly from assumption 3.

PROPOSITION 4. *Under assumption* 3, *it follows with* Q-*probability one that there exists* $t_0 \in \mathbf{N}$ *such that* $\{G\} \subset \{G_i^j\}_{j=1,\dots,g(i),t}$ *for all* $t \geq t_0$.

The following assertions are main massages in this paper.

[5] See for example Capiński, M. and Kopp, E. (1999).

THEOREM 2. *Under assumption 2 and 3, a quasi-Bayesian game is represented by the limit of the extended complete game for agent i with positive Q-probability.*

Proof. From corollary 1 we can define assumption 2 with positive Q-probability. For this reason we verify this result in the same way, and complete this proof. □

PROPOSITION 5. *Under assumption 2 and 3, a complete game is represented by the limit of the extended complete game for agent i with Q-probability one.*

These results tell us that when agents satisfy assumption 2 and 3, they can perceive the same game and agree to this game theoretical model. In short even if they do not have a priori knowledge of the structure of the true game which they play repeatedly, and do not have a common prior, they can recognize the quasi-Bayesian games and complete games. The common prior assumption is justified from this theorem even if individuals do not have common knowledge.

5. DISCUSSIONS AND CONCLUDING REMARKS

In this paper we consider that agents forecast a situation in a subjective way, and show that they can know the true game according to the information accumulating process. Concretely complete games and quasi-Bayesian games are expressed by the limit of the extended complete game. For this reason, we interpret that even if agents do not have common knowledge they can recognize these games. Namely, the Common Prior is characterized by the limit of a subjective probability. We have to say that when the true game is a Bayesian game, the players may know another Bayesian game and cannot recognize the true game. However, the player can know the true game at the positive provability in this case, since a quasi-Bayesian game consists of finite complete games $(G^1, \ldots, G^m)$. To improve this point is one of our future's topics.

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