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WEALTH DISTRIBUTION IN RAMSEY–CASS–KOOPMANS ECONOMIES: SOME FURTHER RESULTS

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Abstract: The paper considers the nature of the long-run distribution of wealth in a Ramsey–Cass–Koopmans economy with CRRA utility and exogenous technical progress. A finite number of agent types exist differentiated by the size of their wealth endowments. The equilibrium growth path for the economy is shown to have an asymptotic distribution of wealth. The more unequal the initial distribution, the more unequal is the distribution of wealth in the long run. Also, the long-run distribution of wealth may be more unequal than the initial distribution and such cases of divergence may be associated with higher long-run rates of growth.

Key words: Ramsey–Cass–Koopmans, growth, inequality, wealth distribution.
J.E.L. Classification Number: D31, D63, O41.

1. INTRODUCTION

The Ramsey–Cass–Koopmans (R–C–K) model of growth (Ramsey (1928), Cass (1965), Koopmans (1965)[1]) is probably the most widely used dynamic model of a market economy. Caselli and Ventura (2000) consider examples of R–C–K economies to illustrate the diverse dynamics of wealth distribution possible within the R–C–K model. These examples demonstrate that even if in a standard R–C–K model (without government intervention) infinitely-lived consumers are differentiated only by their initial wealth endowments, inequality in the distribution of wealth can not only persist in the long run but the degree of inequality can increase over time.

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[1] Ramsey, Cass and Koopmans all presented their models as exercises in the dynamic optimisation of intertemporal social welfare, the solutions to these exercises representing *optimal* growth paths. For standard versions of the model presented as dynamic representations of market economies see, for example, Blanchard and Fischer (1989), Barro and Sala-i-Martin (1995).

These results are significant because a common notion underlying much writing on growth and inequality is that traditional neoclassical models of growth cannot explain why historical inequalities in the distribution of wealth persist in the process of economic growth.[2] Stiglitz (1969) was the first to consider how the distribution of wealth evolves in a neoclassical growth model in which individuals, differentiated only by their wealth endowments, bequeath their savings equally among equal numbers of descendants. For almost all the behavioural savings (bequest) functions considered, Stiglitz obtained strong results indicating long-run convergence to equality in individual endowments of wealth.[3]

The perceived failure of neoclassical growth models to explain the long-run persistence of inequality has also meant that linkages between the long-run rate of growth and the extent of long-run inequality in the distribution of wealth have typically been traced through the presence of capital market imperfections (e.g. Galor and Zeira (1993), Piketty (1997)) or political choices determining public policy (e.g. Bertola (1993), Alesina and Rodrik (1994)). Caselli and Ventura's analysis can, however, be easily extended to show that, in the presence of exogenous technical progress, there exists the possibility of long-run trade-offs between the rate of growth of the economy and the degree of inequality in the distribution of wealth.[4]

The results obtained by Caselli and Ventura for the R–C–K model relate to the case where instantaneous utility is a logarithmic function of the rate of consumption. In this paper we consider the more general case of R–C–K economies with *any* CRRA instantaneous utility function and explicitly allow for exogenous labour augmenting technical progress. The degrees of inequality in different distributions of wealth are compared using the standard Lorenz criterion, extended to take account of possible states of individual indebtedness (that is, 'negative shares' in total wealth). We report three results.

Proposition 1 states that along the equilibrium growth path of the R–C–K economy, the distribution of wealth converges to a unique asymptotic distribution.[5] Not only is this asymptotic distribution unequal in the presence of historical inequalities in the initial distribution of wealth, but Proposition 2 notes that the more unequal is the initial distribution of wealth, the more unequal is the distribution of wealth in the long run. Finally, Proposition 3 implies that the steady state distribution of wealth may not only be

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[2] See, for example, Aghion and Bolton (1992, p. 606) and Aghion, Caroli and Garcia-Penalosa (1999, p. 1621).

[3] Subsequent contributions have shown that if individual savings are a convex function of current income then unegalitarian steady states may not only exist (see Schlicht (1975)) but an unegalitarian steady state may also be Pareto-superior to an egalitarian steady state (see Bourguignon (1981)). However, an obvious limitation of the Stiglitz framework is its failure to take account of individual motives for making bequests.

[4] See, for example, the case of the Cobb-Douglas technology (Caselli and Ventura (2000), p. 918).

[5] Lucas and Stokey (1984) provide sufficient conditions for the existence of a unique stationary distribution of wealth in optimal growth problems with one consumption good and heterogenous agents. The condition of 'increasing marginal impatience' is, however, clearly not applicable to the R–C–K economy.

more unequal than the initial distribution but such cases of divergence may in fact be associated with higher rates of technical progress and steady state growth in the economy, implying a possible trade-off between growth and equity in the long run.

Papers by Becker (1980), Chatterjee (1994) and Sorger (2002) consider the long-run dynamics of the distribution of capital or wealth within variants of the standard R–C–K model. The papers by Becker and Sorger consider the case where individuals have heterogeneous rates of time preference. Chatterjee constructs an economy in which the unique equilibrium path of per capita capital stock is the same as in a standard R–C–K economy but reproducible capital is the only factor of production. Individual incomes comprise entirely of distributed profits and there are no wage earnings. Finally, Ghiglino and Sorger (2002) extend the R–C–K model to include endogenous labour supply and a production externality and demonstrate how the initial distribution of wealth may itself determine the aggregate long-run behaviour of the economy, a possibility ruled out in the standard R–C–K model.

The paper is organised as follows. Section 2 discusses the assumptions and definitions underlying our analysis. Propositions 1 and 2 are then established in section 3. Section 4 discusses Proposition 3 (the formal proof is carried in an appendix to the paper). Some concluding comments are presented in section 5.

2. ASSUMPTIONS AND DEFINITIONS

We consider a closed perfectly competitive economy with a single good and labour. The good can either be consumed or stocks of the good, constituting the capital stock of the economy, can be used in production. To simplify we assume that capital stocks are not subject to depreciation. All capital is privately owned.

There is a constant population of infinitely-lived individuals who are differentiated only by the amount of wealth owned by them at an initial point in time 0. Wealth equals ownership of capital stock less net debts outstanding[6]. Individuals are accordingly classified into a finite number of groups (say, $q \geq 2$). Let l_i denote the constant proportion of population in group $i (i = 1, 2, \ldots, q)$. Each individual is endowed with one unit of labour at each point in time and individuals have perfect foresight about all macroeconomic variables.

The economy is assumed to undergo labour augmenting technical progress at a constant rate $\mu > 0$. The number of efficiency units per natural unit of labour at time $t \geq 0$ is denoted by $m(t)$. The production function for the single good satisfies the standard properties:

$$\forall k \geq 0 : f(k) \geq 0; \forall k > 0 : f'(k) > 0; \forall k > 0 : f''(k) < 0;$$
$$\lim_{k \to 0} f'(k) = \infty; \exists k > 0 : f'(k) = \mu ;$$

where, k is capital per unit of effective labour and $f(k)$ is the average product of effective labour.

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[6] Net debts outstanding is defined as equal to the amount of loans taken (and awaiting repayment) less the amount of loans given (and awaiting repayment).

Let $c(t)$ and $k(t)$ denote respectively the rate of consumption and the capital stock, both per efficiency unit of labour, in the economy at time $t \geq 0$. Let $w(t)$ denote the wage per unit of effective labour and $r(t)$, the rental price of capital in the economy at $t \geq 0$. Given that factor markets are perfectly competitive, factor rentals equal respective marginal products, so that for any $t \geq 0$, $w(t) = f(k(t)) - k(t) \cdot f'(k(t))$; $r(t) = f'(k(t))$. We denote by $W(t)$, the ratio of the present value of the stream of wage earnings earned by an individual time t onwards to the number of efficiency units per natural unit of labour at time t. Therefore, for any $t \geq 0$, $W(t) = \int_t^\infty w(\tau)e^{-\int_t^\tau \{r(v)-\mu\}dv}d\tau$.

Let $c_i(t)$ and $a_i(t)$ denote respectively the rate of consumption and the amount of wealth, both per unit of effective labour, in the ith group at time t ($i = 1, 2, \ldots, q$; $t \geq 0$). Therefore, for all $t \geq 0$, $c(t) = \sum_{i=1}^q l_i c_i(t)$ and $k(t) = \sum_{i=1}^q l_i a_i(t)$. Note that the proportion of the total population in any wealth group is fixed and at any given instant every individual is endowed with the same amount of effective labour. The distribution of wealth in the economy at time $t \geq 0$ can therefore be represented by $(a_1(t), a_2(t), \ldots, a_q(t))$.

Given any two distributions of wealth, $(a_1', a_2', \ldots, a_q')$ and $(a_1'', a_2'', \ldots, a_q'')$, we assume that $(a_1', a_2', \ldots, a_q')$ is *more unequal than* (*as unequal as*) $(a_1'', a_2'', \ldots, a_q'')$ if for all pairs of wealth groups $i, j \in \{1, 2, \ldots, q\}$, it is true that

$$\left|\frac{a_i' - a_j'}{k'}\right| \geq \left|\frac{a_i'' - a_j''}{k''}\right|; \quad k' = \sum_{h=1}^q l_h a_h'; \quad k'' = \sum_{h=1}^q l_h a_h'';$$

and *there exists at least* (*there does not exist even*) one pair of wealth groups for which the inequality holds in the strict sense. The above criterion for comparing degrees of inequality between different distributions of wealth satisfies the standard Lorenz criterion when the latter is extended to allow for negative net worth of individuals and therefore, for 'negative shares' of total wealth.

Individuals can always borrow against their future wage-earnings, but individuals cannot indulge in Ponzi finance. The optimisation problem of individuals is entirely standard. For every $i \in \{1, 2, \ldots, q\}$ and every $\tau \in [0, \infty)$, an individual in group i at time τ chooses a pair $(c_i, a_i)^7$ which solves

PROBLEM P(iτ). $\max \int_\tau^\infty u(c_i(t)m(t))e^{-\theta(t-\tau)}dt$
 subject to: $\dot{a}_i(t) = w(t) + \{r(t) - \mu\}a_i(t) - c_i(t)$, *for all* $t \geq \tau$;
 $\lim_{t\to\infty} a_i(t)e^{-\int_\tau^t \{r(v)-\mu\}dv} \geq 0$; $c_i(t) \geq 0$, *for all* $t \geq \tau$; $a_i(\tau)$ *given*.

The instantaneous utility function $u(.)$ takes the CRRA form:

$$u(c_i(t)m(t)) = \frac{(c_i(t)m(t))^{1-\frac{1}{\sigma}}}{1 - \frac{1}{\sigma}}, \quad \sigma > 0, \quad \sigma \neq 1$$

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[7] c_i must be a piecewise continuous function and a_i a continuous function of the time variable. For the definition of piecewise continuity, see Halkin (1974, p. 268).

$\frac{1}{\sigma}$ is the constant elasticity of marginal utility.[8]

3. THE LONG-RUN DISTRIBUTION OF WEALTH

In this section we show that there exists an asymptotic distribution of wealth for the unique equilibrium growth path of the R–C–K economy. Moreover, the more unequal is the initial distribution of wealth, the greater is the inequality in the distribution of wealth in the long run.

Let us suppose the economy at time 0 has an arbitrary amount of capital per unit of effective labour, $k(0) > 0$, and an arbitrary distribution of wealth $(a_1(0), a_2(0), \ldots, a_q(0))$. Suppose that the rate of time preference $\theta > \mu(1 - \frac{1}{\sigma})$. Let k^* be the value of capital per unit of effective labour at which the marginal product of capital $f'(k^*) = \theta + \frac{\mu}{\sigma}$ and let the rate of consumption per unit of effective labour which keeps the capital-to-effective-labour ratio in the economy unchanged at k^* be given by $c^* = f(k^*) - \mu k^*$.

It is well known[9] that along an equilibrium growth path, the dynamics of aggregate consumption and capital stock must be such that:

If $k(0) = k^*$ then for all $t \geq 0, k(t) = k^*$ and $c(t) = c^*$ (1)

If $k(0) \neq k^*$ then (c, k) lies on the saddle path of the dynamic system:

$$\dot{c}(t) = \sigma c(t) \left\{ f'(k(t)) - \left(\theta + \frac{\mu}{\sigma} \right) \right\}; \dot{k}(t) = f(k(t)) - \mu k(t) - c(t); \quad (2)$$

and converges monotonically to (c^*, k^*)

Moreover, for all $i \in \{1, 2, \ldots, q\}$ *and all* $t \geq 0$,[10]

$$c_i(t) = \frac{a_i(0) + W(0)}{\int_0^\infty e^{\int_0^z \{(\sigma-1)r(v) - \theta\sigma\} dv} dz} e^{\sigma \int_0^t \{r(v) - (\theta + \frac{\mu}{\sigma})\} dv} \quad (3)$$

$$a_i(t) = \frac{a_i(0) + \int_0^t \{w(z) - c_i(z)\} e^{-\int_0^z \{r(v) - \mu\} dv} dz}{e^{-\int_0^t \{r(v) - \mu\} dv}} \quad (4)$$

For a CRRA utility function, the marginal utility of consumption becomes infinitely large as the rate of consumption approaches zero. Therefore, the rate of consumption of any individual along that individual's optimal consumption plan must always be positive. This implies (see (3) above) that an equilibrium growth path can exist only if

$$\forall i \in \{1, 2, \ldots, q\} : -a_i(0) < W(0) \quad (5)$$

where for all $t \geq 0, k(t)$ is defined by (1) and (2). That is, initial levels of indebtedness of individuals must be sufficiently small so that, for the growth path of the aggregate economy defined in (1) and (2), the capitalised value of an individual's lifetime stream

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[8] The subsequent discussion also holds for the case of a logarithmic instantaneous utility function i.e. for the case $\sigma = 1$.

[9] See, for example, Barro and Sala-i-Martin (1995).

[10] Under the assumption that $\theta > \mu(1 - \frac{1}{\sigma})$, the discounted value of an individual's lifetime stream of wage earnings $W(0)$ is finite along the path (for the aggregate capital stock) defined by (1) and (2).

of wage earnings must be greater than the magnitude of that individual's initial net liabilities.

If the initial distribution of wealth is such that condition (5) is satisfied, then—given the strict concavity of the CRRA utility function—it is easily verified that equations (1)–(4) define the unique perfect foresight competitive equilibrium growth path for the economy.

Along the equilibrium growth path, $\lim_{t \to \infty} a_i(t) e^{-\int_0^t \{r(v) - \mu\} dv} = 0$, for all $i \in \{1, 2, \dots, q\}$. Therefore, from (4), $\lim_{t \to \infty} [a_i(0) + \int_0^t \{w(z) - c_i(z)\} e^{-\int_0^z \{r(v) - \mu\} dv} dz] = 0$. Also, (1) and (2) imply that $\lim_{t \to \infty} \{r(v) - \mu\} = \theta - \mu(1 - \frac{1}{\sigma}) > 0$, so that $\lim_{t \to \infty} e^{-\int_0^t \{r(v) - \mu\} dv} = 0$. Therefore, from (4) we find that $\lim_{t \to \infty} a_i(t), i = 1, 2, \dots, q$, is of the form 0/0.

From (1)–(5), given that the rate of consumption per unit of effective labour, $c(t) = \sum_{i=1}^q l_i c_i(t)$ for all $t \geq 0$, it is straightforward to show that $c_i(t), i = 1, 2, \dots, q$, is a monotonic function of t on $[0, \infty)$, and for all $t \geq 0$,

$$\max \left(\frac{c^*}{l_i}, \frac{c(0)}{l_i} \right) \geq c_i(t) \geq 0$$

That is, $c_i(t), i = 1, 2, \dots, q$, is a bounded function of t on $[0, \infty)$. Therefore

$$\forall i \in \{1, 2, \dots, q\} : \lim_{t \to \infty} c_i(t) \text{ exists and is equal to } c_i^* \; (say) \tag{6}$$

From (4), applying L'Hôpital's Theorem, it follows that

$$\forall i \in \{1, 2, \dots, q\} : \lim_{t \to \infty} a_i(t) = \frac{c_i^* - w^*}{r^* - \mu} = a_i^* (say) \tag{7}$$

where $w^* = \lim_{t \to \infty} w(t) = f(k^*) - k^* f'(k^*)$ and $r^* = \lim_{t \to \infty} r(t) = f'(k^*)$.

Therefore, $(a_1^*, a_2^*, \dots, a_q^*)$ is the asymptotic distribution of wealth along the equilibrium growth path.

PROPOSITION 1. *Given an initial distribution of wealth* $(a_1(0), a_2(0), \dots a_q(0))$ *satisfying* (5) *and an arbitrary initial amount* $k(0) > 0$ *of capital per unit of effective labour in the economy, suppose* $\theta > \mu(1 - \frac{1}{\sigma})$. *Then, the unique equilibrium growth path has an asymptotic distribution of wealth.*

Now, note from (7) that for all pairs of wealth groups $i, j \in \{1, 2, \dots, q\}$,

$$a_i^* - a_j^* = \frac{c_i^* - c_j^*}{r^* - \mu}$$

Moreover, from (3) and (6) we obtain, for all $i \in \{1, 2, \dots, q\}$,

$$c_i^* = \frac{a_i(0) + W(0)}{\int_0^\infty e^{\int_0^z \{(\sigma - 1)r(v) - \theta\sigma\} dv} dz} e^{\sigma \int_0^\infty \{r(v) - (\theta + \frac{\mu}{\sigma})\} dv} \tag{8}$$

Hence, for all pairs of wealth groups $i, j \in \{1, 2, \dots, q\}$,

$$\frac{a_i^* - a_j^*}{k^*} = \frac{k(0)}{(r^* - \mu)k^*} \frac{e^{\sigma \int_0^\infty \{r(v) - (\theta + \frac{\mu}{\sigma})\} dv}}{\int_0^\infty e^{\int_0^z \{(\sigma - 1)r(v) - \theta\sigma\} dv} dz} \frac{a_i(0) - a_j(0)}{k(0)} \tag{9}$$

Since, $k(0) > 0$ and $\theta > \mu(1 - \frac{1}{\sigma})$, the first two terms in the product on the R.H.S. of (9) are positive. Therefore, it must be true that, beginning from the same initial ratio of capital to effective labour, a more unequal initial distribution of wealth is associated with a more unequal asymptotic distribution of wealth.

PROPOSITION 2. *Given an arbitrary initial amount $k(0) > 0$ of capital per unit of effective labour in the economy, suppose that $\theta > \mu(1 - \frac{1}{\sigma})$ and $(a'_{10}, a'_{20}, \ldots, a'_{q0})$, $(a''_{10}, a''_{20}, \ldots, a''_{q0})$ are alternative initial distributions of wealth (with the same mean individual wealth holding $k(0)m(0)$) both of which satisfy (5). Let $(a'_{1\infty}, a'_{2\infty}, \ldots, a'_{q\infty})$ and $(a''_{1\infty}, a''_{2\infty}, \ldots, a''_{q\infty})$ be the asymptotic distributions of wealth along the equilibrum growth paths associated respectively with $(a'_{10}, a'_{20}, \ldots, a'_{q0})$ and $(a''_{10}, a''_{20}, \ldots, a''_{q0})$. Then, if $(a'_{10}, a'_{20}, \ldots, a'_{q0})$ is more unequal than $(a''_{10}, a''_{20}, \ldots, a''_{q0})$, $(a'_{1\infty}, a'_{2\infty}, \ldots, a'_{q\infty})$ must be more unequal than $(a''_{1\infty}, a''_{2\infty}, \ldots, a''_{q\infty})$.*

The intuition for the above proposition follows from equation (3) which implies that the rates of consumption of individuals at any instant are proportional to their initial holdings of 'total wealth' (human plus nonhuman wealth). Since the present value at time 0 of the entire stream of consumption of any individual is equal to that individual's initial 'total wealth', this implies that the ratio between the 'total wealth' holdings of any two individuals must be a constant along the equilibrium growth path. Hence, the difference between the 'total wealth' holdings of any two individuals must be proportional to the difference in their initial 'total wealth' holdings.

Since factor markets are perfectly competitive and individuals supply the same amount of labour at any instant, individuals always possess equal amounts of human wealth. Thus the difference between the 'total wealth' holdings of any two individuals is always equal to the difference in their (nonhuman) wealth holdings. It therefore follows that the difference in the wealth holdings of any two individuals is always proportional and positively related to the difference in their initial wealth holdings. The degree of long-run inequality is therefore positively related to the degree of initial inequality in the distribution of wealth.

4. TRADE-OFF BETWEEN GROWTH AND EQUITY: CASE OF THE CES PRODUCTION FUNCTION

In this section we consider the case where the production function has a constant elasticity of substitution between capital and effective labour. We establish the possibility that the long-run distribution of wealth may be more, less or as unequal as a given initial distribution of wealth according as the long-run rate of growth of per capita output (equal to the rate of technical progress, μ) is greater than, less than or equal to some critical value.

Note from (1) that if the ini tial ratio of capital to effective labour $k(0)$ is such that $f'(k(0)) = \theta + \frac{\mu}{\sigma}$, then the aggregate economy is always in a steady state. In other words, suppose we define, for any given $k(0)$ with $f'(k(0)) > \theta$, the quantity $\mu(0) = \sigma\{f'(k(0)) - \theta\} > 0$. Then in case the rate of technical progress μ is equal to $\mu(0)$, the

economy will be on a steady state growth path. Moreover, in this case, one can easily verify from (9) that the asymptotic distribution of wealth is as unequal as the initial distribution.

To consider the intuition behind this result, note from (3) that at any given instant the rate of growth of consumption per efficiency unit of labour is the same for all individuals in the economy. Therefore, for any individual, the rate of consumption per efficiency unit of labour must be constant when the aggregate economy is in a steady state. The wage per efficiency unit of labour and the interest rate are also constant in steady state. Therefore, for any individual, the present discounted values of current and future consumption per efficiency unit of labour and current and future wage earnings per efficiency unit of labour are constant over time. Since the present discounted value of an individual's consumption stream is always equal to that individual's 'total wealth' holding, it follows that for any individual the amount of (nonhuman) wealth held per efficiency unit of labour is also a constant. The degree of inequality in the distribution of wealth therefore remains unchanged when the aggregate economy is in a steady state.

We will now consider the case where $\mu \neq \mu(0)$ assuming that the production function has the following form:

$$f(k) = A[\alpha k^{-\rho} + (1 - \alpha)]^{\frac{1}{\rho}} \tag{10}$$

where $A > 0$, $\rho > -1$, $\alpha \in (0, 1)$ and $\exists k > 0 : f'(k) = \mu$.

The production function has a constant elasticity of substitution $\varepsilon = \frac{1}{1+\rho}$.

Note that for a production function of the above variety, the Inada condition, $\lim_{k \to \infty} f'(k) = \lim_{k \to \infty} [\alpha A \{\alpha + (1 - \alpha)k^\rho\}^{-\frac{1}{\rho} - 1}] = 0$, is not, in general, satisfied. However, when ε is greater than unity ($\rho < 0$), $\lim_{k \to \infty} f'(k) = \alpha^{-\frac{1}{r-1}} A < \alpha A$. In this case, if $\mu \geq \alpha A$ then there does exist a value of k such that $f'(k) = \mu$.

Given that $\theta > \mu(1 - \frac{1}{\sigma})$ and given that the initial distribution of wealth is such that (5) is satisfied, it follows that for suffciently small values of the parameters α and A, a unique equilibrium growth path exists with an asymptotic distribution of wealth. Using the criterion for comparing the degrees of inequality in any two distributions of wealth introduced in Section 2, we know that the asymptotic distribution of wealth $(a_1^*, a_2^*, \ldots, a_q^*)$ is more unequal than the initial distribution $(a_1(0), a_2(0), \ldots, a_q(0))$ if for all pairs (i, j) of wealth groups $(i, j \in \{1, 2, \ldots, q\})$ it is true that

$$\left| \frac{a_i^* - a_j^*}{k^*} \right| \geq \left| \frac{a_i(0) - a_j(0)}{k(0)} \right|$$

and there exists at least one pair of wealth groups for which this inequality holds in the strict sense. The initial distribution is more unequal than the asymptotic distribution if the direction of the above inequalities are reversed.

From (3) we can infer that, along the equilibrium growth path, the consumption function of individuals must be of the form

$$c_i(t)m(t) = \beta(t)W(t)m(t) + \beta(t)a_i(t)m(t), \quad \beta(t) > 0, \quad \text{for all } i, t$$

where $\beta(t) = [\int_t^\infty e^{\int_t^z \{(\sigma-1)r(v)-\theta/\sigma\}dv} dz]^{-1}$ are functions of the sequence of interest rates and of the parameters θ and σ. The consumption of any individual is therefore made up of two components, one component proportional to the individual's human wealth $W(t)m(t)$, the other proportional to the individual's nonhuman wealth $a_i(t)m(t)$. For any individual $\beta(t)$ defines the (constant) propensity to consume out of 'total wealth' at time t.

Now consider any pair of individuals belonging to distinct wealth groups i and j. The difference in their rates of consumption is clearly proportional to the difference in their (nonhuman) wealth holdings. That is, at any point in time t the absolute value of the difference in the wealth holdings of any given pair of individuals must be proportional to the absolute value of the difference in their rates of consumption.

$$|a_i(t)m(t) - a_j(t)m(t)| = \{\beta(t)\}^{-1}|c_i(t)m(t) - c_j(t)m(t)|$$

It follows from above that the rate of growth in the absolute value of the difference in the wealth holdings of any given pair of individuals must equal the rate of growth in the absolute value of the difference in their rates of consumption less the rate of growth in the propensity to consume out of 'total wealth'.

$$\frac{(d/dt)|a_i(t)m(t) - a_j(t)m(t)|}{|a_i(t)m(t) - a_j(t)m(t)|} = \frac{(d/dt)|c_i(t)m(t) - c_j(t)m(t)|}{|c_i(t)m(t) - c_j(t)m(t)|} - \frac{\dot\beta(t)}{\beta(t)}$$

We know that along the equilibrium growth path the rate of growth in the rate of consumption is the same for all individuals in the economy and equal to the rate of growth in the per capita rate of consumption. Therefore, the rate of growth in the absolute value of the difference in the rates of consumption of any given pair of individuals must also be equal to the rate of growth in the per capita rate of consumption in the economy, so that:

$$\frac{(d/dt)|a_i(t)m(t) - a_j(t)m(t)|}{|a_i(t)m(t) - a_j(t)m(t)|} = \frac{(d/dt)\{c(t)m(t)\}}{c(t)m(t)} - \frac{\dot\beta(t)}{\beta(t)}$$

We also know that the per capita rate of consumption in the economy is the product of 'total wealth' per capita in the economy and the common individual propensity to consume out of 'total wealth'.

$$c(t)m(t) = \beta(t)[k(t)m(t) + W(t)m(t)]$$

Since the rate of growth in the absolute value of the difference in the wealth holdings of any given pair of individuals must equal the rate of growth of per capita consumption in the economy less the rate of growth in the propensity to consume out of 'total wealth', it follows that it must also equal the rate of growth of 'total wealth' per capita in the economy.

$$\frac{(d/dt)|a_i(t)m(t) - a_j(t)m(t)|}{|a_i(t)m(t) - a_j(t)m(t)|} = \frac{(d/dt)\{k(t)m(t) + W(t)m(t)\}}{\{k(t)m(t) + W(t)m(t)\}}$$

It follows from above that for all $t \geq 0$ and for all pairs (i, j) of wealth groups $(i, j \in \{1, 2, \ldots, q\})$:

$$\frac{|a_i(t) - a_j(t)|}{k(t)} = \frac{|a_i(0) - a_j(0)|}{k(0)} \frac{k(0)}{k(t)} \frac{k(t) + W(t)}{k(0) + W(0)}$$

Taking the limit on both sides as $t \to \infty$ we get for all pairs (i, j) of wealth groups $(i, j \in \{1, 2, \ldots, q\})$:

$$\frac{|a_i^* - a_j^*|}{k^*} = \frac{|a_i(0) - a_j(0)|}{k(0)} \frac{k(0)}{k^*} \frac{k^* + [w^*/\{r^* - \mu\}]}{k(0) + W(0)}$$

The asymptotic distribution of wealth is therefore more unequal, as unequal or less unequal than the initial distribution according as

$$\frac{[w^*/\{r^* - \mu\}]}{k^*} \gtreqless \frac{W(0)}{k(0)}$$

i.e. according as the asymptotic ratio of human to nonwealth wealth per capita is greater than, equal to or less than the initial ratio of human to nonhuman wealth per capita in the economy.

When $\mu = \mu(0)$, we have $k(0) = k^*$ and $W(0) = [w^*/\{r^* - \mu\}]$, so that (as we have already noted) the asymptotic distribution of wealth is as unequal as the initial distribution.

Now, suppose we are given values of μ and $\mu(0)$ such that $r(0) = f'(k(0)) = \theta + \frac{\mu(0)}{\sigma} > \max(\theta, \mu)$ and $\mu > \mu(0)$. The latter inequality implies that $k^* < k(0)$ and the wage rate per efficiency unit of labour continually decreases and the rate of interest continually increases along the equilibrium growth path. Therefore, $\frac{w(0)}{r(0) - \mu} = \int_0^\infty w(0)e^{-\int_0^\tau \{r(0) - \mu\}dv}d\tau$ provides an upper bound to $W(0)$. Thus, a sufficient condition for the asymptotic distribution of wealth to be more unequal than the initial distribution is that

$$\frac{[w^*/\{r^* - \mu\}]}{k^*} > \frac{[w(0)/\{r(0) - \mu\}]}{k(0)}$$

We can rewrite the above inequality as

$$\frac{1 - \{\mu/r(0)\}}{1 - (\mu/r^*)} > \frac{\{k^*/k(0)\}}{[(w^*/r^*)/\{w(0)/r(0)\}]}$$

For given values of $\mu, \mu(0), r(0) = \theta + \frac{\mu(0)}{\sigma}$ and $r^* = \theta + \frac{\mu}{\sigma}$ the L.H.S. in the above inequality is a constant. Now, consider what happens to the R.H.S. when the constant elasticity of substitution between capital and effective labour approaches infinity. As the elasticity of substitution becomes infinitely large the wage-rental ratio tends to become invariant to changes in the ratio of capital to effective labour. Therefore, the denominator on the R.H.S. must approach unity.

Also, for the production function in (10), the elasticity of the marginal product of capital with respect to the ratio of capital to effective labour k is given by $-[\varepsilon|1 + \{\alpha/(1 - \alpha)\}k^{\{(\varepsilon - 1)/\varepsilon\}}|]^{-1}$. When ε approaches infinity the elasticity of the marginal product of capital approaches zero at any value of k. For given $r(0) = f'(k(0))$ and $r^* = f'(k^*)$ it follows that the absolute magnitude of $[\{k(0) - k^*\}/k^*]/[(r(0) - r^*)/r^*]$ must approach infinity as ε approaches infinity. It follows that, since $k^* < k(0)$, the

numerator on the R.H.S. of the above inequality and therefore the R.H.S. term in the above inequality must approach zero as ε approaches infinity.[11]

Hence, given μ and $\mu(0)$ such that $\mu > \mu(0)$ one would expect the above inequality to hold for sufficiently large values of the constant elasticity of substitution between capital and effective labour. That is, one would expect the asymptotic distribution of wealth to be more unequal than the initial distribution.

Alternatively, if μ and $\mu(0)$ are such that $\mu < \mu(0)$ then the wage rate of effective labour increases and the rate of interest declines continuously along the equilibrium growth path. In this case $\frac{w(0)}{r(0)-\mu} = \int_0^\infty w(0)e^{-\int_0^{\tau [r(0)-\mu]d\nu}}\,d\tau$ provides a lower bound to $W(0)$. Then, a sufficient condition for the asymptotic distribution of wealth to be less unequal than the initial distribution is that

$$\frac{[w^*/\{r^* - \mu\}]}{k^*} < \frac{[w(0)/\{r(0) - \mu\}]}{k(0)}$$

Again, the above inequality can be rewritten as

$$\frac{1 - \{\mu/r(0)\}}{1 - (\mu/r^*)} < \frac{\{k^*/k(0)\}}{[(w^*/r^*)/\{w(0)/r(0)\}]}$$

However, with the given values of μ and $\mu(0)$ such that $\mu < \mu(0)$ one can now show that while the L.H.S. of the above inequality is a constant and the denominator of the R.H.S. term must approach unity as ε approaches infinity, since $k^* > k(0)$, the numerator on the R.H.S. must also approach infinity as ε approaches infinity. Therefore, for sufficiently large values of the elasticity of substitution one would expect the asymptotic distribution of wealth to be less unequal than the initial distribution if $\mu < \mu(0)$.

We can, in fact, state the following proposition.

PROPOSITION 3. *Let the initial amount of capital per unit of effective labour in the economy* $k(0) > 0$ *be such that* $f'(k(0)) = \theta + \frac{\mu(0)}{\sigma} > max(\mu, \theta)$. *Let the initial distribution of wealth* $(a_1(0), a_2(0), \ldots, a_q(0))$ *satisfy* (5). *Let* $\theta > \mu(1 - \frac{1}{\sigma})$ *and let the production function satisfy* (10). *Then, provided* $\mu \geq \alpha A$, *for sufficiently large values of* ε *the asymptotic distribution of wealth along the unique equilibrium growth path is more, less or as unequal as the initial distribution of wealth according as the steady state rate of growth of per capita output* μ *is greater than, less than or equal to the value* $\mu(0)$.

Proof. See Appendix ∎

5. CONCLUSION

Despite the prevalent view that traditional neoclassical growth models cannot explain persistence of historical inequalities in the distribution of wealth, analysis by Caselli and Ventura (2000) indicates that, in the case of a standard R–C–K model with logarithmic

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[11] The R.H.S. term can in fact be shown to be equal to $\{k^*/k(0)\}^{[(\varepsilon-1)/\varepsilon]} = [(r(0)/\alpha A)^{(\varepsilon-1)} - \alpha]/[(r^*/\alpha A)^{(\varepsilon-1)} - \alpha]$. Using L'Hopital's Rule it can be shown that this approaches zero as ε approaches infinity.

utility, historical inequalities may not only persist but be exacerbated in the process of growth. Considering a R–C–K economy in which individual agents are differentiated by their initial endowments of wealth, the paper verifies the possibility of long-run persistence of wealth inequality in the case of any CRRA instantaneous utility function. The paper shows that the distribution of wealth in the economy converges asymptotically to a steady state distribution. The more unequal the initial distribution of wealth endowments, the more unequal is the distribution in the long run. The paper also demonstrates that the distribution of wealth in the long run may be more unequal than the initial distribution and such cases of divergence may be associated with higher rates of exogenous technical progress and higher long-run rates of growth in the economy.

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APPENDIX

Proof (Proposition 3).

For the production function in (10) we can show that

$$\text{If } \mu \geq \alpha A \text{ then for all } \varepsilon > 1 \text{ there exists } k > 0 \text{ such that } f'(k) = \mu \qquad (11)$$

(1) and (2) imply that

$$\lim_{t \to \infty} c(t) = c^* = f(k^*) - \mu k^* = w^* + (r^* - \mu)k^* \qquad (12)$$

where, $w^* = \lim_{t \to \infty} w(t) = f(k^*) - k^* f'(k^*)$ and $r^* = \lim_{t \to \infty} r(t) = f'(k^*) = \theta + \frac{\mu}{\sigma}$.

From (8) we get

$$c^* = \lim_{t \to \infty} c(t) = \sum_{i=1}^{q} l_i (\lim_{t \to \infty} c_i(t)) = \frac{\{k(0) + W(0)\} e^{\sigma \int_0^\infty \{r(v) - (\theta + \frac{\mu}{\sigma})\} dv}}{\int_0^\infty e^{\int_0^z \{(\sigma - 1)r(v) - \theta\sigma\} dv dz}} \qquad (13)$$

From (9), (12) and (13) we get, for all $i, j \in \{1, 2, \ldots, q\}$,

$$\frac{a_i^* - a_j^*}{k^*} = \frac{k(0)}{(r^* - \mu)k^*} \frac{w^* + (r^* - \mu)k^*}{k(0) + W(0)} \frac{a_i(0) - a_j(0)}{k(0)} \qquad (14)$$

Claim 3a. *If $\mu > \mu(0)$ and $\mu \geq \alpha A$ then for sufficiently large values of ε, $(a_1^*, a_2^*, \ldots, a_q^*)$ is more unequal than $(a_1(0), a_2(0), \ldots, a_q(0))$.*

Suppose that $\mu > \mu(0)$. This implies that $k(0) > k^*$.

Since $\theta > \mu(1 - \frac{1}{\sigma})$, therefore $r^* > \mu$. Also $k(0) > 0$. Therefore, from (14) it follows that

$$\forall \text{ distinct } i, j \in \{1, 2, \ldots, q\} : \left| \frac{a_i^* - a_j^*}{k^*} \right| > \left| \frac{a_i(0) - a_j(0)}{k(0)} \right|$$

$$\text{if and only if } \frac{w^*}{(r^* - \mu)} \frac{k(0)}{k^*} + k(0) > k(0) + W(0) \qquad (15)$$

Also, by assumption, $r(0) > \mu$. Since $k(0) > k^*$, from (2) it can be shown that

$$\frac{w(0)}{r(0) - \mu} > W(0) > \frac{w^*}{r^* - \mu} \qquad (16)$$

From (15) and (16) it therefore follows that

$$\text{If } \frac{w^*}{r^* - \mu} \frac{k(0)}{k^*} \geq \frac{w(0)}{r(0) - \mu}$$

$$\text{then } \forall \text{ distinct } i, j \in \{1, 2, \ldots, q\} : \left| \frac{a_i^* - a_j^*}{k^*} \right| > \left| \frac{a_i(0) - a_j(0)}{k(0)} \right| \qquad (17)$$

From above it follows that

$$If \ \frac{[\{f(k^*)/k^*\} - \mu] - (r^* - \mu)}{[\{f(k(0))/k(0)\} - \mu] - \{r(0) - \mu\}} \geq \frac{r^* - \mu}{r(0) - \mu}$$

$$then \ \forall \ distinct \ i, j \in \{1, 2, \dots, q\} : \left|\frac{a_i^* - a_j^*}{k^*}\right| > \left|\frac{a_i(0) - a_j(0)}{k(0)}\right|$$

By assumption, $r(0) > \mu$. Also, from (10), for all $k > 0$, $f(k)/k > f'(k)$. Therefore, $\{f(k(0))/k(0)\} - \mu > r(0) - \mu > 0$. We know that, if x, y, u, v are real numbers such that $v > y > 0$, then $u/v \geq x/y$ implies that $(u - x)/(v - y) \geq x/y$. Therefore, it follows from above that

$$If \ \frac{[\{f(k^*)/k^*\} - \mu]}{[\{f(k(0))/k(0)\} - \mu]} \geq \frac{r^* - \mu}{r(0) - \mu}$$

$$then \ \forall \ distinct \ i, j \in \{1, 2, \dots, q\} : \left|\frac{a_i^* - a_j^*}{k^*}\right| > \left|\frac{a_i(0) - a_j(0)}{k(0)}\right| \qquad (18)$$

From (10), for all $k > 0$, $f(k)/k = A\{f'(k)/\alpha A\}^\varepsilon$.

Also, since $f(k^*)/k^* > f(k(0))/k(0) > f'(k(0)) > \mu$, it can be shown that

$$If \ \frac{A(r^*/\alpha A)^\varepsilon}{A\{r(0)/\alpha A\}^\varepsilon} \geq \frac{r^* - \mu}{r(0) - \mu} \ then \ \frac{A(r^*/\alpha A)^\varepsilon - \mu}{A\{r(0)/\alpha A\}^\varepsilon - \mu} \geq \frac{r^* - \mu}{r(0) - \mu}$$

Since $r^* > r(0) > \mu \geq 0$, given (18), the above implies that

$$If \ \varepsilon \geq \frac{\ln[(r^* - \mu)/\{r(0) - \mu\}]}{\ln\{r^*/r(0)\}}$$

$$then \ \forall \ distinct \ i, j \in \{1, 2, \dots, q\} : \left|\frac{a_i^* - a_j^*}{k^*}\right| > \left|\frac{a_i(0) - a_j(0)}{k(0)}\right| \qquad (19)$$

From (10) and (19) it follows that if $\mu \geq \alpha A$, then

$$There \ exists \ \varepsilon^* = \frac{\ln[(r^* - \mu)/\{r(0) - \mu\}]}{\ln\{r^*/r(0)\}} (> 1) \ such \ that$$

$$if \ \varepsilon \geq \varepsilon^* \ then \ \forall \ distinct \ i, j \in \{1, 2, \dots, q\} : \left|\frac{a_i^* - a_j^*}{k^*}\right| > \left|\frac{a_i(0) - a_j(0)}{k(0)}\right|$$

This proves Claim 3a. In a similar manner we can prove

Claim 3b. *If* $\mu < \mu(0)$ *and* $\mu \geq \alpha A$, *then for sufficiently large values of* ε, $(a_1^*, a_2^*, \dots, a_q^*)$ *is less unequal than* $(a_1(0), a_2(0), \dots, a_q(0))$.

The case $\mu = \mu(0)$ has already been discussed in the text (section 4). Thus, Claim 3a and Claim 3b are sufficient to establish Proposition 3. ∎