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DYNAMIC OLIGOPOLIES WITH POLLUTION TREATMENT COST SHARING

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Abstract: Dynamic oligopolies are analysed in the case when the firms treat pollution directly and share the cost in proportion to their share in the total output. If instantaneous information on the firms' output is available to all firms as well as to the pollution treatment agency, then the equilibrium is locally asymptotically stable. However the presence of information lags introduces the possibility of instability. Local stability analysis of the information lag situation is performed and conditions for local stability/ instability and the birth of limit cycles are obtained.

1. INTRODUCTION

Industrial waste and pollution emerge as a by-product of any kind of production process. One of the regulatory policies commonly employed to reduce pollution levels is a pollution tax. An alternative regulatory policy is to let the firms treat the pollution directly. In the first case, the pollution tax is modelled as an additional cost term depending on the firm's own output. In the alternative case of pollution treatment cost-sharing, the unit treatment cost depends on the total output of the industry, and so it cannot be modelled in the same way. Okuguchi and Szidarovszky (2000) have examined the existence and uniqueness of the Nash-equilibrium in the cost-sharing case by giving sufficient conditions for the existence of a unique positive equilibrium. The equilibria were compared without and with pollution treatment cost sharing, and comparative static analysis was conducted in relation to a change in pollution treatment technology.

In section 2 of this paper the dynamic extensions of the model of Okuguchi and Szidarovszky (2000) will be introduced, and the asymptotic behaviour of the equilibrium will be analysed. Next dynamic models of firm adjustment will be formulated in a fairly general framework that allows for an information lag structure. In section 3, the case

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of the availability of instantaneous information to all firms and to the pollution treatment agency, is first analysed. In section 4 we will assume that the unit treatment cost is determined by the pollution treatment agency based on a weighted average of past output data, and that this is common knowledge to all firms. In section 5 we will assume in addition that there is an information lag experienced by firms in obtaining and implementing information on the output of the rivals. Section 6 concludes.

2. THE GENERAL MODELLING FRAMEWORK

Consider an n-firm Cournot oligopoly, where pollution emerges as a result of firms' production activity and the firms treat jointly the pollution and share the treatment cost in proportion to their share in the total output.

The profit of firm i is then given as

$$\pi_i = x_i f\left(\sum_{j=1}^{n} x_j\right) - c_i(x_i) - x_i \frac{T\left(\sum_{j=1}^{n} x_j\right)}{\left(\sum_{j=1}^{n} x_j\right)}, \tag{1}$$

where x_i is the output of firm i, f is the inverse demand function, c_i is the cost function of firm i, and T is the total pollution treatment cost function. The economic interpretation of these functions implies that f is strictly decreasing, whilst c_i and T are strictly increasing. In the following analysis we assume differentiability of all relevant functions up to the necessary order. A non-cooperative n-person game is now defined: the firms are the players, $[0, \mathcal{N}]$ and π_i are the set of strategies and the payoff function of firm i, respectively.

Introduce the notation

$$X = \sum_{j=1}^{n} x_j, \quad G(X) = \frac{T(X)}{X}, \quad F(X) = f(X) - G(X). \tag{2}$$

For any given $Q_i = \sum_{j \neq i} x_j$, the best response of firm i is the output that maximises

$$\pi_i = x_i F(x_i + Q_i) - c_i(x_i). \tag{3}$$

Excluding a corner optimum, the first order condition is

$$F(x_i + Q_i) + x_i F'(x_i + Q_i) - c_i'(x_i) = 0 \tag{4}$$

and the second order condition is

$$2F'(x_i + Q_i) + x_i F''(x_i + Q_i) - c_i''(x_i) < 0. \tag{5}$$

As in Okuguchi and Szidarovszky (2000), assume that the following conditions hold:
(A) $F'(X) + x_i F''(X) \leq 0$,
(B) $F'(X) < c_i''(x_i)$,
for all x_i and X.

These assumptions imply that π_i is strictly concave and the left hand side of equation (4) is strictly decreasing in x_i with fixed values of Q_i. In Okuguchi and Szidarovszky (2000) sufficient conditions were derived for the existence of a positive equilibrium. We do not make those assumptions here, rather we simply assume the existence of a positive equilibrium, $x^* = (x_1^*, \dots, x_n^*)$. In the neighbourhood of this equilibrium there is a best response function for each firm, $x_i = R_i(Q_i)$, which is the unique solution of equation (4) for x_i. Clearly, at the equilibrium,

$$x_i^* = R_i\left(\sum_{j \neq i} x_j^*\right)$$

for all $i=1,2, \dots, n$. The derivative of R_i can be obtained by differentiating implicitly equation (4) with respect to Q_i:

$$R_i'(Q_i) = -\frac{F'(X) + x_i F''(X)}{2F'(X) + x_i F''(X) - c_i''(x_i)} \,. \tag{6}$$

Assumptions (A) and (B) then imply that

$$-1 < R_i'(Q_i) \leq 0 \,. \tag{7}$$

We now formulate three possible dynamic adjustment regimes for output based on different information sets for the firms and/or the pollution treatment agency:—

(I) Consider first the situation in which the output of each firm is instantaneously available to all firms as well as to the pollution treatment agency, and the pollution treatment cost is computed by using the current output values. It is also assumed that each firm adjusts its output into the direction of its best response resulting in the dynamic equations

$$\dot{x}_i = k_i(R_i(Q_i) - x_i) \tag{8}$$

where $k_i > 0$, the speed of adjustment of firm i, is a given constant for all i.

(II) Consider next the situation in which the pollution treatment agency computes the unit treatment cost based on an average of past output data, and the averaging rule is known by all firms. Firms however continue to have instantaneous information about the output of rival firms. In this case each firm maximises its profit, which now has the form

$$\pi_i = x_i f(x_i + Q_i) - c_i(x_i) - x_i G(X^E) \,, \tag{9}$$

where X^E is an average of the past outputs of the industry. We assume that at each time period

$$X^E(t) = \int_0^t w(t-s,\, T,\, m) \sum_{i=1}^n x_i(s) ds \,, \tag{10}$$

where w is a weighting function known by all firms. As in Invernizzi and Medio (1991), Chiarella and Khomin (1996), and Chiarella and Szidarovszky (2000) we will use weighting functions of the form

$$w(t-s, T, m) = \begin{cases} \dfrac{1}{m!}\left(\dfrac{m}{T}\right)^{m+1} (t-s)^m e^{-\frac{m(t-s)}{T}} & \text{if } m \geq 1 \\ \dfrac{1}{T} e^{-\frac{(t-s)}{T}} & \text{if } m = 0 \,, \end{cases} \tag{11}$$

where m is a non-negative integer and $T > 0$ is a given parameter. Notice that this weighting function has the following properties:

(a) For $m = 0$, weights are exponentially declining with the most weight given to the most current value;
(b) For $m \geq 1$, zero weight is given to the most current value, rising to maximum at $t - s = T$, and declining exponentially thereafter;
(c) As m increases, the weighting function becomes more peaked around $t - s = T$. For sufficiently large values of m the function may for all practical purposes be regarded as very close to the Dirac delta function centered at $t - s = T$.
(d) As $T \to 0$, the weighting function tends to the Dirac delta function centered at zero for all m;
(e) The area under the weighting function is unity for all T and m.

The best response of firm i is obtained by maximising function (9) with given Q_i and X^E. The first order conditions are

$$(f(x_i + Q_i) - G(X^E)) + x_i f'(x_i + Q_i) - c_i'(x_i) = 0 \tag{12}$$

and the second order conditions are

$$2f'(x_i + Q_i) + x_i f''(x_i + Q_i) - c_i''(x_i) < 0\,. \tag{13}$$

If we assume that for all x_i and X,
(C) $f'(X) + x_i f''(X) \leq 0$,
(D) $f'(X) < c_i''(x_i)$,
then the left hand side of (12) is strictly decreasing in x_i, so there is a best response

$$x_i = R_i^{(1)}(Q_i, X^E) \tag{14}$$

of firm i. The derivatives of $R_i^{(1)}$ are obtained by differentiating equation (12) implicitly with respect to Q_i and X^E, so that

$$\frac{\partial R_i^{(1)}}{\partial Q_i} = -\frac{f'(X) + x_i f''(X)}{2f'(X) + x_i f''(X) - c_i''(x_i)} \tag{15}$$

and

$$\frac{\partial R_i^{(1)}}{\partial X^E} = \frac{G'(X^E)}{2f'(X) + x_i f''(X) - c_i''(x_i)}\,. \tag{16}$$

Assumptions (C) and (D) imply that

$$-1 < \frac{\partial R_i^{(1)}}{\partial Q_i} \leq 0\,, \tag{17}$$

however the sign of $\partial R_i^{(1)}/\partial X^E$ is indeterminate.

Assuming again that each firm adjusts its output into the direction of its best response, then we have the dynamic equations:

$$\dot{x}_i = k_i(R_i^{(1)}(Q_i, X^E) - x_i)\,, \tag{18}$$

where X^E is given in equation (10).

(**III**) Finally we consider the case in which the firms, in addition to the pollution treatment agency experience information lags. In particular each firm experiences a time lag in obtaining information about the output of the rest of the industry, and the time lag is continuously distributed. So instead of the instantaneous output Q_i of the rest of the industry, each firm uses the expectation

$$Q_i^E(t) = \int_0^t w(t-s, S_i, l_i) Q_i(s) ds, \tag{19}$$

where $l_i \geq 0$ is an integer, $S_i > 0$ is a parameter, and the weighting function is given in equation (11). In this case the dynamic adjustment of output is modified to

$$\dot{x}_i = k_i (R_i^{(1)}(Q_i^E, X^E) - x_i), \tag{20}$$

where X^E is given in equation (10).

In the following sections the asymptotic behaviour of the equilibrium of the output adjustment processes (8), (18), and (20) will be analysed, particularly with a view to determining the effect on the dynamic behaviour of the different information structures.

3. THE CASE OF INSTANTANEOUS INFORMATION

As we have shown in the previous section, the dynamic behaviour of output in this case is described by the system of ordinary differential equations (8), the asymptotic behaviour of which is examined by linearization around the equilibrium.

The Jacobian of system (8) at the equilibrium has the form

$$\underline{J} = \begin{pmatrix} -k_1 & k_1 r_1^* & \cdots & k_1 r_1^* \\ k_2 r_2^* & -k_2 & \cdots & k_2 r_2^* \\ k_n r_n^* & k_n r_n^* & \cdots & -k_n \end{pmatrix} \tag{21}$$

where $r_i^* = R_i'\left(\sum_{j \neq i} x_j^*\right)$.

Since (21) may be re-expressed as

$$\underline{J} = \underline{D} + \underline{a}\,\underline{1}^T \tag{22}$$

with $\underline{1}^T = (1, 1, \ldots, 1)$ and

$$\underline{D} = \text{diag}\,(-k_1(1+r_1^*), -k_2(1+r_2^*), \ldots, -k_n(1+r_n^*)) \quad \text{and} \quad \underline{a} = \begin{pmatrix} k_1 r_1^* \\ k_2 r_2^* \\ \vdots \\ k_n r_n^* \end{pmatrix},$$

the characteristic polynomial of $\underline{J}$ has the special form

$$\begin{aligned} \varphi(\lambda) &= \det\,(\underline{D} + \underline{a}\,\underline{1}^T - \lambda \underline{I}) = \det\,(D - \lambda \underline{I}) \det\,(\underline{I} + (\underline{D} - \lambda \underline{I})^{-1} \underline{a}\,\underline{1}^T) \\ &= \det\,(\underline{D} - \lambda \underline{I})[1 + \underline{1}^T (\underline{D} - \lambda\, \underline{I})^{-1} \underline{a}] \\ &= \prod_{i=1}^n (-k_i(1+r_i^*) - \lambda) \left[1 + \sum_{i=1}^n \frac{k_i r_i^*}{-k_i(1+r_i^*) - \lambda}\right]. \end{aligned}$$

Let $\rho_1 < \rho_2 < \ldots < \rho_s$ denote the different eigenvalues $-k_i(1+r_i^*)$, and I_j the set of firms having the same eigenvalue $\rho_j = -k_i(1+r_i^*)$, and let m_j denote the number of firms belonging to I_j. Then

$$\varphi(\lambda) = \prod_{i=1}^{s} (\rho_i - \lambda)^{m_j} \left[1 + \sum_{j=1}^{s} \frac{w_j}{\rho_j - \lambda}\right] \tag{23}$$

with

$$w_j = \sum_{i \in I_j} k_i r_i^* .$$

From relation (7) we see that for all j, $\rho_j < 0$ and $w_j \leq 0$. We will next prove that all roots of the characteristic polynomial φ are real and negative implying the following

THEOREM 1. *Under conditions (A) and (B) the equilibrium of system* (8) *representing the case of instantaneous information is always locally asymptotically stable.*

Proof. Since $\rho_j < 0$ for all j, it is sufficient to show that all roots of the equation

$$\sum_{j=1}^{s} \frac{w_j}{\rho_j - \lambda} = -1 \tag{24}$$

are real and negative. We may assume that $w_j \neq 0$ for all j, otherwise we would consider only the nonzero terms of the left hand side with a smaller value of s. Let $g(\lambda)$ denote the left hand side. Clearly

$$\lim_{\lambda \to \pm\infty} g(\lambda) = 0 , \quad \lim_{\lambda \to \rho_j \pm 0} g(\lambda) = \pm\infty$$

and

$$g'(\lambda) = \sum_{j=1}^{s} \frac{w_j}{(\rho_j - \lambda)^2} < 0 .$$

The graph of function g is shown in Figure 1. Notice that equation (24) is equivalent to a polynomial equation of degree s, and there is a root below ρ_1, and one root between each pair ρ_j and ρ_{j+1} of poles for j =1,2, ... , $s-1$. Hence all roots are real and negative. ■

It is known (see for example, Okuguchi and Szidarovszky, 1999) that in the situation without pollution treatment cost sharing dynamic oligopolies under conditions (A) and (B) are always locally asymptotically stable. Therefore the introduction of pollution treatment cost sharing merely changes the location of the equilibrium in general, but the equilibrium still remains locally asymptotically stable.

4. THE CASE OF INFORMATION LAG ONLY FOR THE POLLUTION TREATMENT AGENCY

We have shown in section 2 that in the case in which the pollution treatment agency bases its calculation on an average of past output, the dynamic adjustment of output is given by equation (18). The asymptotic behaviour of the resulting dynamics will be examined by linearisation around the equilibrium.

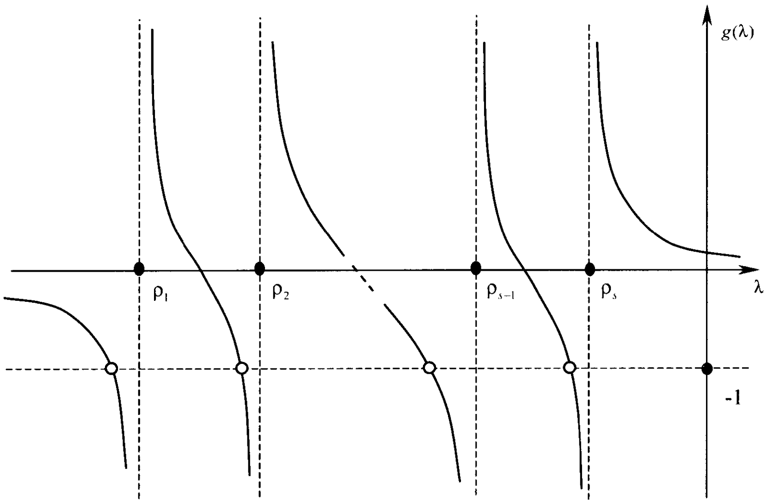

Figure 1. Graph of Function g.

The linearised equation has the form

$$\dot{x}_{i\delta} = k_i \left(\frac{\partial R_i^{(1)}}{\partial Q_i}(Q_i^*, X^*) Q_{i\delta} + \frac{\partial R_i^{(1)}}{\partial X^E}(Q_i^*, X^*) X_\delta^E - x_{i\delta} \right) \tag{25}$$

where $x_{i\delta}$, $Q_{i\delta}$, and X_δ^E denote the deviation of x_i, Q_i, and X^E from their equilibrium levels. Equations (15) and (16) imply that this equation can be rewritten as

$$\dot{x}_{i\delta} = k_i \left(-\frac{\alpha_i}{\beta_i} \sum_{j \neq i} x_{j\delta} + \frac{\gamma}{\beta_i} \int_0^t w\,(t-s, T, m) \sum_{j=1}^n x_{j\delta}(s) ds - x_{i\delta} \right)^{\cdot} \tag{26}$$

with

$$\alpha_i = f'(X^*) + x_i^* f''(X^*),$$
$$\beta_i = 2f'(X^*) + x_i^* f''(X^*) - c_i''(x_i^*),$$

and

$$\gamma = G'(X^*).$$

We seek a solution to the integro-differential equation system (26) in the form

$$x_{i\delta} = v_i e^{\lambda t}, \quad (i = 1, 2, \ldots, n).$$

Substituting this form into equation (26) and allowing $t \to \infty$ we have

$$\begin{aligned} &\left(\lambda + k_i \left(1 - \frac{\gamma}{\beta_i} \int_0^\infty w(s, T, m) e^{-\lambda s} ds \right) \right) v_i \\ &\quad + k_i \left(\frac{\alpha_i}{\beta_i} - \frac{\gamma}{\beta_i} \int_0^\infty w(s, T, m) e^{-\lambda s} ds \right) \sum_{j \neq i} v_j = 0. \end{aligned} \tag{27}$$

By introducing

$$A_i(\lambda) = \lambda + k_i \left(1 - \frac{\gamma}{\beta_i}\left(\frac{\lambda T}{r} + 1\right)^{-(m+1)}\right)$$

and

$$B_i(\lambda) = k_i \left(\frac{\alpha_i}{\beta_i} - \frac{\gamma}{\beta_i}\left(\frac{\lambda T}{r} + 1\right)^{-(m+1)}\right)$$

with

$$r = \begin{cases} m & \textit{if } m \geq 1 \\ 1 & \textit{if } m = 0, \end{cases}$$

equation (27) becomes

$$\det \begin{pmatrix} A_1(\lambda) & B_1(\lambda) & \cdots & B_1(\lambda) \\ B_2(\lambda) & A_2(\lambda) & \cdots & B_2(\lambda) \\ \vdots & \vdots & \vdots & \vdots \\ B_n(\lambda) & B_n(\lambda) & \cdots & A_n(\lambda) \end{pmatrix} = 0 .$$

Notice that this determinant has a similar structure to that of the Jacobian (21) examined in the case of instantaneous information. By using the same method that was used to derive equation (23) we find that this determinantal equation can be rewritten as

$$\prod_{i=1}^{n} (A_i(\lambda) - B_i(\lambda)) \left[1 + \sum_{i=1}^{n} \frac{B_i(\lambda)}{A_i(\lambda) - B_i(\lambda)}\right] = 0 . \tag{28}$$

Notice that

$$A_i(\lambda) - B_i(\lambda) = \lambda + k_i \left(1 - \frac{\alpha_i}{\beta_i}\right) ,$$

and from relation (17) we see that $k_i\left(1 - \frac{\alpha_i}{\beta_i}\right)$ is positive for all i. Hence the roots of the first product are all negative. The other roots are the solutions of the equation

$$1 + \sum_{i=1}^{n} \frac{1}{\lambda + k_i\left(1 - \frac{\alpha_i}{\beta_i}\right)} k_i \left(\frac{\alpha_i}{\beta_i} - \frac{\gamma}{\beta_i}\left(\frac{\lambda T}{r} + 1\right)^{-(m+1)}\right) = 0 ,$$

which can be rewritten as

$$\left(\frac{\lambda T}{r} + 1\right)^{m+1} \left\{1 + \sum_{i=1}^{n} \frac{k_i \frac{\alpha_i}{\beta_i}}{\lambda + k_i\left(1 - \frac{\alpha_i}{\beta_i}\right)}\right\} = \gamma \sum_{i=1}^{n} \frac{\frac{k_i}{\beta_i}}{\lambda + k_i\left(1 - \frac{\alpha_i}{\beta_i}\right)} . \tag{29}$$

In the general case numerical methods are needed to locate the roots of (29), so in order to obtain analytic results special cases will need to be considered.

For the sake of algebraic simplicity consider the symmetric case, when $k_1 = \cdots = k_n = k, \alpha_1 = \cdots = \alpha_n = \alpha, \ \beta_1 = \cdots = \beta_n = \beta$. Then equation (29) simplifies to

$$\left(\frac{\lambda T}{r} + 1\right)^{m+1} \left(\lambda + k\left(1 + (n-1)\frac{\alpha}{\beta}\right)\right) = \frac{nk\gamma}{\beta} . \tag{30}$$

Assume first that m=0. Then (30) reduces to the quadratic equation

$$\lambda^2 T + \lambda\left[1 + kT\left(1 + \frac{(n-1)\,\alpha}{\beta}\right)\right] + k\left[1 + (n-1)\frac{\alpha}{\beta} - \frac{n\gamma}{\beta}\right] = 0\,. \qquad (31)$$

In this case we can establish the following result:

THEOREM 2. *Assume* $m = 0$. *If*

$$\gamma > \frac{\beta + (n-1)\,\alpha}{n}\,,$$

then the equilibrium of the output dynamics (18) *is locally asymptotically stable. If*

$$\gamma < \frac{\beta + (n-1)\,\alpha}{n}\,,$$

then the equilibrium is unstable, and if

$$\gamma = \frac{\beta + (n-1)\,\alpha}{n}\,,$$

then no conclusion about the stability of the equilibrium can be drawn.

Proof. Notice first that the linear coefficient is always positive. If

$$\gamma > \frac{\beta + (n-1)\alpha}{n},$$

then the constant term is also positive demonstrating the asymptotic stability of the equilibrium. If

$$\gamma = \frac{\beta + (n-1)\alpha}{n},$$

then one root is negative, the other is zero. Hence no conclusion can be drawn about the stability of the equilibrium. If

$$\gamma < \frac{\beta + (n-1)\alpha}{n},$$

then one root is positive and the other is negative implying the instability of the equilibrium. ■

Notice that without pollution treatment cost sharing $G(x) \equiv 0$, so $\gamma = 0$. Since both α and β are negative, $\beta + (n-1)\alpha$ is always negative, so when $\gamma = 0$ the condition of local asymptotic stability of the equilibrium is always satisfied. Hence the introduction of pollution cost sharing (when $\gamma \neq 0$ necessarily) may make the equilibrium unstable, so that the asymptotic behaviour of the equilibrium becomes much richer. If $G'(x^*)$ is a large negative number, then instability in fact occurs.

Assume next that m=1. Then (30) reduces to the cubic equation

$$\lambda^3 T^2 + \lambda^2\left[2T + kT^2\left(1 + \frac{(n-1)\alpha}{\beta}\right)\right] + \lambda\left[1 + 2Tk\left(1 + \frac{(n-1)\alpha}{\beta}\right)\right] + k\left(1 + \frac{(n-1)\alpha - n\gamma}{\beta}\right) = 0\,. \qquad (32)$$

In this case we can establish the following result:

THEOREM 3. *Assume $m = 1$. The equilibrium of the output dynamics* (18) *is locally asymptotically stable if*

$$\frac{\beta + (n-1)\alpha}{n} < \gamma < \frac{-2\beta(1 + kTA)^2}{nkT},$$

where $A = 1 + \frac{(n-1)\alpha}{\beta} \geq 1.$

If

$$\gamma < \frac{\beta + (n-1)\alpha}{n} \quad \text{or} \quad \gamma > \frac{-2\beta(1 + kTA)^2}{nkT},$$

then the equilibrium is unstable.

If

$$\gamma = \frac{\beta + (n-1)\alpha}{n}$$

then no conclusion about the stability of the equilibrium can be drawn.

If

$$\gamma = \frac{-2\beta(1 + kTA)^2}{nkT},$$

then a Hopf bifurcation occurs, so there is a limit cycle in the neighbourhood of the equilibrium.

Proof. Notice first that since α and β are negative, all coefficients of (23) are positive if and only if

$$\gamma > \frac{\beta + (n-1)\alpha}{n}.$$

All eigenvalues have negative real parts if and only if this relation holds and

$$\left[2T + kT^2\left(1 + \frac{(n-1)\alpha}{\beta}\right)\right]\left[1 + 2Tk\left(1 + \frac{(n-1)\alpha}{\beta}\right)\right] > T^2k\left(1 + \frac{(n-1)\alpha - n\gamma}{\beta}\right)$$

as a consequence of the Routh-Hurwitz stability criterion (see for example, Szidarovszky and Bahill, 1997).

By introducing the quantity

$$A = 1 + \frac{(n-1)\alpha}{\beta} \geq 1$$

the above inequality can be rewritten as

$$\gamma < \frac{-2\beta(1 + kTA)^2}{nkT}.$$

Assume next that

$$\gamma < \frac{\beta + (n-1)\,\alpha}{n},$$

then the constant term of the cubic (32) is negative implying the existence of a positive root.

If

$$\gamma > \frac{-2\beta\,(1 + kTA)^2}{nkT}$$

then the Routh-Hurwitz criterion shows the existence of a root with positive real part. In both cases the equilibrium is unstable.

If

$$\gamma = \frac{\beta + (n-1)\,\alpha}{n},$$

then equation (32) has a zero root and the other two roots have negative real parts. We will finally show that if

$$\gamma = \frac{-2\beta\,(1 + kTA)^2}{nkT},$$

then there is a pair of pure complex roots and Hopf bifurcation occurs.

Assume that $\lambda = ia$ is a pure complex root, then by equating the real and imaginary parts to zero we have

$$-a^3T^2 + a\left[1 + 2Tk\left(1 + \frac{(n-1)\alpha}{\beta}\right)\right] = 0$$

and

$$-a^2\left[2T + kT^2\left(1 + \frac{(n-1)\alpha}{\beta}\right)\right] + k\left(1 + \frac{(n-1)\alpha - n\gamma}{\beta}\right) = 0$$

implying that

$$a^2 = \frac{1 + 2Tk\left(1 + \frac{(n-1)\alpha}{\beta}\right)}{T^2} = \frac{k\left(1 + \frac{(n-1)\alpha - n\gamma}{\beta}\right)}{2T + kT^2\left(1 + \frac{(n-1)\alpha}{\beta}\right)}. \tag{33}$$

We select γ as the bifurcation parameter. This relation gives for the critical value of γ the simple expression

$$\gamma^* = \left\{kT\left(1 + \frac{(n-1)\alpha}{\beta}\right) - \left(2 + kT\left(1 + \frac{(n-1)\alpha}{\beta}\right)\right) \times \left(1 + 2Tk\left(1 + \frac{(n-1)\alpha}{\beta}\right)\right)\right\} \Big/ \left(\frac{nkT}{\beta}\right).$$

Notice that the middle part of equation (33) is always positive, so there is always real solution for a.

The critical γ^* value can be further simplified to

$$\gamma^* = \frac{-2(1+kTA)^2\beta}{nkT},$$

which is always positive. Differentiating equation (32) implicitly with respect to γ we have $(\dot{\lambda} \equiv d\lambda/d\gamma)$

$$3\lambda^2T^2\dot{\lambda} + 2\lambda\dot{\lambda}\left[2T + kT^2\left(1 + \frac{(n-1)\alpha}{\beta}\right)\right] + \dot{\lambda}\left[1 + 2Tk\left(1 + \frac{(n-1)\alpha}{\beta}\right)\right] - \frac{nk}{\beta} = 0,$$

implying that

$$\dot{\lambda} = \frac{\frac{nk}{\beta}}{3\lambda^2T^2 + 2\lambda\left[2T + kT^2A\right] + [1 + 2TkA]},$$

which at $\lambda = ia$ has the real part

$$Re\left(\dot{\lambda}\right)\Big|_{\lambda=ia} = \frac{-\frac{2a^2T^2nk}{\beta}}{\left(2a\left(2T + kT^2A\right)\right)^2 + \left(2a^2T^2\right)^2} > 0.$$

Hence there is a limit cycle in the neighbourhood of the equilibrium as a consequence of the Hopf bifurcation theorem (see for example, Guckenheimer and Holmes, 1983). ■

With fixed values of β, n, k and T Figure 2 shows the stability region in the (γ, α) plane. Notice that $\gamma = 0$ is always in the stability region showing again that without pollution treatment cost sharing the equilibrium is always locally asymptotically stable. Therefore in the presence of time lag in pollution treatment the asymptotic behaviour of the equilibrium becomes richer. In this case the critical value for γ (at which the birth of a limit cycle occurs) as a function of α is represented as the upper parabolic boundary of the stability region shown in Figure 3. It is clear from Figure 1 that the asymptotic stability of the equilibrium is preserved if $\left|G'(x^*)\right|$ is sufficiency small.

5. THE CASE OF INFORMATION LAGS FOR FIRMS AND THE POLLUTION TREATMENT AGENCY

In section 2 we have shown that in this case the dynamic adjustment of output is given by equation (20), the asymptotic behaviour of which will be examined by linearisation. The linearised equation now has the form

$$\dot{x}_{i\delta} = k_i\left(\frac{\partial R_i^{(1)}}{\partial Q_i}\left(Q_i^*, X^*\right)Q_{i\delta}^E + \frac{\partial R_i^{(1)}}{\partial X^E}\left(Q_i^*, X^E\right)X_\delta^E - x_{i\delta}\right), \tag{34}$$

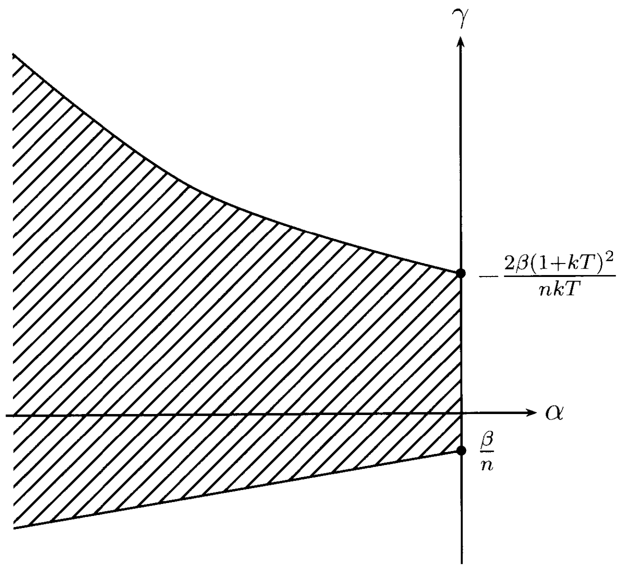

Figure 2. Information lag only for the pollution treatment agency: The stability regtion in the (γ, α) plane for the case $m = 1$.

where $x_{i\delta}$, $Q_{i\delta}^E$, and X_δ^E are the deviation of x_i, Q_i^E, and $X^E$from their equilibrium levels. Equations (15) and (16) imply that this equation can be rewritten as

$$\begin{aligned}\dot{x}_{i\delta} =& k_i \left(-\frac{\alpha_i}{\beta_i} \int_0^t w\left(t-s, S_i, l_i\right) \sum\nolimits_{j \neq i} x_{j\delta}(s)\, ds \right. \\ & \left. + \frac{\gamma}{\beta_i} \int_0^t w\left(t-s, T, m\right) \sum\nolimits_{j=1}^{n} x_{j\delta}(s)\, ds - x_{i\delta} \right)\end{aligned} \tag{35}$$

where α_i, β_i and γ are the same as in the previous section. We again seek the solution to (35) in the form

$$x_{i\delta} = v_i e^{\lambda t} \quad (i = 1, 2, \ldots, n).$$

Substituting this form into equation (35) and allowing $t \to \infty$ we obtain the characteristic equation

$$\begin{aligned}&\left(\lambda + k_i \left(1 - \frac{\gamma}{\beta_i} \int_0^\infty w(s, T, m) e^{-\lambda s} ds \right) \right) v_i \\ &+ k_i \left(\frac{\alpha_i}{\beta_i} \int_0^\infty w(s, S_i, l_i) e^{-\lambda s} ds - \frac{\gamma}{\beta_i} \int_0^\infty w(s, T, m) e^{-\lambda s} ds \right) \sum\nolimits_{j \neq i} v_j = 0 .\end{aligned} \tag{36}$$

Introduce now functions

$$A_i(\lambda) = \lambda + k_i\left(1 - \frac{\gamma}{\beta_i}\left(\frac{\lambda T}{r} + 1\right)^{-(m+1)}\right),$$

and

$$B_i(\lambda) = k_i\left(\frac{\alpha_i}{\beta_i}\left(\frac{\lambda S_i}{q_i} + 1\right)^{-(l_i+1)} - \frac{\gamma}{\beta_i}\left(\frac{\lambda T}{r} + 1\right)^{-(m+1)}\right),$$

with

$$q_i = \begin{cases} l_i, & \text{if } l_i \geq 1, \\ 1, & \text{if } l_i = 0, \end{cases}$$

to see that equation (36) is equivalent to the same type of determinantal equation as in the previous section, therefore it is equivalent to equation (28) with the above defined functions $A_i(\lambda)$ and $B_i(\lambda)$.

Consider first the equation

$$A_i(\lambda) - B_i(\lambda) = 0,$$

which now has the form

$$\lambda + k_i\left(1 - \frac{\alpha_i}{\beta_i}\left(\frac{\lambda S_i}{q_i} + 1\right)^{-(l_i+1)}\right) = 0. \tag{37}$$

This is equivalent to the polynomial equation

$$(\lambda + k_i)\left(\frac{\lambda S_i}{q_i} + 1\right)^{l_i+1} - \frac{k_i \alpha_i}{\beta_i} = 0.$$

Notice that this equation is similar to equation (30), which was examined in section 4. However in this case we can prove that all roots have negative real parts. The proof is by contradiction; assume that for a root, Re $\lambda \geq 0$, then

$$|\lambda + k_i| \geq k_i \quad \text{and} \quad \left|\frac{\lambda S_i}{q_i} + 1\right| \geq 1,$$

so that

$$\left|(\lambda + k_i)\left(\frac{\lambda S_i}{q_i} + 1\right)^{l_i+1}\right| \geq k_i.$$

However assumptions (A) and (B) (see section 2) imply that $\left|\frac{\alpha_i}{\beta_i}\right| < 1$, therefore

$$\left|\frac{k_i \alpha_i}{\beta_i}\right| < k_i,$$

which is an obvious contradiction. Hence all roots must have negative real parts.

The other eigenvalues are obtained by solving the equation

$$1+\sum_{i=1}^{n} \frac{k_i\left(\frac{\alpha_i}{\beta_i}\left(\frac{\lambda S_i}{q_i}+1\right)^{-(l_i+1)}-\frac{\gamma}{\beta_i}\left(\frac{\lambda T}{r}+1\right)^{-(m+1)}\right)}{\lambda+k_i\left(1-\frac{\alpha_i}{\beta_i}\left(\frac{\lambda S_i}{q_i}+1\right)^{-(l_i+1)}\right)}=0,$$

which can be rewritten as

$$\left(\frac{\lambda T}{r}+1\right)^{m+1}+\sum_{i=1}^{n} \frac{k_i\left(\frac{\alpha_i}{\beta_i}\left(\frac{\lambda T}{r}+1\right)^{m+1}-\frac{\gamma}{\beta_i}\left(\frac{\lambda S_i}{q_i}+1\right)^{l_i+1}\right)}{(\lambda+k_i)\left(\frac{\lambda S_i}{q_i}+1\right)^{l_i+1}-\frac{k_i\alpha_i}{\beta_i}}=0. \tag{38}$$

In the general case numerical methods are needed to locate the eigenvalues. In order to obtain analytic results we will examine some special cases. Thus we consider the symmetric case, when $k_1=\cdots=k_n=k, \alpha_1=\cdots=\alpha_n=\alpha$, $\beta_1=\cdots=\beta_n=\beta$, $l_1=\cdots=l_n=l$ (therefore $q_1=\cdots=q_n=q$), and $S_1=\cdots=S_n=S$. Equation (38) then simplifies to

$$\begin{aligned}(\lambda+k)\left(\frac{\lambda T}{r}+1\right)^{m+1}\left(\frac{\lambda S}{q}+1\right)^{l+1}+(n-1)\frac{k\alpha}{\beta}\left(\frac{\lambda T}{r}+1\right)^{m+1}&\\ -\frac{n\gamma k}{\beta}\left(\frac{\lambda S}{q}+1\right)^{l+1}&=0.\end{aligned} \tag{39}$$

In the most simple case of $m=l=0$, this equation becomes the cubic

$$\begin{aligned}\lambda^3 TS+\lambda^2[T+S+kTS]+\lambda\left[1+k\left(T+S+\frac{(n-1)\alpha}{\beta}T-\frac{n\gamma}{\beta}S\right)\right]&\\ +k\left[1+\frac{(n-1)\alpha}{\beta}-\frac{n\gamma}{\beta}\right]&=0.\end{aligned} \tag{40}$$

The local asymptotic stability of the equilibrium can be again examined by using the Routh-Hurwitz criterion. The details are not given here. We are however interested in the possibility of the birth of limit cycles. In fact, the following result is true:

THEOREM 4. *Consider the output adjustment dynamics* (20) *and assume* $m=l=0$. *If*

$$\gamma=\{(1+k(S+TA))(T+S+kTS)-kAST\}\Big/\left(\frac{knS^2(1+kT)}{\beta}\right)$$

with $A=1+\frac{(n-1)\alpha}{\beta}$, *then a Hopf bifurcation occurs, so that a limit cycle is born around the equilibrium.*

Proof. Assume that $\lambda=ia$ is a pure complex root, then by equating the real and imaginary parts to zero we obtain

$$-a^3TS+a\left[1+k\left(T+S+\frac{(n-1)\alpha}{\beta}T-\frac{n\gamma}{\beta}S\right)\right]=0,$$

and

$$-a^2\left[T+S+kTS\right]+k\left[1+\frac{(n-1)\alpha}{\beta}-\frac{n\gamma}{\beta}\right]=0\,,$$

implying that

$$a^2=\frac{1+k\left(T+S+\frac{(n-1)\alpha}{\beta}T-\frac{n\gamma}{\beta}S\right)}{TS}=\frac{k\left[1+\frac{(n-1)\alpha}{\beta}-\frac{n\gamma}{\beta}\right]}{T+S+kTS}\,. \tag{41}$$

We again select γ as the bifurcation parameter. Then this relation gives a simple expression for the critical value of γ viz.

$$\begin{aligned}\gamma^*=&\left\{\left(1+k\left(T+S+\frac{(n-1)\alpha}{\beta}T\right)\right)(T+S+kTS)\right.\\&\left.-k\left(1+\frac{(n-1)\alpha}{\beta}\right)TS\right\}\Big/\left(\frac{knS^2\left(1+kT\right)}{\beta}\right),\end{aligned} \tag{42}$$

which is always negative, since all negative terms cancel out in the numerator. Simple calculation shows that from equation (41) the value of a^2 becomes positive if and only if

$$(S-T)Ak>1+kS\,. \tag{43}$$

In this case there is a pair of pure complex roots. At the critical value, equation (42) can be rewritten as

$$\lambda^3TS+\lambda^2\left(T+S+kTS\right)+\lambda TS^2a+\left(T+S+kTS\right)a^2=0\,,$$

where we used equation (41). The left hand side can be factored as

$$\left(\lambda TS+\left(T+S+kTS\right)\right)\left(\lambda^2+a^2\right)\,,$$

so the third root is negative.

Differentiating equation (40) implicitly with respect to γ we have

$$\begin{aligned}&3\lambda^2\dot{\lambda}TS+2\lambda\dot{\lambda}\left[T+S+kTS\right]\\&\quad+\dot{\lambda}\left[1+k\left(T+S+\frac{(n-1)\alpha}{\beta}T-\frac{n\gamma}{\beta}S\right)\right]-\lambda\frac{nSk}{\beta}-\frac{kn}{\beta}=0\end{aligned}$$

implying that

$$\dot{\lambda}=\frac{\lambda\frac{nSk}{\beta}+\frac{kn}{\beta}}{3\lambda^2TS+2\lambda\left[T+S+kTS\right]+\left[1+k\left(T+S+\frac{(n-1)\alpha}{\beta}T-\frac{n\gamma}{\beta}S\right)\right]}$$

which at $\lambda=i\alpha$ has the real part

$$Re\left(\dot{\lambda}\right)\big|_{\lambda=i\alpha}=\frac{2a^2\frac{nk}{\beta}S^2\left(1+kT\right)}{\left(2a^2TS\right)^2+\left(2a\left(T+S+kTS\right)\right)^2}\,,$$

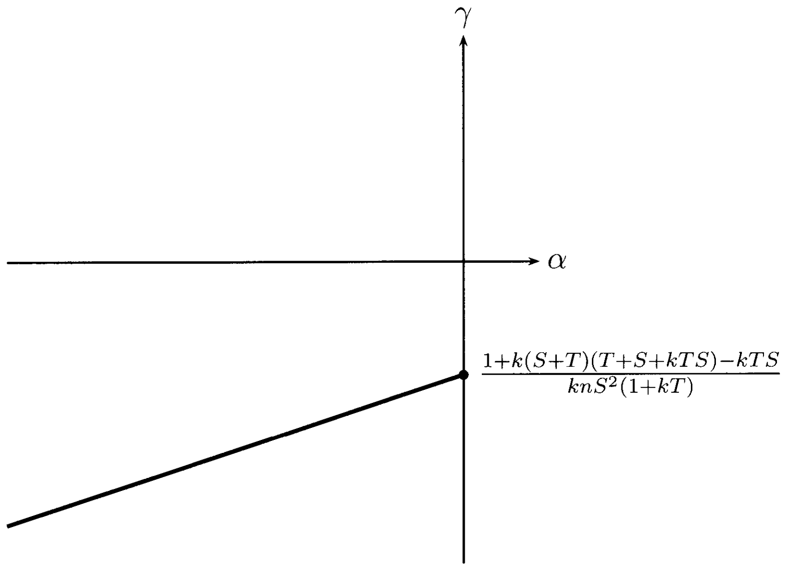

Figure 3. Information lags for both firms and the pollution treatment agency: The critical curve in the (γ, α) plane for the case $m = l = 0$.

which is always negative. Under this condition there is a limit cycle in the neighbourhood of the equilibrium as a consequence of the Hopf bifurcation theorem. ■

Relation (42) is satisfied if the information lag of the firms is much larger than the information lag of the pollution treatment agency. This condition is realistic, if the firms directly report their output to the agency, but not to each other, or the firms treat pollution immediately.

With fixed values of β, n, k, T and S, Figure 3 shows the curve of the critical values of γ as a function of α. It is a linear function with positive slope and negative γ- intercept. Therefore the critical value is always negative as has been already pointed out.

6. CONCLUSIONS

In this paper dynamic oligopolies have been examined in the case when the firms treat pollution directly and share the cost in proportion to their share in the total output. The models discussed here are the dynamic extensions of the static model introduced earlier by Okuguchi and Szidarovszky (2000).

Three particular models were considered. First we proved that in the case of the availability of instantaneous information to all firms as well as to the pollution treatment agency, the equilibrium is always locally asymptotically stable. However instability may occur in the presence of information lags. If instantaneous information is available to the firms, but the pollution treatment agency experiences a time lag in obtaining and implementing information on the total output of the industry, or the unit pollution treatment cost is based on an average of past outputs, then a nonlinear integro-differential equation for the evolution of output is obtained. We have considered two special symmetric cases characterised by the way in which past output information is weighted . In the first case, when the weights decline exponentially, the eigenvalues are real. Conditions were derived for the asymptotic stability of the equilibrium depending on the particular value of the model's parameters. In the second case, when the weighting function rises to maximum and then declines exponentially, in addition to stability conditions the possibility of the birth of limit cycles was demonstrated. A more complicated integro-differential equation for output was obtained when all firms as well as the pollution treatment agency experienced information lags. In the most simple symmetric case, when $m = l = 0$, the possibility of the birth of limit cycle was examined. The conditions for such an outcome require that the information lag of the firms be much longer than the information lag of the pollution treatment agency.

It is known from earlier studies that without information lags and pollution treatment cost sharing the equilibrium is always locally asymptotically stable. The introduction of information lags, in the pollution treatment agency and in the response of the firms may make the equilibrium unstable, and in the case of bifurcation limit cycles may be born. Hence the asymptotic behaviour of the equilibrium becomes much richer than in the instantaneous information case.

In setting up the dynamic output adjustment processes we have assumed that at each time period each firm adjusts its output into the direction of its best response. Alternative models can be obtained by assuming that the firms adjust their outputs proportionally to their marginal profits. The analysis of these alternative output adjustment processes are left to future research.

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