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CONSISTENT STABILITY IN GENERALISED ASSIGNMENT MODELS

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Abstract: We consider a generalised assignment model where the pay-offs, individual, as well as pairwise, depend on the number of matchups that take place. We develop an appropriate notion of stability, called consistent stability, and derive conditions under which existence is ensured. We provide examples to demonstrate that if these conditions are dropped then a consistently stable outcome may not exist.

Key words: Assignment model, stability, externalitics.

1. INTRODUCTION

In this paper we seek to develop an appropriate solution concept for assignment models with externalities.

Assignment models are two-sided matching games where only the agents on the opposite sides of the market, denoted the p- and the q-agents, can, by coming together, create value. Examples of such models include the assignment of worker to firms, the pairing of men and women in marriage, the assignment of students to colleges, to name only a few.

Recently there have been some attempts at generalising various matching models. Sonmez [13], [14], for example, examine a class of generalised matching problems such that both the marriage problem and the housing market emerge as special cases. Li [10] considers matching models with externalities. Another paper is by Chowdhury [6], who examines assignment model with externalities.

In this paper also we examine assignment models with externalities. Such externalities are likely to be of interest whenever the agents, after the completition of the matching process, indulge in some activity where the out-come depends on the earlier matching. As examples we can mention labour, as well as marriage markets. In labour

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markets, for example, the profits of a firm may depend on which worker is hired by the rival firms.

The standard solution concept for assignment games is that of stability, developed by Shapley and Shubik [12]. An outcome is said to be stable if it cannot be blocked by individual or pairwise coalitions. It is clear that in the presence of externalities the notion of stability also needs to be generalised. The problem lies with evaluating the expected deviation payoffs following a break-up. The expected deviation payoffs will depend on what the deviating agents, following a deviation, expect the other agents to do. One approach is to assume that the agents are very pessimistic, so that following any deviation they assume that the outcome is goint to be the worst possible in terms of their payoffs. Chowdhury [5] demonstrates that under this assumption a stable outcome always exists. Despite this attractive technical property, however, it is clear that this assumption is *ad hoc* in nature. In this paper we develop an alternative notion of stability that we call consistent stability (c-stability), where the conjectures are derived endogenously within the model in a consistent manner.

The idea is to use reduced form games to build up the conjectures consistently. Begin with the smallest reduced form games consisting of one agent each from both sides of the market, and assume that the assignment of the other agents are fixed. Extending the notion of stability to this game is straightforward. Next consider a reduced form game consisting of two agents from one side of the market and one agent from the other side. We solve for the stable outcome of this game when the deviation conjectures are such that they are consistent with outcomes in the smaller reduced form games solved earlier. We can proceed inductively in this manner to find a consistent set of conjectures for the whole game. Any stable outcome where the conjectures are consistent in the above sense, is said to be consistently stable.

We provide sufficient conditions under which a symmetric game would have a consistently stable outcome. We also provide two examples to demonstrate that if the sufficient conditions are dropped then a c-stable outcome may not exist. It appears that a consistently stable outcome is more likely to exist if the externalities are of the increasing returns type, rather than if they are of the decreasing returns type.

We then relate our paper to the existing literature on the subject. Chowdhury [6] is another paper that examines an assignment model with externalities. He sets up a two-stage non-cooperative game and examines sub-game perfect equilibria of this game. Thus, in contrast to the present paper, Chowdhury [6] adopts a strictly non-cooperative approach to the problem. Li [10] considers a matching model where he defines a rational expectations equilibrium, such that the expectations are consistent with the equilibrium. He shows that under the appropriate assumptions existence is ensured. He, however, analyses a matching, rather than an assignment model, the focus of our interest in this paper. Finally, Sonmez [13], [14] is not concerned with developing equilibrium notions. Rather he focuses on the issue of strategy-proofness in the class of models defined by him.

In the next section we set up the basic model. The notion of consistent stability is analysed in section 3.

2. THE MODEL

We examine a two-sided assignment model with one-to-one matching. There are two disjoint set of agents, denoted P and Q, consisting of n agents each. Members of P and Q are called the p- and the q-agents respectively. Following the practice in Roth and Sotomayor [11], we reserve the symbol i for agents belonging to P, and the symbol j for agents belonging to Q. The pairwise payoff from matching a p-agent, i, to a q-agent, j, when the total number of matchups equals k, is denoted by $x_{ij}(k)$. Let $X(k)$ denote the matrix whose typical element is $x_{ij}(k)$. The payoff from remaining single is $p_i(k)$ for a p-agent. i, and $q_j(k)$ for a q-agent, j, where, as before, k denotes the number of matchups. We assume that $x_{ij}(k)$, $p_i(k)$ and $q_j(k)$ are non-negative.

We now introduce a series of definitions, mostly adapted from Roth and Sotomayor [11], that we require for the analysis. We begin by providing a formal definition of an assignment.

DEFINITION. An assignment σ is an one-to-one correspondence from the set $P \cup Q$ onto itself of order two such that if, for some $i \in P$, $\sigma(i) \neq i$, then $\sigma(i) \in Q$ and if, for some $j \in Q$, $\sigma(j) \neq j$, then $\sigma(j) \in P$. An assignment σ that involves k pairs of matchups is denoted by σ_k.

The fact that σ is of order two, implies that if i is matched to j, then j is matched to i. The definition also requires that agents who are not single, be matched to agents from the opposite set.

We next define an optimal assignment for a matrix X, whose typical elements is x_{ij}. Consider the following linear programming problem:

$$\text{Maximise} \quad \sum_{i,j} \alpha_{ij} x_{ij}$$

$$\text{Subject to} \quad (a)\ \sum_i \alpha_{ij} \leq 1\,, \quad (b)\ \sum_j \alpha_{ij} \leq 1\,, \quad (c)\ \alpha_{ij} \leq 0\,. \tag{1}$$

It can be shown that there is a solution, σ, of the above problem that only involves values of zeros and ones (see Dantzig [8], pp. 318). Any such assignment, σ, is called an optimal assignment of X.

DEFINITION. A vector (u, v), with $(u, v) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$, is called a feasible vector for a matrix X, if there is an assignment σ_k such that

$$\sum_{i \in P} u_i + \sum_{j \in Q} v_j = X_{\sigma_k}\,, \tag{2}$$

where X_{σ_k} denotes the aggregate payoff arising from the assignment σ_k. In that case $((u, v), \sigma_k)$ is called a feasible outcome.

In the sequel we will often use the notation $((\bar{u}, u, \bar{v}, v), \sigma_k)$ instead of $((u, v), \sigma_k)$ to denote an outcome. Here $\bar{u}_i$ denotes the payoff of a p-agent who is matched under the assignment and u_i denotes the payoff of a p-agent who is not. The definitions of $\bar{v}_j$ and v_j are symmetric and hence omitted.

3. CONSISTENT STABILITY

With the basic notations and definitions in place, we now proceed to the task of extending the notion of stability to the generalised framework.

As in Gale and Shapley [9] we concentrate on notions of stability that respect the importance of pairwise coalitions in assignment models. (Pairwise coalitions, in this paper, always refer to agent pairs where the two agents belong to the opposite sides of the market.) We, therefore, restrict attention to individual and pairwise deviations, rather than consider deviations by all possible coalitions. The important question is to what extent is it possible for the larger coalitions to make binding commitments. In some cases there may be legal restrictions on the formation of large coalitions e.g. in the context of a model of technology transfer such coalitions may violate anti-trust legislations. Alternatively, we can assume that coordination across relatively larger coalitions is prohibitively costly in terms of search or negotiation costs, or perhaps even ruled out by the prevalent social norms.

Clearly, the problem in extending the notion of stability to this generalised model is to define, in an intuitively acceptable manner, exactly how the players perceive their possibilities following a deviation. Depending on the perception of the players, we obtain different notions of stability. We seek to develop a notion that we call consistent stability, where these conjectures are endogenously developed within the framework of the model. We then use this notion to analyse some simple examples with symmetric payoffs.

The basic idea is to use the notion of reduced form games to build up the conjectures inductively. To begin with, consider a reduced form game consisting of one p- and one q-agent. Assume that the assignment of the other agents are given and involves $\tilde{k}$ matchings. Given $\tilde{k}$, we can examine the stable outcome (in the sense of Shapley and Shubik [12]) for this reduced form game. Since this reduced form game involves no externalities, the definition of stability in this context is unambiguous.

DEFINITION. Let $(\tilde{k}, p, q)$ denote a reduced form game consisting of p p-agents and q q-agents, where the number of matchups by the other agents (who are not members of the reduced form game) is $\tilde{k}$.

We can next examine the stable outcome for a game consisting of two p-agents and one q-agent. Now, however, externality issues crop up. In particular consider an outcome that involves exactly one matchup. Suppose the assigned p-agent wants to decide whether to stick to the outcome, or to deviate and remain single. The question is what is his reservation payoff from remaining single? Consider the reduced form game consisting of the remaining one p- and one q-agent. The stable outcome for this reduced form game is known. Hence the p-agent can calculate the expected number of matchups if he deviates. The p-agent can now use this information to find out his reservation payoff and make his decision accordingly.

We then examine a reduced form game consisting of two p- and two q-agents. We look for a stable outcome of this game such that the conjectures are consistent with the stable outcomes for the smaller reduced form games examine earlier. Proceeding inductively in this manner we can derive a consistent set of conjectures for the original game. We then seek a stable outcome where the conjectures are consistent in the above sense. Such an outcome, if it exists, is called consistently stable.

Of course, the idea of imposing some kind of consistency condition while defining equilibrium notions is not new in the literature. Game theorists have used this idea in many different contexts. We can mention, among other equilibrium notions too numerous to mention, the concept of coalition-proof Nash equilibria developed by Bernheim, Peleg and Whinston [3], [4].

Next turning to the formal analysis, we simplify by assuming that the payoffs are symmetric, so that $x_{ij}(k) = x(k)$, $p_i(k) = p(k)$ and $q_j(k) = q(k)$, $\forall i, j$ and k. The symmetry assumption is not required for the definition of consistent stability, though it does help in simplifying the notation. However, since we use this concept to analyse games with symmetric payoffs, we impose this condition at the outset. We then introduce a series of notations and definitions that we require for the analysis.

Note that if $((\bar{u}, u, \bar{v}, v), \sigma_k)$ is feasible then for the symmetric case we have that

$$\sum u_i + \sum \bar{u}_i + \sum v_j + \sum \bar{v}_j = kx(k) + (n-k)p(k) + (n-k)q(k)\,.$$

Thus the payoffs of the agents exactly exhaust the aggregate output.

Let P' and Q' be subsets of P and Q respectively. Let μ denote an assignment for the reduced form game consisting of agents belonging to P' and Q'. Similarly, an outcome for this reduced form game is denoted by $((\bar{u}, u, \bar{v}, v), \mu_k)$. We begin by defining the stable outcome for two agent reduced form games.

First consider a reduced form game consisting of exactly one p-agent and one q-agent. Since the reduced form game has exactly one p-agent and one q-agent, consistency is not a problem in this case.

We first provide the definition before interpreting it.

DEFINITION. Consider a reduced form game such that $|P'| = |Q'| = 1$. Assume that the other agents have formed $\tilde{k}$ matchings, where $\tilde{k} \leq n-1$. An outcome $((\bar{u}, u, \bar{v}, v), \mu_{k'(\tilde{k},1,1)})$ is said to be c-stable for $(\tilde{k}, 1, 1)$ if the assignment involves $k'(\tilde{k}, 1, 1)$ matchups and the payoffs satisfy:

(i) $\bar{u}_i + \bar{v}_j \geq x(\tilde{k} + k'(\tilde{k}, 1, 1))$, for $\mu(i) = j$,
(ii) $u_i + v_j \geq x(\tilde{k} + 1)$,
(iii) $\bar{u}_i \geq p(\tilde{k})$, $\bar{v}_j \geq q(\tilde{k})$,
(iv) $u_i \geq p(\tilde{k})$, $u_j \geq q(\tilde{k})$.

Note that the second and third arguments of $k'(\tilde{k}, \cdot, \cdot)$ and c-stability for $(\tilde{k}, \cdot, \cdot,)$ denote, respectively, the number of p- and q-agents in the reduced form game under consideration.

This definition is a straightforward extension of the notion of stability. Suppose $k'(\tilde{k}, 1, 1) = 1$. Then, in the reduced form game, the p-agent and the q-agent matchup, the total number of matchups is $\tilde{k}+1$ and their payoffs will be denoted by $\bar{u}_i$ and $\bar{v}_j$ respectively. Conditions (i) and (iii) are the natural stability conditions for this case.

Next consider the case where $k'(\tilde{k}, 1, 1) = 0$. Then, in the reduced form game, the p-agent and the q-agent remain single, the total number of matchups is $\tilde{k}$ and their payoffs will be denoted by u_i and v_j respectively. Conditions (ii) and (iv) provide the stability conditions for this case. Condition (ii), for example, simply states that the sum of payoffs for the two (unassigned) agents must be at least equal to what they would obtain from matching up.

We then consider a reduced form game consisting of two p-agents and one q-agent. This leads us to a definition of $k'(\tilde{k}, 2, 1)$.

DEFINITION. Suppose that $|P'| = 2$, $|Q'| = 1$ and let $\tilde{k}$ be defined as usual. An outcome $((\bar{u}, u, \bar{v}, v), \mu_{k'(\tilde{k},2,1)})$ is said to be c-stable for $(\tilde{k}, 2, 1)$ if the assignment involves $k'(\tilde{k}, 2, 1)$ matchups and the payoffs satisfy:

(i) $\bar{u}_i + \bar{v}_j \geq x(\tilde{k} + k'(\tilde{k}, 2, 1))$, for $\mu(i) = j$,

(ii) $u_i + v_j \geq x(\tilde{k} + 1)$,

(iii) $u_i + \bar{v}_j \geq x(\tilde{k} + 1)$,

(iv) $\bar{u}_i \geq p(\tilde{k} + k'(\tilde{k}, 1, 1))$, $\bar{v}_j \geq q(\tilde{k})$,

(v) $u_i \geq p(\tilde{k} + k'(\tilde{k}, 2, 1))$, $v_j \geq q(\tilde{k} + k'(\tilde{k}, 2, 1))$.

Again there ae two cases to consider. First, when $k'(\tilde{k}, 2, 1) = 1$ and second when $k'(\tilde{k}, 2, 1) = 0$. Conditions (i), (iii) and (iv) are concerned with the first case, while conditions (ii) and (v) are concerned with the second.

In this case, however, consistency issues crop up. Consider the case where one of the p-agents decides to remain single. In that case what is the reservation payoff of this p-agent? Note that the reduced form game now consists of exactly one p-agent and one q-agent. The numbeer of match-ups in this reduced form game is exactly $k'(\tilde{k}, 1, 1)$. We can now appeal to the previous definition to solve for $k'(\tilde{k}, 1, 1)$. Thus the total number of matchups should be $\tilde{k} + k'(\tilde{k}, 1, 1)$ and the reservation payoff of the p-agent should be $q(\tilde{k} + k'(\tilde{k}, 1, 1))$. This is formalised in the first part of condition (iv), which states that the reservation payoff of an assigned p-agent, in case he decides to remain single, is $p(\tilde{k} + k'(\tilde{k}, 1, 1))$. Thus he must obtain at least this much if he is not going to deviate. Furthermore, observe that $k'(\tilde{k}, 1, 1)$ may be non-unique. In that case we assume that any arbitrary selection from the set of possible values of $k'(\tilde{k}, 1, 1)$ is equally admissible. However, since we do not want to complicate the notations any further, we shall use $k'(\tilde{k}, 1, 1)$ to denote this selection as well.

The interpretations of the other conditions do not involve any consistency issues. Hence we do not elaborate on them.

Clearly, $k'(\tilde{k}, 1, 2)$ can be defined analogously. We can then use $k'(\tilde{k}, 1, 1)$, $k'(\tilde{k}, 1, 2)$ and $k'(\tilde{k}, 2, 1)$, to define $k'(\tilde{k}, 2, 2)$. The definition for $k'(\tilde{k}, p, q)$ for general p and q can now be constructed inductively.

DEFINITION. Suppose that $|P'| = p$, $|Q'| = q$ and let $\tilde{k}$ be defined as usual. An outcome $((\bar{u}, u, \bar{v}, v), \mu_{k'(\tilde{k},p,q)})$ is said to be c-stable for $(\tilde{k}, p, q)$ if the assignment involves $k'(\tilde{k}, p, q)$ matchups and the payoffs satisfy:

(i) $\bar{u}_i + \bar{v}_j \geq x(\tilde{k} + k'(\tilde{k}, p, q))$, for $\mu(i) = j$,
(ii) $\bar{u}_i + \bar{v}_j \geq x(\tilde{k} + 1 + k'(\tilde{k} + 1, p - 1, q - 1))$, for $\mu(i) \neq j$,
(iii) $u_i + v_j \geq x(\tilde{k} + 1 + k'(\tilde{k} + 1, p - 1, q - 1))$,
(iv) $u_i + \bar{v}_j \geq x(\tilde{k} + 1 + k'(\tilde{k} + 1, p - 1, q - 1))$,
(v) $\bar{u}_i + v_j \geq x(\tilde{k} + 1 + k'(\tilde{k} + 1, p - 1, q - 1))$,
(vi) $\bar{u}_i \geq p(\tilde{k} + k'(\tilde{k}, p - 1, q))$, $\bar{v}_j \geq q(\tilde{k} + k'(\tilde{k}, p, q - 1))$,
(vii) $u_i \geq p(\tilde{k} + k'(\tilde{k}, p, q))$, $v_j \geq q(\tilde{k} + k'(\tilde{k}, p, q))$.

In order to make this definition more transparent let us re-state some of the above conditions.

Consider condition (i). Suppose that under the outcome $((\bar{u}, u, \bar{v}, v), \mu_{k'(\tilde{k},p,q)})$ agent p_i is matched to agent q_j in the reduced game. Then the sum total of their assigned payoffs must be at least as much as what the pair obtains under this assignment, which is $x(\tilde{k} + k'(\tilde{k}, p, q))$.

Next consider condition (ii). Suppose that under the assignment in the reduced form game agent p_i and agent q_j are both matched, but not to each other. Then, for the outcome to be stable, the sum total of their assigned payoffs must be at least as much as what the pair obtains if they decide to deviate and matchup with each other. Note that in that case the total number of matchups is $\tilde{k} + 1$ plus $k'(\tilde{k} + 1, p - 1, q - 1)$, where the last term is the number of matchups in the reduced form game $(\tilde{k} + 1, p - 1, q - 1)$. Thus the deviation payoff of this pair is $x(\tilde{k} + 1 + k'(\tilde{k} + 1, p - 1, q - 1))$. Hence condition (ii) follows.

Next consider condition (iii). Suppose an unassigned p-agent and an unassigned q-agent decide to matchup, what is the joint payoff that they can expect? Following their matchup, the remaining agents in the reduced form game take $(\tilde{k} + 1)$ matchups as given. Hence in the reduced form game consisting of $(p - 1)$ p-agents and $(q - 1)$ q-agents, the number of matchups is going to be $k'(\tilde{k} + 1, p - 1, q - 1)$. Thus the total expected number of matchups following the deviation is $\tilde{k} + 1 + k'(\tilde{k} + 1, p - 1, q - 1)$. Hence the expected payoff is $x(\tilde{k} + 1 + k'(\tilde{k} + 1, p - 1, q - 1))$.

Conditions (iv) and (v) can be interpreted similarly.

Next consider the first part of condition (vi). What is the reservation payoff of a p-agent who is assigned? Given that the p-agent has decided to remain unassigned the reduced form game is now of the form $(\tilde{k}, p - 1, q)$. Hence the reservation payoff of the p-agent is $p(\tilde{k} + k'(\tilde{k}, p - 1, q))$. For stability, the payoff of the p-agent must at least equal this.

The rest of the conditions can be interpreted similarly.

We are finally in a position to define the notion of consistent stability. Notice that the deviation payoffs are derived by using conjectures that are consistent in our sense.

DEFINITION. An outcome $((\bar{u}, u, \bar{v}, v), \sigma_k)$ is said to be c-stable if:

(i) $\bar{u}_i + \bar{v}_j \geq x(k)$, for $\sigma(i) = j$,

(ii) $\bar{u}_i + \bar{v}_j \geq x(1 + k'(1, n-1, n-1))$, for $\sigma(i) \neq j$,

(iii) $u_i + v_j \geq x(1 + k'(1, n-1, n-1))$,

(iv) $u_i + \bar{v}_j \geq x(1 + k'(1, n-1, n-1))$,

(v) $\bar{u}_i + v_j \geq x(1 + k'(1, n-1, n-1))$,

(vi) $\bar{u}_i \geq p(k'(0, n-1, n))$, $\bar{v}_j \geq q(k'(0, n, n-1))$,

(vii) $u_i \geq p(k)$, $v_j \geq q(k)$.

We now interpret these conditions.

Condition (i) is entirely standard.

Next consider condition (ii). Suppose that under the outcome $((\bar{u}, u, \bar{v}, v), \sigma_k)$ agent p_i and agent q_j are both matched, but not to each other. Then, for the outcome to be stable, the sum total of their assigned payoffs must be at least as much as what the pair would obtain if they decide to deviate and matchup with each other. Note that in that case the total number of matchups would be $1+k'(1, n-1, n-1)$, where the last term is the number of matchups in the reduced form game $(1, n-1, n-1)$. Thus the deviation payoff of this pair is $x(1 + k'(1, n-1, n-1))$. Hence condition (ii) follows.

Next consider condition (iii). Suppose an unassigned p-agent and an unassigned q-agent decide to matchup, what is the joint payoff that they can expect? Following their matchup, the remaining agents in the reduced form game take 1 matchup as given. Hence in the reduced form game consisting of $(n-1)$ p-agents and $(n-1)$ q-agents, the number of matchups is going to be $k'(1, n-1, n-1)$. Thus the total expected number of matchups following the deviation is $1 + k'(1, n-1, n-1)$. Hence the expected payoff for this deviating pair of agents is $x(1 + k'(1, n-1, n-1))$.

Conditions (iv) and (v) carry similar interpretations.

Next consider the first part of condition (vi). What is the reservation payoff of a p-agent who is assigned? Given that the p-agent has decided to remain unassigned the reduced form game is now of the form $(0, n-1, n)$. Hence the reservation payoff of the p-agent is $p(k'(0, n-1, n))$. For stability, the payoff of the p-agent must at least equal this.

The rest of the conditions can be interpreted similarly.

It is worthwhile to try and understand the conceptual basis of these conditions. Consider, for example, condition (iii). The essential point is that we are considering a situation where none of the agents have as yet committed to the prescribed outcome $((\bar{u}, u, \bar{v}, v), \sigma_k)$. Consider the decision problem facing a p-agent and a q-agent, who are unassigned under the prescribed outcome. If they decide to remain unassigned, then, since they think that the prescribed outcome is c-stable, they expect that the others will also follow this outcome. However, if they decide to matchup, then the situation changes drastically. Now the prescribed outcome $((\bar{u}, u, \bar{v}, v), \sigma_k)$ is no longer relevant. The other agents are faced with a situation where one p-agent and one q-agent has already matched up, so that the other agents face a reduced form game of the form $(1, n-1, n-1)$. The number of matchups in this reduced form game is given by $k'(1, n-1, n-1)$. That is why the reservation payoff of this pair is $x(1+k'(1, n-1, n-1))$. Moreover, there is no necessary relation between the matchups in this reduced form game, and those originally prescribed. It is conceptually possible that agents who were matched up in the original matchups remain single in the reduced form game.

We then apply the notion of c-stability to solve the symmetric assignment game with externalities. We begin by introducing some more definitions:

DEFINITION. $f(k) = x(k) - p(k-1) - q(k-1), \forall k \in \{1, 2, \cdots, n\}$.

$f(k)$ can be interpreted as the incentive for a single p-agent and a single q-agent to matchup if there are $k-1$ existing matchups and, following this matchup, no additional matchups are expected to take place.

DEFINITION. $g(k) = x(k) - p(k) - q(k)$, $\forall k \in \{1, 2, \cdots, n-1\}$ and $g(n) = x(n) - p(n-1) - q(n-1)$.

Thus for $k \in \{1, 2, \cdots, n-1\}$, $g(k)$ can be interpreted as the incentive for a single p-agent and a single q-agent to matchup if the number of matchups is expected to be k irrespective of whether they matchup or not. $g(n)$ carries the same interpretation as $f(n)$.

DEFINITION. k_j (respectively k_g) be the minimum integer such that $f(k)$ (respectively $g(k)$) is non-negative.

k_f can be interpreted as follows. Consider a hypothetical situation where the agents are naive in the sense that if there are k existing matchups and if a single p-agent and a single q-agent matches up, then the expect that the ultimate number of matchups is going to be exactly $k+1$.[1] In such a situation any assignment must involve at least k_f matchups. This is because for any assignment with less than k_f matchups, it is the case that $f(k) < 0$, so that such an assignment cannot be stable.

k_g can be given a related interpretation. Consider another hypothetical situation where the agents are naive in the sense that if there are k $(< n)$ existing matchups and if a single p-agent and a single q-agent matches up, then they expect that the ultimate number of matchups is going to be exactly k. In such a situation any assignment must involve at least k_g matchups. Clearly, for any assignment with less than k_g matchups, $g(k) < 0$. Hence such an assignment cannot be stable.

Next we introduce an assumption that we require in the proof of Proposition 1 below.

ASSUMPTION (I).

(i) $p(k), q(k), f(k)$ and $g(k)$ are all (weakly) increasing in k.

(ii) $k_f = k_g = \bar{k}$.

Let us try to interpret assumption I.

[1] Of course, in our model the agents are much more sophisticated. The alternative assumption is just for expositional purposes.

First consider assumption I(i). The conditions that $p(k)$ and $q(k)$ are increasig in k essentially states that the payoffs of the unassigned agents exhibit increasing returns to scale in the numbers of matchups. While the conditions that $f(k)$ and $g(k)$ are increasing in k ensure that, in some sense, the incentive to form matchups are increasing in k. So that under these conditions there would be a tendency for consistently stable outcomes to involve n matchups.

Next consider assumption I(ii). Recall that k_f and k_g can be interpreted as the minimum number of matchups for two different kinds of conjectures by the agents regarding the outcome following a matchup. Thus the condition that $k_g = k_f$ is essentially a consistency condition that irrespective of the conjectures the minimum number of matchups is the same.

The proof of the following proposition can be found in the appendix.

PROPOSITION 1. *Under assumption I a c-stable outcome exists.*

(i) *If* $\bar{k} \leq n-1$, *then all outcomes such that there are n matchups,* $\bar{u}_i + \bar{v}_j = x(n)$, $\bar{u}_i \geq p(n-1)$ *and* $\bar{v}_j \geq p(n-1)$, *are c-stable.*

(ii) *If* $\bar{k} = n$, *then all outcomes such that there are n matchups,* $\bar{u}_i + \bar{v}_j = x(n)$, $\bar{u}_i \geq p(0)$ *and* $\bar{v}_j \geq p(0)$, *are c-stable.*

Thus in the presence of increasing returns to scale a c-stable outcome is likely to exist. The proof is constructive.

Let us take a look at the c-stable outcomes described in Proposition 1. Note that all are Pareto optimal in the sense that they involve exactly n matchups. This, of course, is expected given our assumption that the incentive to form matchups is increasing in the number of matchups, i.e. $f(k)$ and $g(k)$ are increasing in k. There are two cases. First consider the case where $\bar{k} \leq n-1$. In this case if an agent decides to remain single, then, in the reduced form game, there are going to be $n-1$ matchups. Hence we find that $\bar{u}_i \geq p(n-1)$ and $\bar{v}_j \geq p(n-1)$. Next consider the case where $\bar{k} = n$. If an agent decides to remain single, then in the reduced form game all the agents are going to remain unassigned. Hence in this case $\bar{u}_i \geq p(0)$ and $\bar{v}_j \geq p(0)$.

Why do we require that both $f(k)$ and $g(k)$ be increasing? In the actual proof the consistent conjectures are constructed inductively starting from the smallest reduced form games. The condition that $f(k)$ be increasing is required to construct the conjectures in reduced form games with an equal number of p-agents and q-agents. Whereas for reduced form games with an unequal number of p-agents and q-agents, we require the condition that $g(k)$ be increasing. As we mentioned before, assumption I(ii) is essentially a consistency condition. Formally, what it does is to ensure that reduced form games with an unequal number of p-agents and q-agents have a solution. For such reduced form games we need that $f(k)$ and $g(k)$ should have the same sings for all k. However, that is possible when $k_f = k_g$, not otherwise.[2]

[2] Example 2 later illustrates some of the possible problems when $k_f \neq k_g$.

We then provide two examples to show that if we drop either one of the two conditions in assumption I, then a c-stable outcome may not exist.

EXAMPLE 1. We first consider an example where $n = 2$. $x(k)$, $p(k)$ and $q(k)$ are all strictly decreasing in k, and $x(2) - p(1) - q(1) < 0 < x(1) - p(0) - q(0)$. Clearly, assmption I(i) fails to hold. Notice, however, that $k_f = k_g = 1$, thus assumption I(ii) holds.

We then examine if this game has a c-stable outcome. To begin with it is easy to see that $k'(0, 1, 1) = 1$ and $k'(1, 1, 1) = 0$. We then claim that $k'(0, 2, 1) = 1$. Consider an outcome that involves one matchup, $u_i = \bar{u}_i = p(1)$ and $\bar{v}_j = x(1) - p(1)$. We then check that $\bar{v}_j \geq q(0)$ i.e. $x(1) - p(1) - q(0) \geq 0$. This follows since $x(1) - p(0) - q(0) > 0$ and $p(0) > p(1)$. (We can similarly argue that $k'(0, 1, 2) = 1$.)

Finally let us examine $k'(0, 2, 2)$. There are three different cases to consider.

Case (i). We first examine if $l'(0, 2, 2)$ can equal zero. In that case $u_i \geq p(0)$ and $v_j \geq q(0)$, $\forall i, j$. In fact we can argue, following Roth and Sotomayor ([11], lemma 8.5)), that $u_i = p(0)$ and $v_j = q(0)$, $\forall i, j$. Since $k'(1, 1, 1) = 0$, the c-stability condition is that $u_i + v_j \geq x(1)$. This in turn implies that $p(0) + q(0) \geq x(1)$, which is a contradiction.

Case (ii). We then check if $k'(0, 2, 2)$ can take a value of one. The payoffs in this case must satisfy $u_i = p(1)$, $v_j = q(1)$ and $\bar{u}_i + \bar{v}_j = x(1)$. Since $k'(\tilde{k}, 1, 1) = 0$, the c-stability condition is that $u_i + v_j \geq x(1)$, which in turn implies that $x(1) - p(1) - q(1) \leq 0$. But this violates the condition that $x(1) - p(0) - q(0) > 0$ (since $p(k)$ and $q(k)$ are decreasing in k).

Case (iii). Finally we examine if $k'(0, 2, 2)$ can assume a value of two. In that case the payoffs must involve $\bar{u}_i + \bar{v}_j = x(2)$ whenever $\sigma(i) = j$. If $\sigma(i) \neq j$, then the c-stability condition yields that $\bar{u}_i + \bar{v}_j \geq x(1)$. Summing up we obtain $2x(2) = \sum u_i + \sum v_j \geq 2x(1)$ i.e. $x(2) \geq x(1)$. This, however, is a contradiction.

Thus in the presence of decreasing returns to scale a c-stable outcome may not exist.

EXAMPLE 2. We then provide an example where assumption I(i) holds, but I(ii) fails. Assume that $n = 2$, and $x(k)$, $p(k)$ and $q(k)$ are increasing in k. Furthermore assume that $x(2) - p(1) \geq x(1) - p(0) - q(0) > 0$ and $x(1) - p(1) - q(0) < 0$. Observe that under these assumptions $k_f = 1 < 2 = k_g$. (This follows because if $k_g = 1$, then this would imply that $x(1) - p(1) - q(0) \geq x(1) - q(1) \geq 0$, which is a contradiction.)

We then argue that we cannot find a set of consistent conjectures for this game.

To begin with, it is obvious that $k'(\tilde{k}, 1, 1) = 1$, $\forall \tilde{k} \in \{0, 1\}$. We then examine if $k'(0, 2, 1)$ is defined. It is easy to see that $k'(0, 2, 1)$ cannot equal zero. If it does, then the payoffs must involve $u_i = p(0)$ and $v_j = q(0)$, $\forall i, j$. But in that case one p- and one q-agent can matchup. Since $k'(1, 10) = k'(1, 0, 1) = 0$, their joint payoff is going to be $x(1)$, which is greater than the sum of their payoffs $p(0) + q(0)$. (This follows since $x(1) - p(0) - q(0) > 0$.)

Finally, we examine if $k'(0, 2, 1)$ can take a value of one. Suppose that it can. In that case the payoffs must satisfy $u_i = p(1)$, $\bar{u}_i + \bar{v}_j = x(1)$, $\bar{u}_i \geq p(1)$ and $\bar{v}_j \geq (0)$.

But the last three conditions together imply that $x(1) - p(1) - q(0) \geq 0$, which is a contradiction.

Thus $k'(0, 2, 1)$ is not defined, and a c-stable outcome does not exist.

We then demonstrate that c-stable outcomes do not necessarily belong to the core. Consider an example where $n = 2$, $x(2) = 3$, $x(1) = 5$, $p(1) = q(1) = 1$, and $p(0) = q(0) = 0$. Notice that assumption (I) is satisfied and thus, from Proposition 1, there is a c-stable outcome that involves two matchups, $\bar{u}_i = 1$ and $\bar{v}_j = 2$. $\forall i, j$. Observe, however, that

$$x(1) + p(1) + q(1) = 7 > 2x(2) = 6 > 2p(0) + 2q(0) = 0\,.$$

Thus the grand coalition can deviate and make a gain. Hence the given outcome is c-stable, but not in the core. (This also implies that c-stable outcomes need not be coalition proof Nash equilibrium.)

This is interesting because the sufficient condition given by Li [10] for the existence of a rational expectations equilibrium, that the problem have 'weak externalities', leads, in a matching context, to a core outcome. Thus in some sense the condition provided by Li [10] is too strong. In contrast assumption (I) does allow for non-core outcomes. Of course, assumption (I) is for symmetric games only, while Li [10] allows for non-symmetric matching games also.

Finally, recall that Chowdhury [6] also examines assignment models with externalities and defines a non-cooperative solution concept, that of a 'perfect' equilibrium. It is natural to examine if the notion of 'perfect' equilibrium is related to that of c-stability developed in this paper.

We first claim that a c-stable outcome need not be 'perfect'. Consider a standard assignment game with no externalities. It is clear that the set of stable and c-stable outcomes coincide. From Proposition 5 in Chowdhury [6], however, it follows that only the p-optimal outcome can be sustained as a 'perfect' equilibrium. Thus the other c-stable outcomes are not 'perfect'.

We then argue that the converse is also true, i.e. a 'perfect' equilibrium need not be c-stable. Consider the following symmetric game where $x(2) = 5$, $x(1) = 10$, $p(1) = q(1) = 3$ and $p(0) = q(0) = 4$. Observe that this game satisfies the conditions given in example 1 above. Thus this game has no c-stable outcome. Next consider the following strategies. Both p_1 and p_2 offers the vector (3, 7) to q_1. They make no other offers, q_1 accepts the offer made by p_1, say. It is easy to see that these strategies are 'perfect'. (See Chowdhury [6], pp. 347, for the definition of 'perfect'ness.)

Thus the set of c-stable outcomes and that of 'perfect' equilibrium are not nested.

To summarise, in this paper we study a natural extension of the standard assignment model and formulate an appropriate equilibrium concept for this class of models. We also establish an useful existence result. We then provide some examples to demonstrate that if the sufficient conditions are violated, then a c-stable outcome may not exist.

4. APPENDIX

PROOF OF PROPOSITION 1. We first argue that if $\tilde{k} \geq \max\{\bar{k}-1,0\}$, then $k'(\tilde{k},1,1) = 1$. Otherwise, it takes a value of zero. If $\tilde{k} \geq \max\{\bar{k}-1,0\}$ then the outcome that involves one matchup, $\bar{u}_i = p(\tilde{k})$ and $\bar{v}_j = x(\tilde{k}+1) - p(\tilde{k})$ is c-stable for $(\tilde{k},1,1)$. If, however, $\tilde{k} < \max\{\bar{k}-1,0\}$, then the outcome that involves no matchups, $u_i = p(0)$ and $v_j = q(0)$ is c-stable for $(\tilde{k},1,1)$. The proof is obvious and follows the fact that $f(k) \geq 0$, if and only if $k \geq \bar{k}$.

We then argue that if $\tilde{k} \geq \max\{\bar{k}-1,0\}$, then $k'(\tilde{k},2,1)$ equals one. Otherwise, it takes a value of zero. To begin with consider the case where $\tilde{k} < \max\{\bar{k}-1,0\}$. Consider the outcome where all the agents remain unassigned and the payoffs inolve $u_i = p(\tilde{k})$ and $v_j = q(\tilde{k})$, $\forall i,j$. This outcome is c-stable for $(\tilde{k},2,1)$ for the reduced form game since $u_i + v_j = p(\tilde{k}) + q(\tilde{k}) < x(\tilde{k}+1)$.

We then examine the case where $\tilde{k} \geq \max\{\bar{k}-1,0\}$. Consider the outcome where $u_i = p(\tilde{k}+1) = \bar{u}_i$ and $\bar{v}_j = x(\tilde{k}+1) - p(\tilde{k}+1)$. Clearly the c-stability condition for the reduced form game $(\tilde{k},2,1)$ is that $\bar{v}_j \geq q(\tilde{k})$, i.e. $x(\tilde{k}+1) - p(\tilde{k}+1) - q(\tilde{k}) \geq 0$. (This follows since $g(\tilde{k}+1) \geq 0$ and $q(k)$ is increasing in k.)

It is clear that $k'(\tilde{k},1,2)$ has the same form as $k'(\tilde{k},2,1)$.

We then consider the general case where $p \geq q$. We demonstrate that $k'(\tilde{k},p,q)$ equals q if $\tilde{k} \geq \max\{\bar{k}-q,0\}$. Otherwise it assumes a value of zero.

The proof is through induction.

Inductive hypothesis: Let p' and q' be such that $p' \leq p$ and $q' \leq q$, and at least one of the inequalities is strict. Then $k'(\tilde{k},p',q')$ equals $\min\{p',q'\}$, if $\tilde{k} \geq \max\{\bar{k} - \min\{p',q'\},\}$. Otherwise it assumes a value of zero.

We first consider the case where $p > q$ and $\tilde{k} = \max\{\bar{k}-1,0\}$. Consider the outcome that involves q matchups, $u_i = p(\tilde{k}+q) = \bar{u}_i$ and $\bar{v}_j = x(\tilde{k}+q) - p(\tilde{k}+q)$. Notice that if a q-agent decides to remain single then, from the induction hypothesis, $k'(\tilde{k},p,q-1) = 0$. The c-stability condition for $(\tilde{k},p,q)$ therefore is that $\bar{v}_j \geq q(\tilde{k})$ i.e. $x(\tilde{k}+q) - p(\tilde{k}+q) - q(\tilde{k}) \geq 0$. This follows since $g(\tilde{k}+q) \geq 0$ and $q(k)$ is increasing in k.

We then consider the case where $p > q$ and $\tilde{k} > \max\{\bar{k}-q,0\}$. Consider an outcome that involves q matchings and $u_i = p(\tilde{k}+q) = \bar{u}_i$ and $\bar{v}_j = x(\tilde{k}+q) - p(\tilde{k}+q)$. In this case $k'(\tilde{k},p,q-1) = q-1$, and hence the c-stability condition for the reduced form game is that $\bar{v}_j \geq q(\tilde{k}+q-1)$, i.e. $x(\tilde{k}+q) - p(\tilde{k}+q) - q(\tilde{k}+q-1)$. As in the previous case, this follows since $g(\tilde{k}+q) \geq 0$ and $q(k)$ is increasing in k.

Finally, if $\tilde{k} < \max\{\bar{k}-1,0\}$, then it is easy to see that an outcome that involves no matchups, $u_i = p(0)$ and $v_j = q(0)$, $\forall i,j$, is c-stable.

The proof for the case where $p = q$ is similar and hence omitted.

We are now in a position to solve for the c-stable outcome of this game. There are two cases to consider.

Let us first examine the case where $\bar{k} \leq n-1$. Consider the outcome where there are n matchups, $\bar{u}_i + \bar{v}_j = x(n))$, for $\sigma(i) = j$, $\bar{u}_i = p(n-1)$ and $\bar{v}_j = x(n) - p(n-1)$. Since $\bar{k} \leq n-1$, $k'(0,n,n-1) = n-1$ and thus, the expected number of matchups if an agent decides to remain single, is $(n-1)$. The c-stability conditions thus become $\bar{u}_i + \bar{v}_j \geq x(n)$ (for $\sigma(i) \neq j$), and $\bar{v}_i \geq p(n-1)$. That the first condition is satisfied is obvious and the second condition implies that $x(n) - p(n-1) - q(n-1) \geq 0$, which follows from the fact that $f(n) \geq 0$.

We then consider the case where $\bar{k} = n$. Consider the outcome where there are n matchups, $\bar{u}_i + \bar{v}_j = x(n)$, for $\sigma(i) = j$, $\bar{u}_i = p(0)$ and $\bar{v}_j = x(n) - p(0)$. Since $\bar{k} = n$, the expected number of matchups, if an agent remains single, is zero. Thus the c-stability condition is that $\bar{v}_j \geq q(0)$ i.e. $x(n) - p(0) - q(0) \geq 0$. This follows from the fact that $f(n) \geq 0$ and $p(k)$ and $q(k)$ are increasing in k. ∎

REFERENCES

Aumann, R. J. 1963. A survey of cooperative games without side payments, in: *Essays in Mathematical Economics in Honour of Oskar Morgenstern* ed. Shubik, M. Princeton University Press, Princeton, 1963.

Balinski, M. L. and D. Gale, 1990, On the core of assignment games, in: *Game theory and applications* eds. Ichiishi, T. Neyman A. and Tauman, Y. Academic Press, London, 1990.

Bernheim, B. D., B. Peleg and M. D. Whinston, Coalition-proof Nash equilibria: Concepts, Journal of Economic Theory 42 (1987), 1–12.

Bernheim, B. D., B. Peleg and M. D. Whinston, Coalition-proof Nash equilibria: Applications, Journal of Economic Theory 42 (1987), 13–29.

Chowdhury, P. R. Stability in generalised assignment models: Some concepts with an application to technology transfer, Discussion paper No. 9344, CORE (1993).

Chowdhury, P. R. Generalised assignment models: With an application to technology transfer, Economic Theory 10 (1997) 335–360.

Chowdhury, P. R. Assignment models and reneging, Economic Theory 14 (1999) 393–415.

Dantzig, G. B. *Linear programming and extensions*, Princeton University Press, Princeton, 1963.

Gale, D. and L. S. Shapley, College admissions and the stability of marriage, American Mathematical Monthly 69 (1962) 9–15.

Li, S. Competitive matching equilibria and multiple principal-agent models, mimeo, University of Minnesota (1993).

Roth, A. E. and M. Sotomayor, *Two-sided matching: A study in game theoretic modelling and analysis*, Cambridge University Press, Cambridge, 1990.

Shapley, L. S. and M. Shubik, The assignment game 1: The core, International Journal of Game Theory 1 (1972) 111-130.

Sonmez, T. Strategy-proofness and singleton cores in generalised matching problems, Working paper, University of Rochester (1994).

Sonmez, T. Implementation in generalised matching problems, Journal of Mathematical Economics 26 (1996) 429–439.