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A SUBJECTIVE PROBABILITY CONVERGES TO THE OBJECTIVE PROBABILITY

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Abstract: In this paper we formulate an uncertain situation for an agent and investigate a learning process which is formed from his or her knowledge structure. The purpose of this paper is to prove that his or her subjective probability converges to the true distribution which governs the uncertain states, whenever he or she updates his or her forecast by a rational learning. This result means that even if individuals do not have the same prior, they will come to have a common posterior.

Key words: Subjective probability, objective probability, rational learning
JEL Classification Number: D81, D83

1. INTRODUCTION

The common prior assumption plays an important role in modern microeconomics and game theory. It states that two individuals having access to the same information will necessarily form the same subjective probability. In this paper we shall formulate a situation in which each individual has a subjective probability and can access to the same information; and then give a justification of this substantial common prior assumption.

The expected utility theory, which was proposed by von Neumann and Morgenstern, has been widely used in game theory and decision theory under uncertainty and risk. However, it is also known that there are several problems with that concept. One of them is that probabilities to represent risk or uncertainty are objectively defined. In other words decision-makers assign the same prior probabilities to the risk. Thus, we will have a difficulty when agents do not know the objective probability. If the decision-maker does not know the probabilities of events, how does he or she make a decision under this uncertain situation? Savage (1954), and Anscombe and Aumann (1963) proposed subjective expected utility models to solve this problem. This probability is called a subjective probability in the sense that it is derived from an individual's preference. Anscombe and Aumann defined a subjective probability on horse lotteries, where prizes

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were roulette lotteries, under three reasonable axioms. Edwards (1962), Machina (1989) and Schmeidler (1989) also considered related problems.

These subjective probabilities do not generally coincide with the true probability by which all states are governed uniformly in the expected utility theory. However, an agent may update his or her own forecast by experience if he or she faces the same situation many times. That is, he or she may learn the objective probabilities gradually in terms of the long-run frequencies. The idea of learning is not new in economic theory. In macro economic theory the rational expectation has been used for a long time (for example, see Muth 1961). Recent developments of these approaches include Blackwell and Dubins (1962), Feldman (1987), Kalai and Lehrer (1993), Cho (1998) and Sagara (1999). Other types of learning models are formulated in game theory, and summarized in Fudenberg and Levine (1998) and Young (1998).

In this paper we formulate the learning process which is derived from an individual's knowledge, and clarify the relationship between the expected utility theory and the subjective expected utility theory. Little has been known about this relationship. One of the main contributions is to characterize the common prior assumption studying this relation. More precisely, we show that an individual, who does not know the true probability, can find it out using his or her past observation and knowledge. In other words, his or her forecast, which is derived from a subjective probability, converges to the true distribution in accordance with a rational learning. Therefore we can state that even if individuals do not have the same prior, they will come to have the common posterior.

The rest of this paper is organized as follows. In Section 2, we define a rational learning and a structure of information. In Section 3, we show that the subjective probability of an agent converges to the objective probability whenever he or she updates his or her subjective probability, with respect to which the objective probability is absolutely continuous. The last section presents concluding remarks and extensions.

2. RATIONAL LEARNING AND OBJECTIVE PROBABILITY

2.1. *Subjective Probabilities and the Objective Probability*

Initially, we define the objective probability and introduce subjective probabilities. Let $\Omega_t = \{\omega_1, \dots, \omega_s\}$ be the set of states at period t, and $X \subset \boldsymbol{R}^l$ be the set of consequences. An element x in a subset of X^{Ω_t} is called an action, and means that an agent obtains $x(\omega_j)$ when $\omega_j \in \Omega_t$ occurs at this period.

If an agent knows the prior probability Q_t on Ω_t in this setting, it is known from the expected utility theorem that a preference on a subset of X^{Ω_t} is represented by $\sum_{i=1}^{s} Q_t(\omega_i)u(x(\omega_i))$, where $u : X \to \boldsymbol{R}$ is a utility function for an agent. This probability Q_t is called the *objective probability* on Ω_t in the sense that it is independent of his or her preference relation and experience. For example, we know from experience the probability distributions of dice, roulettes and a coin tossing. Besides $Q_t(\omega_i) > 0$ for all ω_i. Namely, it is implicitly assumed that $Q_t(\omega_i) > 0$ for all ω_i in many situations.

DEFINITION 1. The *objective probability* Q_t on Ω_t is the probability measure on $\Omega_t = \{\omega_1, \dots, \omega_s\}$ which satisfies $Q_t(\omega_i) > 0$ for all $i = 1, \dots, s$.

Even if an agent does not know the prior probability on Ω_t, it is also known that he or she has a probability measure P_t on Ω_t under reasonable axioms.[1] This probability measure P_t is called a *subjective probability* in the sense that it is generated from his or her preference relation. In the same way as the expected utility theorem, a preference on a subset on X^{Ω_t} is represented by $\sum_{i=1}^{s} P_t(\omega_i)u(x(\omega_i))$. In this paper we assume that a subjective probability on Ω_t is derived from some axioms.

2.2. *The Structure of Information and Learning*

We suppose the situation in which the states in Ω_t are determined by the objective probability Q_t. Even if an agent does not know this probability he or she can form a subjective probability. If he or she is confronted with the similar situation repeatedly and observes past information, he or she may have another forecast. That is, he or she can learn from his or her past experience and revises his or her prediction.

Pratt, Raiffa and Schlaiffer (1964) proved that an agent updated his or her own subjective probability by using a conditional probability under certain axioms. But it is not clear what the relationship between the objective probability and a subjective probability is. Here, we will clarify this relationship using a learning approach. To this end, we introduce a dynamic model where an agent is assumed to be faced with the same situations for several times, and in which he or she is supposed to be informed of the states. Since an agent is considered to be faced with the same situation Ω_t for each period, we use the product set of Ω_t as a state set throughout periods.

Let $\Omega := \prod_{t=1}^{\infty} \Omega_t$ be the product set of $\Omega_t = \{\omega_1, \ldots, \omega_s\}$ and $\boldsymbol{P}$ be the product measure over Ω which is constructed from P_t. To characterize an agent's information we adopt a partition of Ω.[2]

$\mathcal{P}_t$ describes a partition of Ω at period t. $\mathcal{P}_t(\omega)$ denotes the element of $\mathcal{P}_t$ which contains $\omega \in \Omega$. Moreover we assume that these partitions satisfy

$$\mathcal{P}_t(\omega) \supset \mathcal{P}_{t+1}(\omega) \quad \text{for all } t = 1, 2, \ldots .$$

This supposition describes that an agent's information is refined and accumulated by observing past information. According to this refinement process, we interpret that an agent knows the realized state and learns from his or her past experience. In addition, a sequence of σ-algebras $\{\mathcal{F}_t\}$ generated by the sequence of the partitions $\{\mathcal{P}_t\}$ also forms an information increasing class. In other words, the sequence of σ-algebras $\{\mathcal{F}_t\}$ satisfies

$$\mathcal{F}_t \subset \mathcal{F}_{t+1} \quad \text{for all } t = 1, 2, \ldots .$$

We summarize the previous notions and formulate a learning process which is derived from an individual's knowledge.

DEFINITION 2. The *structure of information and learning* for an agent is defined as the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \boldsymbol{P})$, where

(1) Ω is the product set of Ω_t,

(2) $\mathcal{F}$ is the σ-algebra on Ω and satisfies $\mathcal{F}_t \subset \mathcal{F}$ for all $t = 1, 2, \ldots$,

[1] See Anscombe and Aumann (1963) and Savage (1954).

[2] This partition approach is sometimes adopted to describe knowledge.

(3) $\{\mathcal{F}_t\}$ is a filtration, and

(4) $\boldsymbol{P}$ is the product probability measure which is derived from a subjective probability P_t on Ω_t.

From this specification, the σ-algebra $\mathcal{F}_t$ of the conditional probability $\boldsymbol{P}(\cdot \mid \mathcal{F}_t)$ indicates an agent's information and observation until the period t. For this reason *the marginal conditional probability* $\boldsymbol{P}_t(\cdot \mid \mathcal{F}_t)$ expresses his or her forecast for states at period $t+1$, noting that $\boldsymbol{P}$ is the product measure. In the similar way his or her preference on the subset of $X^{\Omega_{t+1}}$ is represented by $\sum_{i=1}^{S} \boldsymbol{P}_t(\omega_i \mid \mathcal{F}_t) u(x(\omega_i))$ using the marginal probability $\boldsymbol{P}_t(\cdot \mid \mathcal{F}_t)$. Furthermore, an agent is assumed to revise his or her own prediction $\boldsymbol{P}(\cdot \mid \mathcal{F}_t)$ by Bayesian learning in this definition. We would like to emphasize that this definition has a decision theoretical background.

Finally we define the product measure of the objective probability Q_t. $\boldsymbol{Q}$ denotes the product measure on Ω and constructed from the objective probability Q_t on Ω_t. $(\Omega, \mathcal{F}, \boldsymbol{Q})$ is also the probability space. Note that if $Q_t(\omega_i)$ for all $\omega_i \in \Omega_t$, $\boldsymbol{Q}(A) > 0$ for all $A \in \mathcal{F}$. This implies that if Q_t is the objective probability, $\boldsymbol{Q}$ is also one.

3. MAIN RESULTS

We clarify the relationship between the expected utility theory and the subjective expected utility. We shall point out an important result, which was proved by Blackwell and Dubins (1962) about the relationship between two probabilities on the same space. They showed that if two probability measures are predictive and absolutely continuous, two conditional probability measures merge almost everywhere as time goes to infinity.

In this paper, we make a progress one step ahead of their result. We prove a stronger result than theirs, focusing on a subjective probability and the objective probability, which are related to the decision theory. More concretely, we employ the product measure space which has a basis of a subjective probability and the objective probability, and the structure of information and learning which has also a decision theoretical background. We prove that if the objective probability on Ω_t is absolutely continuous with respect to a subjective probability on Ω_t, the conditional subjective probability converges uniformly to the conditional objective probability. Roughly speaking our result is the uniform convergence instead of the almost everywhere convergence in the mathematical sense. Since it is not generic, we can interpret it more easily. It indicates that an agent's subjective expected utility at period t, $\sum_{i=1}^{S} \boldsymbol{P}_t(\omega_i \mid \mathcal{F}_t) u(x(\omega_i))$, converges to the expected utility $\sum_{i=1}^{S} \boldsymbol{Q}_t(\omega_i \mid \mathcal{F}) u(x(\omega_i))$ as the outcome of his or her learning process. Furthermore we can interpret that individuals form the same subjective probability even if they have different knowledge and private forecasts at the initial point. To state our results we prepared for the following assumption and lemmas.

Assumption 1. $Q_t \ll P_t$ for all $t = 1, 2, \ldots$.[3]

[3] Let P and Q be the probability measures on a set Ω, and Σ be a σ-algebra of subsets of Ω. Now, $Q \ll P$ denotes that Q is absolutely continuous with respect to P, and means that $Q(A) > 0$ implies $P(A) > 0$ for all $A \in \Sigma$.

Assumption 1, which is called the *absolutely continuous condition*, means that $Q_t(A) > 0$ implies $P_t(A) > 0$ for all $A \subset \Omega_t$, so that the prediction by Q_t is also predicted by P_t. At the same time we also permit that an agent can have a wrong expectation. This assumption is usually adopted in Bayesian learning literatures, (for example, see Feldman (1987), Kalai and Lehrer (1993) and Nyarko (1998)).

LEMMA 1. *If $Q_t \ll P_t$ for all $t = 1, 2, \ldots$ then $Q \ll P$.*

Proof. We will show that for all $A \in \mathcal{F}_\infty$, $Q(A) > 0$ implies $P(A) > 0$. Let $A = \prod_{l=1}^{\infty} A_l$ be the product set in Ω, where $A_k = \Omega_k$ for all except finitely many k.

$$\begin{aligned} \forall A \in \mathcal{F}_\infty, Q(A) > 0 &\Rightarrow \forall t = 1, 2, \ldots Q_t(A_l) > 0 \text{ for all } l = 1, 2, \ldots \\ &\Rightarrow \forall t = 1, 2, \ldots P_t(A_l) > 0 \text{ for all } l = 1, 2, \ldots \\ &\Rightarrow P(A) > 0 . \end{aligned}$$

∎

Using Lemma 1, we may interpret Assumption 1 in another way. If Assumption 1 holds for some individuals, we can consider that they have a common accessible information which is described by Q_t. The following result is useful to prove Theorem 1.

LEMMA 2. *For all $f \in L^1(\Omega, \mathcal{B}, P)$ and for every monotone increasing sequence $\{\mathcal{B}_n\}$ of σ-algebras converging to a σ-algebra $\mathcal{B}$,*

$$\lim_{n\to\infty} \mathrm{E}[f \mid \mathcal{B}_n] = \mathrm{E}[f \mid \mathcal{B}] \text{ almost everywhere} .$$

Proof. The proof is basically the same as Theorem 35.6 in Billingsley (1995). ∎

The following theorem is the main result in this paper. Intuitively, it states that a subjective probability converges to the objective probability.

THEOREM 1. *Under Assumption 1*

$$Q(A \mid \mathcal{F}_\infty) = \lim_{t\to\infty} P(A \mid \mathcal{F}_t) \text{ uniformly}$$

for all $A \in \mathcal{F}_\infty$.[4]

Proof. From Lemma 1, $Q \ll P$ follows. We can see from Radon-Nikodym's theorem that

$$\exists f(\omega) : \mathcal{F}_\infty\text{-measurable s.t.} \int_{\mathcal{P}} f(\omega) dP(\omega) = Q(\mathcal{P}) \tag{1}$$

for all $\mathcal{P} \in \mathcal{F}_\infty$. Using the definition of conditional expectation and Lemma 2, we examine the following two cases, for all $\mathcal{P}_{t-1} \in \mathcal{F}_t$.

Case I. $P(\mathcal{P}_{t-1}(\omega)) \neq 0$.

$$\mathrm{E}_P[f(\omega) \mid \mathcal{P}_{t-1}(\omega)] = \frac{1}{P(\mathcal{P}_{t-1}(\omega))} \int_{\mathcal{P}_{t-1}(\omega)} f(\omega) dP(\omega)$$

[4] We can prove the similar result without the product set approach.

$$= \frac{Q(\mathcal{P}_{t-1}(\omega))}{P(\mathcal{P}_{t-1}(\omega))} \qquad \text{(from (1))}$$

$$\to \frac{Q(\mathcal{P}_{\infty}(\omega))}{P(\mathcal{P}_{\infty}(\omega))} \quad (a.e.). \qquad \text{(Lemma 2)}$$

Case II. $P(\mathcal{P}_{t-1}(\omega)) = 0$.

In this case, Theorem 1 is trivial by $Q \ll P$. Let us consider the Case I as follows.

Since $\mathcal{P}_{\infty}(\omega) \subset \mathcal{P}_{t-1}(\omega)$,

$$\lim_{t\to\infty} \frac{P(\mathcal{P}_{\infty}(\omega) \mid \mathcal{P}_{t-1}(\omega))}{Q(\mathcal{P}_{\infty}(\omega) \mid \mathcal{P}_{t-1}(\omega))} = 1 \quad (a.e.).$$

This equation indicates that

$$\exists F : P\text{-null set s.t. } \forall \varepsilon > 0, \ \forall \omega \in \Omega, \ \exists t_0 \in N;$$

$$t \geq t_0 \Rightarrow \left| \frac{P(\mathcal{P}_{\infty}(\omega) \mid \mathcal{P}_{t-1}(\omega))}{Q(\mathcal{P}_{\infty}(\omega) \mid \mathcal{P}_{t-1}(\omega))} - 1 \right| < \varepsilon$$

on $\Omega \setminus F$. Then,

$$t \geq t_0 \Rightarrow 1 - \varepsilon < \frac{P(\mathcal{P}_{\infty}(\omega) \mid \mathcal{P}_{t-1}(\omega))}{Q(\mathcal{P}_{\infty}(\omega) \mid \mathcal{P}_{t-1}(\omega))} < 1 + \varepsilon.$$

Therefore

$$t \geq t_0 \Rightarrow |P(\mathcal{P}_{\infty}(\omega) \mid \mathcal{P}_{t-1}(\omega)) - Q(\mathcal{P}_{\infty}(\omega) \mid \mathcal{P}_{t-1}(\omega))| < \varepsilon Q(\mathcal{P}_{\infty}(\omega) \mid \mathcal{P}_{t-1}(\omega)).$$

Noting that $0 \leq Q(\mathcal{P}_{\infty}(\omega) \mid \mathcal{P}_{t-1}(\omega)) \leq 1$, we can conclude

$$\exists F : P\text{-null set s.t. } \forall \varepsilon > 0, \ \forall \omega \in \Omega, \ \exists t_0 \in N;$$

$$t \geq t_0 \Rightarrow |P(\mathcal{P}_{\infty}(\omega) \mid \mathcal{P}_{t-1}(\omega)) - Q(\mathcal{P}_{\infty}(\omega) \mid \mathcal{P}_{t-1}(\omega))| < \varepsilon$$

on $\Omega \setminus F$. Let A be an atom of $\mathcal{F}_{\infty}$. We obtain that

$$t \geq t_0 \Rightarrow |P(A \mid \mathcal{F}_t) - Q(A \mid \mathcal{F}_{\infty})| < \varepsilon.$$

In fact since A is not contained in $\mathcal{P}_t(\omega)$, it follows that $P(A \mid \mathcal{F}_t) = 0$ and $Q(A \mid \mathcal{F}_t) = 0$. Hence

$$\lim_{t\to\infty} P(A \mid \mathcal{F}_t) = Q(A \mid \mathcal{F}_{\infty}) \quad a.e.$$

Noting that Q is the objective probability, we can conclude from Egroff's theorem that

$$\lim_{t\to\infty} P(A \mid \mathcal{F}_t) = Q(A \mid \mathcal{F}_{\infty}) \quad \text{uniformly}.$$

for all $A \in \mathcal{F}_{\infty}$. The proof is completed. ∎

An interpretation of Theorem 1 is that if an agent has a subjective probability with respect to which the objective probability Q_t is absolutely continuous, and updates his or her subjective probability according to a rational learning then he or she can find out the objective probability. As an agent updates, the subjective expected utility which he or she forms under the subjective probability converges to the expected utility which he or she forms under risk or uncertain situations. Another interpretation is that even if individuals have different knowledge and private forecasts, they have a common posterior. In this sense, this result may be similar to Aumann (1976). Let we consider that two individuals have different information which are described by subjective probabilities. Apply Theorem 1, if their probabilities are equivalent they can agree throughout learning without the common prior and common knowledge.

PROPOSITION 1. *If* $\lim_{t\to\infty} \boldsymbol{P}(A \mid \mathcal{F}_t) = \boldsymbol{Q}(A \mid \mathcal{F}_\infty)$, *then the conditional probability distribution of* $\boldsymbol{Q}$ *is absolutely continuous with respect to the conditional probability distribution of* $\boldsymbol{P}$.

Proof. By assumptions,

$$\forall \varepsilon > 0 \; \exists t_0 \in \boldsymbol{N} : t \geq t_0 \Rightarrow \boldsymbol{P}(A \mid \mathcal{F}_t) = 0 \text{ and } \boldsymbol{Q}(A \mid \mathcal{F}_t) < \varepsilon .$$

Since the value of $\boldsymbol{Q}(A \mid \mathcal{F}_t)$ does not depend on ε, we can conclude that $\boldsymbol{P}(A \mid \mathcal{F}_t) = 0$ and $\boldsymbol{Q}(A \mid \mathcal{F}_t) = 0$ for all $t \geq t_0$. Then, the conditional probability of $\boldsymbol{Q}$ is absolutely continuous with respect to the conditional probability of $\boldsymbol{P}$. ∎

4. CONCLUDING REMARKS AND EXTENSIONS

We have clarified the relationship between the expected utility theory and the subjective utility theory. We introduce a filtered probability space as a representation of learning which is derived from agent's knowledge. Our main result, Theorem 1, is that a conditional subjective probability converges to the conditional objective probability.

Our result is one step ahead of Blackwell and Dubins's result, focusing on the relationship between a subjective probability and the objective probability. To put it briefly, our result is the uniform convergence instead of the almost everywhere convergence. Moreover its interpretation may be similar to Aumann (1976). Roughly speaking his result is that if agents with different partitions have common knowledge and the same priors, then these posteriors are equal, that is, they agree. It may be said that a similar result holds by introducing a rational learning approach without assuming any common knowledge and the same priors. In this sense, even if agents do not have the common prior they will come to have the common posterior.

One of our future researches is to deal with multi agents. In particular, if each agent has different knowledge, what we can conclude about the convergence? Another way is to apply our model to game theory. For example, we may take the state set Ω as a type space in Bayesian game. Under this situation it will be shown that even if players do not have common knowledge of their type spaces, they can find out their opponents' true type spaces by learning.

By this extension, it may also be possible to apply to Harsanyi's doctrine. Namely, if it is assumed that players revise their own subjective forecasts by a rational learning, they may come up with the (objective) Bayesian game, without the common prior assumption.

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