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PATH INDEPENDENCE AND CHOICE ACYCLICITY PROPERTY

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Abstract: The paper begins by exploring the Path Independence Property. The possibility of a lower and upper approximation of a choice function satisfying Path Independence is dealt with in the paper. A significant property implied by Path Independence is Outcasting. We propose in the paper a unique characterisation of a choice function, called the batch choice function, which satisfies Outcasting. However, the relevant characterisation theorem requires a property stronger than Outcasting called the Choice Acyclicity Property. In an appendix to the paper, we provide a simple proof (without using Zorn's Lemma), of the fact that satisfaction of a property (a*) by a choice function is equivalent to the existence of a utility function, whose maximizers on a feasible set are always chosen. This result is originally due to Deb [1983]. This theorem is used in our paper to prove the existence of Path Independent lower approximations.

1. INTRODUCTION

In choice theory, a decision maker is assumed to be equipped with a decision rule or a choice function which associates with each non-empty finite subset of a universal set, the set of all chosen points from the given set of feasible alternatives. The purpose of choice theory is to characterise choice functions satisfying desirable properties and also to establish interrelations between properties.

One such property is Path Independence, due to Plott [1973]. Along with a property called Concordance (which basically says that if a point is chosen from two sets, then it would also be chosen from their union), Path Independence is necessary and sufficient for a choice function to be rationalised by a quasi-transitive, reflexive and complete binary relation. Path Independence implies the Superset Property of Blair, Bordes, Kelly and Suzumura (1976) and has been shown to be equivalent to the simultaneous satisfaction of this latter property and Chernoff's Axioms.

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A slight strengthening of the Superset Property is the celebrated Outcasting axiom due to Nash [1950]. The Superset property says that if two sets are given with the first contained in the second and if the chosen points of the second are contained in the set of chosen points of the first, then the two sets of chosen points coincide. Outcasting on the other hand requires that if the chosen points of the second set are contained in the first set, then the two sets of the chosen points coincide. Theorem 9 of Aizerman (1985) [Theorem 4.6 of Aizerman and Aleskerov (1995)] asserts that Path Independence is equivalent to the simultaneous satisfaction of Chernoff's Axiom and Outcasting.

In this paper we study lower and upper approximations of choice functions and propose a necessary and sufficient condition for a choice function to have a Path Independent lower approximation. On lower approximations, the condition is shown by property (a*) presented in Deb [1983]. Property (a*) means that given any feasible set, there exists a chosen point from that set, such that if this chosen point continues to belong to a subset, then it is necessarily chosen from the latter set. However, for upper approximations, the scene is a lot more dismal. Satisfaction of well known and reasonable conditions, fail to guarantee the existence of path independent upper approximations. The concept of lower and upper approximations is due to Litvakov [1981]. Given a choice function, the lower approximation of the choice function satisfying a set of properties is the union of all choice functions contained in the given choice function and satisfying, the stated properties. The upper approximation on the other hand is the intersection of all choice functions containing the given choice function and satisfying the stated properties. The natural question is: whether the lower approximations and the upper approximation continue to satisfy the stated properties? To prove our results we make significant use of Theorem 9 of Aizerman (1985). An alternative, proof of Theorem 9 is also provided in the paper (see Theorem 4.6 of Aizerman and Aleskerov [1995]). In an appendix to this paper we provide a simple proof of Theorem 2.10 in Deb [1983] in the case when the universal set is finite. This theorem says, that if a choice function satisfies Property (a*), then there exists a real-valued function defined on the universal set, such that its maximizers from a feasible set are always chosen by the given choice function. We use this theorem to establish the existence of Path Independent lower approximations. Our proof does not require the use of Zorn's Lemma, which has been used in Deb [1983] to prove Theorem 2.10. Our method of proof is entirely constructive.

In a final section of this paper, we explore another concept related to Outcasting. This property is called Choice Acyclicity Property (CAP). This property is stronger than Outcasting. It turns out that choice functions satisfy the Choice Acyclicity Property if and only if they are batch choice functions [see Aizerman and Aleskerov (1995)]. A batch choice function assigns to each set the unique subset which maximises a real valued function (defined on all non-empty subsets of the universal set) amongst all subsets of the given set. Although a batch choice function satisfies Choice Acyclicity Property and hence Outcasting, it is shown in this paper that it may fail to satisfy Path Independence Hence, a batch choice function need not coincide with the set of maximizers (from a feasible set) of a utility function (defined on the universal set). This latter consequence would of necessity require the satisfaction of Path Independence. This interesting result gives additional appeal to choice theory.

To the extent that our batch choice functions are choice functions rationalized by reflexive, complete and transitive binary relations on the set of all non-empty subsets of a given set, the analysis reported here is closely related to the modest yet significant literature on "freedom of choice". In the "freedom of choice" literature, the principal problem is to define a binary relation on non-empty subsets of a given set, so as to formalize the notion of "preference for freedom" which any non-empty set of alternatives provides to a decision maker. Presumably, the idea is to use this binary relation to rank opportunity sets and arrive at decisions on the basis of such a ranking. This field has been pioneered by Pattanaik and Xu [1990], with subsequent contributions by Pattanaik and Xu [1997, 1998], Arrow [1995], Carter [1996], Puppe [1996], Sen [1990, 1991], Rosenbaum [1996], Van Hees [1998, 1999] and Van Hees and Wissenburg [1999]. In this paper we are interested in the converse problem given a choice function, is there anything akin to a "preference for freedom" (however, queer that may be) which rationalizes the observed behaviour of a decision maker?

It is interesting to note that the Choice Acyclicity Property is implied by a property called Functional Acyclicity [see, Aizerman and Aleskerov (1995), Lahiri (1999)]. In Aizerman and Aleskerov (1995), it is asserted that a choice function satisfies Functional Acyclicity if and only if there is a real valued function defined on the universal set and a real valued function defined on all nonempty subsets of the universal set, such that the chosen points from each feasible set of alternatives coincides exactly with those feasible points whose value according to the first function is greater than or equal to the real number assigned by the second function to the given feasible set. Such choice functions are called threshold rationalizable in Lahiri [1999]. In Lahiri [1999] a correct proof of this equivalence is available.

2. THE MODEL

Let X be a non-empty universal set and let $[X]$ denote the set of all non-empty finite subsets of X. A choice function is a function $C : [X] \to [X]$ such that $C(S) \subset S \ \forall S \in [X]$.

A choice function C is said to satisfy Chernoff's Axiom (CA) if $S, T \in [X]$, $S \subset T$ implies $C(T) \cap S \subset C(S)$.

A choice function C is said to satisfy Outcasting (O) if $C(T) \subset S \subset T \in [X]$ implies $C(S) = C(T)$.

A choice function C is said to satisfy Superset Property (Su) if $C(T) \subset C(S) \subset S \subset T \in [X]$ implies $C(S) = C(T)$.

It is easy to see that Outcasting implies Superset Property.

A choice function C is said to satisfy Path Independence (PI) if $\forall S, T \in [X]$, $C(S \cup T) = C(C(S) \cup C(T))$.

Suzumura [1983] proves that a choice function satisfies Path Independence (PI) if and only if it satisfies Chernoff's Axiom and the Superset Property.

We will proceed (for the sake of completeness and for the purpose of being self contained) to provide a complete proof of Theorem 9 in Aizerman [1985], which is Theorem 1 in our paper.

LEMMA 1. *C satisfies* PI$\leftrightarrow$ $C(S \cup T) = C(C(S) \cup T)$ $\forall S, T \in [X]$.

LEMMA 2. *C satisfies* CA$\leftrightarrow$ $C(S \cup T) \subset C(S) \cup C(T)$ $\forall S, T \in [X]$.

Proof. Let C satisfy CA. Thus for $S, T \in [X]$,

$$C(S \cup T) \cap S \subset C(S),$$
$$C(S \cup T) \cap T \subset C(T).$$

Hence $C(S \cup T) \subset C(S) \cup C(T)$.
Conversely suppose, $C(S \cup T) \subset C(S) \cup C(T)$ $\forall S, T \in [X]$.
Let $S, T \in [X]$ with $S \subset\subset T$.

$$C(T) \subset C(S) \cup C(T \setminus S).$$

If $x \in C(T) \cap S$, then $x \notin C(T \setminus S)$.
Thus $x \in C(S)$.
Thus $C(T) \cap S \subset C(S)$. Q.E.D.

LEMMA 3. *C satisfies* PI *implies* $C(C(S)) = C(S)$ $\forall S \in [X]$.

Proof. Simply put $S = T$ in the definition of PI. Q.E.D.

THEOREM 1. *C satisfies* PI$\leftrightarrow$ *C satisfies* CA *and* O.

Alternative Proof. Suppose C satisfies CA and O.
By Lemma 2, $\forall S, T \in [X]$,

$$C(S \cup T) \subset C(S) \cup C(T) \subset S \cup T.$$

By 'O', $C(S \cup T) = C(C(S) \cup C(T))$.
Thus C satisfies PI.
Now suppose C satisfies PI.
Thus $\forall S, T \in [X]$,

$$C(S \cup T) = C(C(S) \cup C(T)) \subset C(S) \cup C(T).$$

By Lemma 2, C satisfies CA.
Let $C(T) \subset S \subset T \in [X]$.
By CA, $C(T) = C(T) \cap S \subset C(S)$.
Now,

$$\begin{aligned} C(T) = C(T \cup S) &= C(C(T) \cup C(S)) \text{ by PI} \\ &= C(C(S)) \\ &= C(S) \text{ by Lemma 3}. \end{aligned}$$

Thus C satisfies O. Q.E.D.

3. LOWER APPROXIMATIONS

Let $C : [X] \to [X]$ be a choice function and let $Q(\neq \phi)$ be a class of choice functions on $[X]$.

Say that a choice function C' is contained in C if $C'(S) \subset C(S)\ \forall S \in [X]$. In such a situation we write $C' \subset C$.

The lower approximation of C given Q is the choice function

$$C^L : [X] \to [X] \text{ such that } C^L(S) = \bigcup_{C' \subset C, C' \in Q} C'(S).$$

Suppose X is finite. A choice function C is said to satisfy Property (a*) [see Deb [1983]] if $\forall S \in [X]$, there exists $X_o \in C(S)$ such that $(\phi \neq)T \subset S$, $x_o \in T$ implies $x_o \in C(T)$.

LEMMA 4. *If C' satisfies* CA *and $C' \subset C$ then C satisfies Property* (a*).

Proof. Given $S \in [X]$, let $x_o \in C'(S) \subset C(S)$.
Now $T \subset S$ implies $C'(S) \cap T \subset C'(T)$ by CA.
Therefore $x_o \in T$ implies $x_o \in C'(T) \subset C(T)$.
Thus C satisfies Property (a*). Q.E.D.

Actually, Deb [1983] proves that choice function C on $[X]$ contains a choice function C' satisfying Arrow's Axiom (Arrow [1959]) (which implies CA, CON and O) if and only if C satisfies Property (a*).

THEOREM 2. *Let Q be the set of all Path Independent choice function. A choice function C has a lower approximation in Q if and only if C satisfies Property* (a*).

Proof. Suppose C satisfies Property (a*). Then by the main result in Deb [1983], and Theorem 1, there exists C' in Q such that $C' \subset C$. Let $C^L : [X] \to [X]$ be defined as the lower approximation of C from Q. We have to show C^L is in Q.

Let $(\phi \neq)T \subset S$. Then

$$C^L(S) \cap T = \bigcup_{C' \subset C, C' \in Q} [C'(S) \cap T] \subset \bigcup_{C' \subset C, C' \in Q} C'(T) = C^L(T),$$

where we appeal to CA for C'.
Thus C^L satisfies CA.
Let $C^L(T) \subset S \subset T \in [X]$.
Thus, $\bigcup_{C' \subset C', C' \in Q} C'(T) \subset S \subset T \in [X]$.
Therefore $C'(T) \subset S \subset T \in [X]\ \forall C' \subset C, C' \in Q$.
By 'O' applied to C', $C'(T) = C'(S)$.
Therefore, $\bigcup_{C' \subset C, C' \in Q} C'(T) = \bigcup_{C' \subset C, C' \in Q} C'(S)$.
Therefore, $C^L(T) = C^L(S)$.
Thus, C^L satisfies O.
Thus, C^L is in Q.

Conversely, suppose C has a lower approximation C^L in Q. Since C^L satisfies CA, by the previous lemma C satisfies Property (a*). Q.E.D.

A choice function C on $[X]$ is said to satisfy Concordance (CON) if $\forall S, T \in [X]$, $C(S) \cap C(T) \subset C(S \cup T)$.

THEOREM 3. *Let Q be the set of all choice functions satisfying* CA. *A choice function C has a lower approximation in Q if and only if C satisfies Property* (a*).

The above analysis is considerably different from a similar analysis reported in Litvakov (1981), Aizerman (1985) and Aizerman and Aleskerov (1995), because we require all our choice functions to be non-empty valued. This makes a lot of difference in the analysis.

A related question posed in Litvakov (1981), Aizerman (1985) and Aizerman and Aleskerov (1995) is about the existence of upper approximations.

Let $C : [X] \to [X]$ be a choice function and let $Q(\neq \phi)$ be a class of choice functions on $[X]$. Say that a choice function C' contains C if $C(S) \subset C'(S)\ \forall S \in [X]$. In such a situation we write $C \subset C'$. The upper approximation of C given Q is the choice function $C : [X] \to [X]$ such that

$$C^U(S) = \bigcap_{C \subset C', C' \in Q} C'(S).$$

The question that naturally arises is if C satisfies CA (which is considerably stronger than Property (a*)) does it have an upper approximation which satisfies Path Independence? The answer is in the negative.

EXAMPLE 1. Let $X = \{x, y, z\}$, $C(X) = \{x\}$ and $C(S) = S$ otherwise. Let $C'(X) = \{x, y\}$ and $C'(S) = S$ otherwise. Let $C''(X) = \{x, z\}$ and $C''(S) = S$ otherwise. Here C satisfies CA but not O. Both C' and C'' contain C and satisfy Path Independence. However, $C'(S) \cap C''(S) = C(S)\ \forall S \in [X]$. Thus C has no upper approximation satisfying Path Independence.

In fact we can show that even if C satisfies CA and CON, it may fail to have an upper approximation which satisfies PI (and CON).

EXAMPLE 2. Let $X = \{x, y, x\}$. Let $C(X) = \{x\}$, $C(\{x, y\}) = \{x\}$, $C(\{x, z\}) = \{x, z\}$, $C(\{y, z\}) = \{y\}$ and $C(\{a\}) = \{a\}\ \forall a \in X$. C satisfies CA and CON but does not satisfy PI: $\{x\} = C(X) = C(\{x, y\}) \cup (\{x, y\}) \neq \{x, z\} = C(C(\{x, y\}) \cup C(\{x, z\})$. Let $C'(X) = \{x, y\}$, $C'\{x, y\} = \{x, y\}$, $C'(\{x, z\}) = \{x, z\}$, $C'(\{y, z\}) = \{y\}$, $C'(\{a\}) = \{a\}\ \forall a \in X$. C' satisfies PI and CON. Let $C''(X) = \{x, z\}$, $C''(\{x, y\}) = \{x\}$, $C''(\{x, z\}) = \{x, z\}$, $C''(\{y, z\}) = \{y, z\}$, $C''(\{a\}) = \{a\}\ \forall a \in X$. C'' satisfies PI and CON.

Observe $C \subset C'$ and $C \subset C''$. However, $C(S) = C'(S) \cap C''(S)\ \forall S \in [X]$. Thus C has no upper approximation satisfying PI (and CON).

4. CHOICE FUNCTIONS SATISFYING CHOICE ACYCLICITY

Let $f : [X] \to \mathfrak{R}$ satisfy the property: [$\forall S \in [X]$, $\{T \subset S / f(T) \geq f(T') \; \forall T' \subset S\}$ is a singleton]. Then the choice function $C_f : [X] \to [X]$ defined by

$$C_f(S) = \underset{T \subset S, T \in [X]}{\arg\max} \, [f(T)],$$

is called a batch choice function. Aizerman and Aleskerov (1995) show that batch choice functions satisfy Outcasting. Batch choice functions satisfy an even stronger property:

Choice Acyclicity Property (CAP). There does not exist a positive integer k and sets $S_1, \ldots, S_k \in [X]$, all distinct such that: (1) $C(S_i) \neq C(S_j)$ for some i, j $(i \neq j)$; (2) $C(S_i) \subset S_{i+1}, i = l, \ldots, k-1$; (3) $C(S_k) \subset S_1$.

If $k = 2$ and $S_1 \subset S_2$, we get Outcasting.

Let $C_f : [X] \to [X]$ be a batch choice function. Suppose there exists $S_1, S_2, \ldots, S_k \in [X]$ such that $C(S_i) \subset S_{i+1} \; \forall i = 1, \ldots, k-1$, $C(S_k) \subset S_1$ and $C(S_i) \neq C(S_j)$ for some i, j $(i \neq j)$. Then $f(C(S_{i+1})) \geq f(C(S_i)), i = 1, \ldots, k-1$ and $f(C(S_1)) \geq f(C(S_k))$ with at least one strict inequality. But this is impossible.

We can now prove the converse result, that every choice function satisfying Choice Acyclicity Property is a batch choice function.

Assume in what follows that X is a non-empty finite set.

THEOREM 4. *A choice function is a batch choice function if and only if it satisfies* CAP.

Proof. The fact that a batch choice function satisfies CAP has been established above. Hence, let us assume that C satisfies CAP.

Given $S \in [X]$, let $[S]$ denote the set of all non-empty subsets of S and let $2^S = [S] \cup \{\phi\}$. Given $\phi \neq Y \subset [X]$, $[S] \subset Y$ will be said to be maximal for Y if there does not exist $[T] \subset Y$ with $S \subset\subset T$ and $C(T) \subset S$. Given $(\phi \neq) Y \subset [X]$, let $[T_i]$, $i = 1, \ldots, m$ be maximal for Y (: provided there exists at least one $[S]$ maximal for Y). Note: If $Y = [S]$, for some $S \in [X]$, then $[S]$ is the only one of its kind maximal for Y. Consider T_1. If $i \neq 1$ implies $[C(T_1)$ not a subset of T_i or $C(T_1) = C(T_i)]$, let $F(Y) = C(T_1)$. If there exists $T \in \{T_i\}_{i \neq 1}$ such that $C(T_1) \subset T$, $C(T_1) \neq C(T)$, then consider $C(T)$. Say $T = T_2$. By CAP, $C(T_2)$ is not a subset of T_1. If $i \neq 2$ implies $[C(T_2)$ not a subset of T_i or $C(T_2) = C(T_i)]$, let $F(Y) = C(T_2)$. If not there exists T_3 say such that $C(T_2) \subset T_3$, $C(T_2) \neq C(T_3)$. Repeat the above procedure for $C(T_3)$. By CAP, and the finiteness of $[X]$, there exists $C(T_i)$ such that for $j \neq i$, either $C(T_i) = C(T_j)$ or $C(T_i)$ not a subset of T_j. Let $F(Y) = C(T_i)$.

If there does not exist any $T \in [X]$ such that $[T] \subset Y$, then put $F(Y) = \phi$.

Now consider $S_1, \dots, S_k$ such that

$$F([X]) = C(X) = S_1 ,$$
$$F([x] \setminus \{S_1\}) = S_2 ,$$
$$F([X] \setminus \{S_1, S_2\}) = S_3 ,$$
$$\vdots$$
$$F([X] \setminus \{S_1, S_2, \dots , S_k\}) = \phi ,$$

where k is the first positive integer to satisfy the above property.

Since C satisfies Outcasting, F satisfies the following property: $[S] \subset Y \subset [X]$, $F(Y) \in [S]$ implies $F([S]) = C(S) = F(Y)$. The reasoning goes thus:

If $[S]$ is not maximal in Y, then there exists $[S']$ maximal in Y such that $S \subset\subset S'$, $C(S') \subset S$. Thus by Outcasting $C(S') = C(S)$. If $F(Y) \neq C(S')$, then there exists $[T]$ maximal in Y such that $F(Y) = C(T) \subset S \subset S'$, contradicting definition of $F(Y)$. Hence we get the assertion.

Let

$$f(S_i) = k - i + 1 \quad \forall i = 1, \dots , k$$

and

$$f(S) = -1 \text{ if } S \in [X] \setminus \{S_1, \dots , S_k\} .$$

Thus $f : [X] \to \Re$ is well defined.

Let $S \in [X]$. Clearly $S \subset X$. If $S_1 \subset S$, then by O,

$C(S) = S_1 = \arg\max_{T \subset S, T \in [X]} [f(T)]$.

If S_1 is not a subset of S, then $[S] \subset [X] \setminus \{S_1\}$.

If $F([X] \setminus \{S_1\}) \in [S]$, then $F([X] \setminus \{S_1\}) = F([S]) = C(S)$.

Therefore $C(S) = S_2 = \arg\max_{T \subset S, T \in [X]} [f(T)]$.

If S_2 is not a subset of S, then $[S] \subset [X] \setminus \{S_1, S_2\}$ and we repeat the above process again. At each new step we need to undertake, because S_{i-1} is not a subset of S, either $S_i \subset S$, so that

$$C(S) = S_i = \underset{T \subset S, T \in [X]}{\arg\max} [f(T)] ,$$

(since $[S] \subset [X] \setminus \{S_1, S_2, \dots , S_{i-1}\}$) or S_i is not a subset of S.

Since $[X]$ is finite, there exists 'i' such that S_{i-1} is not a subset of S and $S_i \subset S$. Thus $C(S) = S_i = \arg\max_{T \subset S, T \in [X]} [f(T)]$. Q.E.D.

Note. In the above iterative construction of $S_1, S_2, \dots , S_k$, when S_i is removed, then only those $[S]$'s are affected for which $[S] \subset [X] \setminus \{S_1, S_2, \dots , S_{i-1}\}$ and $S_i \in [S]$. But then, by the construction of F, $C(S) = S_i$.

EXAMPLE 3. Let $X = \{x, y\}$, $C(X) = \{x\}$, $C(\{a\}) = \{a\}\ \forall a \in X$. $[X] = \{\{x, y\}, \{x\}, \{y\}\}$. Thus $F[X] = \{x\}$, $F(\{\{x, y\}, \{y\}\}) = \{y\}$, $F(\{\{x, y\}\}) = \phi$.

EXAMPLE 4. O does not imply CAP: Let $X = \{x, y, z\}$, let $C(X) = X$, $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, $C(\{x, z\}) = \{z\}$, $C(\{a\}) = \{a\}\ \forall a \in X$. C satisfies O vacuously. However, let $S_1 = \{x, y\}$, $S_2 = \{x, z\}$ and $S_3 = \{y, z\}$. Then $C(S_1) \subset S_2$, $C(S_2) \subset S_3$, $C(S_3) \subset S_1$. Further $C(S_i) \neq C(S_j)$ for $i \neq j$. Thus C violates CAP.

EXAMPLE 5. A batch choice function need not satisfy PI: Let $X = \{x, y, z\}$. Define $f : [X] \to \mathfrak{R}$ as follows: $f(X) = 0$, $f(\{x, y\}) = 5$, $f(\{x, z\}) = f(\{y, z\}) = 1$, $f(\{x\}) = 2$, $f(\{y\}) = 4$, $f(\{z\}) = 3$. Let C be the batch choice function generated by f. Thus, $C(X) = C(\{x, y\}) = \{x, y\}$, $C(\{x, z\}) = \{z\}$, $C(\{y, z\}) = \{y\}$, $C(\{a\}) = \{a\}\ \forall a \in X$. Let $S = \{x, z\}$ and $T = \{y, z\}$. Thus, $S \cup T = X$. $C(X) = \{x, y\} \neq \{y\} = C(C(S) \cup C(T))$.

It is instructive to note that batch choice functions satisfy Outcasting and hence the Superset property. However, as shown here, we require the stronger choice acyclicity property to uniquely characterise all batch choice functions. Batch choice functions are obtained by maximising a utility function defined on subsets of a given set. The decision maker is now in a position to compare not only alternatives (which are singletons) but finite sets of alternatives.

The choice acyclicity property says that it is not possible to find an arrangement of a collection of feasible sets (all distinct) such that the corresponding sets of chosen alternatives are also all distinct and the chosen alternatives from a set are contained in the succeeding one, so as to form a cycle.

It is worthwhile to consider the analogue of the weak ordinality result due to Deb [1983] in the present context. Towards that goal we formulate the following property:

Weak Choice Acyclicity Property (W.CAP): For all $S \in [X]$, there exists a non-empty subset $T(S) \subset C(S)$ for which the following holds: it is not possible to find a positive integer k and sets $S_1, \ldots, S_k$ in $[X]$ such that $T(S_i) \neq T(S_j)$ for some i and j and $T(S_i) \subset S_{i+1}$, $i = 1, \ldots, k-1$, with $T(S_k) \subset S_1$.

The proof of the following theorem is analogous to the proof of Theorem 4.

THEOREM 5. *Suppose X is finite. Then there exists a function $f : [X] \to \mathfrak{R}$ satisfying*

(a) $\{T \subset S / f(T) \geq f(T')\ \forall T' \subset S\}$ *is a singleton* $\forall S \in [X]$,

(b) $\arg\max_{T \subset S, T \in [X]} [f(T)] \subset C(S)\ \forall S \in [X]$,

if and only if C satisfies W.CAP.

APPENDIX

In this appendix we provide a simple proof of Theorem 2.10 in Deb [1983] in the case where the universal set is finite. Our proof does not require the use of Zorn's Lemma, which has been used in Deb [1983] to prove Theorem 2.10.

Let X be a finite set of alternatives. A choice function C is said to satisfy Property (a*) if $\forall S \in [X]$, there exists $x_o \in S$ such that $T \in [X]$, $T \subset S$, $x_o \in T$ implies $x_o \in C(T)$.

It is easy to see that this x_o (which depends on S) must belong to $C(S)$. [Simply take $T = S$ in the definition.]

THEOREM. *Given a choice function C, there exists a function* $\psi : X \to \Re$ *such that* $\forall S \in [X]$,

$$\{x \in S/\psi(x) \geq \psi(y) \quad \forall y \in S\} \subset C(S) \tag{$*$}$$

if and only if C satisfies Property (a*).

Proof. Let C be a choice function satisfying $(*)$. Given $S \in [X]$, pick $x_o \in \{x \in S/\psi(x) \geq \psi(y)\ \forall y \in S\}$. Then $T \in [X]$, $T \subset S$, $x_o \in T$ implies $\psi(x_o) \geq \psi(y)\ \forall y \in T$.

Thus $x_o \in C(T)$.

Now suppose C satisfies Property (a*).

Let $x_1 \in C(X)$ satisfy the conditions of Property (a*) for X.

Let $x_2 \in C(X \setminus \{x_1\})$ satisfy the conditions of Property (a*) for $X \setminus \{x_1\}$.

Having selected $x_1, x_2, \ldots, x_{i-1}$ choose $x_i \in C(X \setminus \{x_1, \ldots, x_{i-1}\})$ satisfying the conditions of Property (a*) for $X \setminus \{x_1, \ldots, x_{i-1}\}$.

Since X is finite, $X = \{x_1, x_2, \ldots, x_s\}$ for some positive integer 's'.

Define, $\psi(x_i) = s - i + 1, i = 1, \ldots, s$.

Let $S = \{y_1, \ldots, y_m\} \in [X]$. Assume that the elements of S are ordered in such a way that $y_i = x_k$, $y_j = x_n$ and $i < j$ implies $k < n$. Suppose $y_1 = x_h$.

Now, $\{x \in S/\psi(x) \geq \psi(y)\ \forall y \in S\} = \{x_h\}$.

Further $S \subset X \setminus \{x_1, x_2, \ldots, x_{h-1}\}$ and $x_h \in S$.

By Property (a*) and construction of x_h, $x_h \in C(S)$.

Therefore, $\{x \in S/\psi(x) \geq \psi(y)\ \forall y \in S\} \subset C(S)$. Q.E.D.

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