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DYNAMIC INEFFICIENCY IN A WORLD ENDING WITH PROBABILITY ONE

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Abstract: The equivalence of standard overlapping-generations models and overlapping-generations models where the world ends after each period with positive probability is formally established for a very general class of models. This shows that even in a model where the world ends with probability one in finite time, there may be dynamically inefficient competitive equilibria. Considering a simple stationary OG-economy, it is shown that introducing a possible end of the world after each period generally raises equilibrium interest rates. It is argued therefore that, albeit possible, dynamic inefficiency is less relevant if one takes into account the possible end of the world.

JEL-classification: D91, D51.
Key words: overlapping generations, end of the world.

1. INTRODUCTION

In order to deal with intertemporal allocation problems, two classes of models are widely used. The first one consists of Arrow–Debreu economies with finite numbers of consumers and commodities. Under the assumption of perfect competition, equilibria of these economies are always Pareto-optimal. The second class of models embraces overlapping generations (OG-) economies with an unbounded time horizon (Samuelson 1958). In these economies, competitive equilibria need not be Pareto-optimal. As has become clear by now, this inefficiency is due to the assumption of a double infinity of consumers and commodities (cf Shell (1971) and Geanakoplos (1987)). An overlapping generations model with a bounded time horizon, however, is analytically equivalent to an Arrow–Debreu model; in particular, all equilibria are Pareto-optimal.

All studies dealing with the problem of Pareto-optimality in intertemporal models[1] have concentrated on the two polar cases of

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[1] For a modern treatment and an overview, cf. Homburg (1992).

—economies with a bounded horizon, or
—economies with an infinite horizon.

The real world, however, seems to be somewhere in between: No precise date of termination is known. Yet science suggests that the earth will definitely not survive infinitely, although it is uncertain when it ends. In view of this, the descriptive power of the OG-model seems to be limited. Its implications for applied issues such as public pension systems, public debt or monetary policy appear questionable since they mostly rely on the assumption of an infinite time horizon. A way to enhance the realism of OG-models is to assume that after each period the world ends with some probability which is positive but less than one. In this case, the question arises whether or not dynamic inefficiencies may occur in an economy with a stochastic lifetime modelled this way. More generally, it is interesting to know whether the results obtained for the OG-model carry over to models which have been modified such that the end of the world is possible after each period.

In this respect, a common conjecture by most economists who work with the OG-model[2] is that one can simply re-interpret the OG-model as a model with a possible end of the world without changing the formal structure. The present paper confirms this conjecture. Hence, the reader may not be surprised by its conclusion. However, to our knowledge, there is no formal proof of it as yet, and we feel that it is sufficiently important in order to deserve a proof. In doing so, the paper examines a very general setting which allows for varying numbers of consumers and commodities per period, time-dependent probabilities for the world's survival, and preferences which do not satisfy the von Neumann-Morgenstern hypothesis. We proceed by constructing a one-to-one map between the set of pure exchange OG-models in which the world ends after each period with some probability and the set of pure exchange OG-models in which the world continues with certainty after each period. Using this map, it becomes obvious that any economy in the first set is formally equivalent to an economy in the second set and vice versa. Considering any economy with a possible end of the world and its image economy which goes on with certainty forever, Theorem 1 states that the sets of competitive equilibria in both economies are equivalent. Completing the analysis, Theorem 2 then shows that the same holds true for the sets of Pareto-optimal allocations.

In the second part of the paper, we go beyond this general result and discuss some of the economic consequences of a possible end of the world. We examine the set of stationary equilibria of Samuelson's original model with one good and one consumer per period. If a possible end of the world is introduced, the interest rate generally increases because the return on savings is uncertain. For given interest rates, consumers would reduce their savings, if they suddenly notice that the world may end soon. In order to restore equilibrium, the interest rate must increase. This is best illustrated for

[2] As an example, we may cite McCandless and Wallace (1991, p. 6–7): "Even if no one believes that time will go infinitely into the future (...), the fact that this date is far into the future and is unknown may be sufficient to allow infinity to be a good approximation of that future, unknown end point."

the special case in which the risk of the world's end has the same effect as an additional discounting of future utilities. However, there also exists an extreme case where the consumers' behavior does not change at all, if they take the finiteness of the world into account. On the other end of the scale, there are economies for which the inefficient stationary equilibrium vanishes once the probability for the world's end is sufficiently high.

In an OG-model which ends after every period with the same positive probability, the probability of an actually infinite stream of available endowments is zero. Nevertheless, the model exhibits the characteristic features of the infinite model and not the finite one. However, it is not the quantity of goods but the total value of them which is relevant for efficiency. Since a possible end of the world generally increases the interest rate, it tends to reduce the value of endowments. Hence, inefficient equilibria become less likely. However, if consumers are very patient, this is not sufficient to eliminate inefficient equilibria. In this case, the consumers' demand for future goods will be too strong although it is uncertain whether these goods can be consumed at all.

These results are related to the work by Blanchard (1985) and Blanchard and Fischer (1989, p. 115–126). There, a continous time model is analyzed where each agent faces a constant probability of death per unit of time. Also in that model, the risk of losing one's saving through death increases the interest rate and reduces the scope of dynamic inefficiency. However, in Blanchard's model, individual lifetimes are uncertain, while in the present paper, the whole economy may end. In Blanchard's model, economic activity continues with certainty at any point in time. No matter who died and who survived from older generations, new cohorts are born at every point in time. In the present paper, if the world ends, all old agents die and no new ones are born any more.

The paper is organized as follows. In section 2, we define the two classes of OG-economies. Section 3 specifies the map between both sets of economies and proves the two main theorems. Following this, the stationary one-good, one-consumer model is discussed. The paper concludes by summarizing the results. Proofs are relegated to the appendix.

2. TWO CLASSES OF ECONOMIES

The first class $\mathcal{E}$ defined in subsection 2.1 contains standard OG-economies. With probability one there exists no end of the world.[3] The second class $\mathcal{F}$ which is described in subsection 2.2 consists of OG-economies such that every period is associated with a possibility of the world ending after this period, while not precluding its continuation with certainty. An economy $F \in \mathcal{F}$ is a special case of a stochastic overlapping generations (SOG-)economy with an incomplete markets system as analyzed by

[3] This class of models has been analyzed in a systematic way by Balasko et al. (1980), Balasko and Shell (1980) and Balasko and Shell (1981).

Schmachtenberg (1990).[4]

2.1 Never Ending Overlapping Generations Economies

In an OG-economy $E \in \mathcal{E}$, with time indexed by $t=1, 2, \ldots$, the set of consumers entering the economy in period $t \geq 1$ is denoted by H_t, the set of old consumers in period 1 is H_0, and the set of commodities available in period $t=1, 2, \ldots$, is K_t. All these sets are assumed to be finite and nonempty. To simplify the notation, H_t, K_t etc. also refer to the cardinality of the respective sets. The consumption set of a consumer $h \in H_t$ is $X^h \subset R_+^{K_t} \times R_+^{K_{t+1}}$, while an old consumer $h \in H_0$ has a consumption set $X^h \subset R_+^{K_1}$. For all $h \in H_t$, $t=0, 1, 2, \ldots$, the utility function of consumer h is denoted by $u^h : X^h \rightarrow R$ and the endowment vector by $e^h \in X^h$. The system of commodity prices is given by $p = \{p_t\}_{t=1}^{\infty}$, with $p_t \in R_+^{K_t}$ for all $t=1, 2, \ldots$. In every period $t=1, 2, \ldots$, there exists one asset priced at one unit of account and yielding a return of one unit of account in period $t+1$. Hence, the asset is nominal and serves as numéraire. Consumers $h \in H_0$ obtain a transfer of $\alpha^h \in R$ units of account. The government is assumed to maintain the aggregate net debt of $A = \sum_{h \in H_0} \alpha^h$ forever. Thus, we do not consider monetary or debt policies other than a lump sum transfer in the first period. Let consumption bundles be denoted by $x^h = (x^{0h}, x^{1h}) \in X^h$, if $h \in H_t$, $t \geq 1$, and $x^h = x^{1h} \in X^h$, if $h \in H_0$. Similarly, asset holdings of a young consumer $h \in H_t$, $t \geq 1$, are given by $a^h \in R$. Then the budget set $\phi^h(p)$ of consumer $h \in H_t$, $t \geq 1$, consists of all $(x^h, a^h) \in X^h \times R$ satisfying

$$p_t(x^{0h} - e^{0h}) + a^h = 0 \tag{1}$$

$$p_{t+1}(x^{1h} - e^{1h}) = a^h. \tag{2}$$

Notice that there is no restriction on asset holdings. Thus, the market system is complete. The budget set $\phi^h(p)$ of a consumer $h \in H_0$ consists of all $x^h \in X^h$ such that $p_1(x^h - e^h) = \alpha^h$. An economy $E \in \mathcal{E}$ is defined by $E = (\{K_t\}_{t=1}^{\infty}, \{H_t\}_{t=0}^{\infty}, \{(X^h, u^h, e^h)_{h \in H_t}\}_{t=0}^{\infty})$, leaving the transfer policy unspecified. Fixing the vector $\alpha = (\alpha^h)_{h \in H_0} \in R_+^{H_0}$, we have the economy (E, α) with the transfer policy α. In the following, let $x := \{(x^h)_{h \in H_t}\}_{t=0}^{\infty}$ and $a := \{(a^h)_{h \in H_t}\}_{t=1}^{\infty}$ refer to an array of consumption bundles and asset holdings, respectively, and define $H := \cup_{t=0}^{\infty} H_t$.

DEFINITION 1. (i) *(x, a) is a feasible allocation in (E, α), if for all $h \in H$, $x^h \in X^h$, and for all $t=1, 2, \ldots$:*

$$\sum_{h \in H_t} (x^{0h} - e^{0h}) + \sum_{h \in H_{t-1}} (x^{1h} - e^{1h}) = 0 \tag{3}$$

$$\sum_{h \in H_t} a^h = A. \tag{4}$$

(ii) *x is called a feasible allocation in E, if for all $h \in H$, $x^h \in X^h$ and* (3) *holds for all $t=1, 2, \ldots$*

[4] In the remainder of the paper, we will shortly refer to economies in $\mathcal{E}$ as OG-economies, and to economies in $\mathcal{F}$ as SOG-economies.

DEFINITION 2. *Let p be a price system and* (x, a) *a feasible allocation in* (E, α). *Then* (x, a, p) *is a competitive equilibrium of* (E, α), *if for all* $h \in H_t$, $t=1, 2, \ldots$, *(resp.* $h \in H_0$), (x^h, a^h) *(resp.* x^h) *maximizes* u^h *over the budget set* $\phi^h(p)$.

DEFINITION 3. *Let* $\overline{x} = \{(\overline{x}^h)_{h \in H_t}\}_{t=0}^{\infty}$ *and x both be feasible allocations in E. Then* $\overline{x}$ *Pareto-dominates x in E, if for all* $h \in H$:

$$u^h(\overline{x}^h) \geq u^h(x^h), \tag{5}$$

where at least one inequality is strict. If there is no allocation which Pareto-dominates x in E, x is called Pareto-optimal in E.

2.2 Overlapping Generations Economies with a Possible End of the World

An SOG-economy $F \in \mathcal{F}$ consists of an infinite sequence of time periods $t=1, 2, \ldots$, a nonempty, finite set of commodities L_t for each $t=1, 2, \ldots$, a nonempty, finite set of consumers I_t for each $t=1, 2, \ldots$, and one asset in every period $t=1, 2, \ldots$. Again, L_t and I_t also identify the numbers of commodities and of consumers in period t. The stochastic structure of the economy is given by an event tree which is depicted in Fig. 1.

The first node of the tree represents period 1. Then, after every period $t=1, 2, \ldots$ two states of nature are possible: either the world ends after t or it continues in period $t+1$. Once the world's end has occurred, economic activity stops. Correspondingly, the elements of the sets I_t and L_t are interpreted as consumers and commodities who enter the economy in the beginning of period t conditional on the event that the world still exists. Each consumer $i \in I_t$, $t=0, 1, 2, \ldots$, is characterized by her consumption set Y^i, her endowment vector $f^i \in Y^i$, and her utility function $v^i : Y^i \to R$. We assume $Y^i \subset R_+^{L_t} \times R_+^{L_{t+1}} \times \{\underbrace{(0, \ldots, 0)}_{L_{t+1} \text{ zeroes}}\}$, if $i \in I_t$, $t \geq 1$, and $Y^i \subset R_+^{L_1}$, if $i \in I_0$. Given the consumption vector $y^i = (y^{0i}, y^{1i}, 0) \in Y^i, i \in I_t, t \geq 1$, $y^{0i} \in R_+^{L_t}$ represents youth consumption, $y^{1i} \in R_+^{L_{t+1}}$ stands for old age consumption conditional on the event that the world still exists in period $t+1$, and $0 \in R_+^{L_{t+1}}$ is old age consumption conditional on

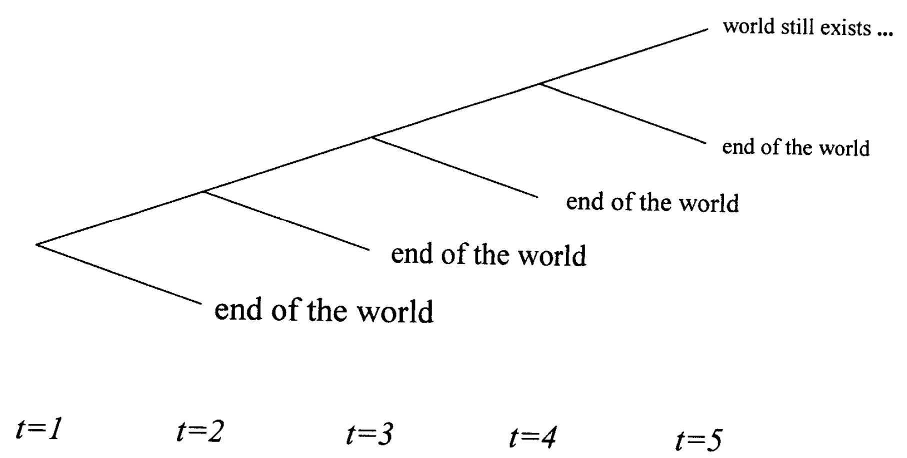

Fig. 1. The event tree in an economy with a possible end of the world.

the event that the world has perished after period t. The same decomposition can be applied to the endowment vector of a consumer $i \in I_t, t \geq 1$, yielding $f^i = (f^{0i}, f^{1i}, 0)$.[5]

The prices of commodities in period $t=1, 2, \ldots$ are denoted by $q_t \in R_+^{L_t}$, and the price system is $q = \{q_t\}_{t=1}^{\infty}$. The asset of period $t=1, 2, \ldots$ is traded only if the world exists in period t. It is priced at one unit of account in period t and yields one unit of account in period $t+1$, if the world still exists, and 0 units of account after the world's end. The market system is incomplete because there are two states of nature but only one asset. The quantity of the asset held by an agent $i \in I_t$, $t=1, 2, \ldots$ is denoted by $b^i \in R$ and her budget set $\chi^i(q)$ consists of all $(y^i, b^i) \in Y^i \times R$ such that

$$q_t(y^{0i} - f^{0i}) + b^i = 0 \tag{6}$$

$$q_{t+1}(y^{1i} - f^{1i}) = b^i. \tag{7}$$

A consumer $i \in I_0$ obtains a transfer of $\beta^i \in R$ units of account from the government. We assume that the government maintains a net position $\sum_{i \in I_0} \beta^i = B$ forever. The budget set $\chi^i(q)$ of a consumer $i \in I_0$ consists of all consumption bundles $y^i \in Y^i$ satisfying $q_1(y^i - f^i) = \beta^i$. The set $\mathcal{F}$ consists of economies $F = (\{L_t\}_{t=1}^{\infty}, \{I_t\}_{t=0}^{\infty}, \{(Y^i, v^i, f^i)_{i \in I_t}\}_{t=0}^{\infty})$. Appending a vector of transfers $\beta = (\beta_i)_{i \in I_0}$ to a given economy $F \in \mathcal{F}$ yields an economy (F, β). As in the preceding section, let $y := \{(y^i)_{i \in I_t}\}_{t=0}^{\infty}$ and $b := \{(b^i)_{i \in I_t}\}_{t=1}^{\infty}$ be arrays of consumption bundles and of asset holdings, respectively. Finally, define $I := \cup_{t=0}^{\infty} I_t$.

DEFINITION 4. (i) *(y, b) is a feasible allocation in (F, β) if for all $i \in I$, $y^i \in Y^i$ and for all $t=1, 2, \ldots$:*

$$\sum_{i \in I_t} (y^{0i} - f^{0i}) + \sum_{i \in I_{t-1}} (y^{1i} - f^{1i}) = 0 \tag{8}$$

$$\sum_{i \in I_t} b^i = B. \tag{9}$$

(ii) *y is a feasible allocation in F if for all $i \in I$, $y^i \in Y^i$ and* (8) *holds for all $t=1,2,\ldots$*

DEFINITION 5. *Let q be a price system and (y, b) a feasible allocation in (F, β). Then (y, b, q) is a competitive equilibrium of (F, β) if for all $i \in I_t$, $t=1,2,\ldots$, (resp. $i \in I_0$), (y^i, b^i) (resp. y^i) maximizes v^i over the budget set $\chi^i(q)$.*

DEFINITION 6. *Let $\bar{y} = \{(\bar{y}^i)_{i \in I_t}\}_{t=0}^{\infty}$ and y both be feasible allocations in F. Then $\bar{y}$ Pareto-dominates y in F if for all $i \in I$:*

$$v^i(\bar{y}^i) \geq v^i(y^i), \tag{10}$$

where at least one inequality is strict. If there is no allocation which Pareto-dominates

[5] It may be convenient to drop the zero entries in consumption bundles. We prefer not to do so in order to highlight the correspondence to general SOG-models, and, more generally, to models of incomplete financial markets. The aim of the present paper is to show that two sorts of models with different interpretations are formally equivalent. This only makes sense if in either case, the notation follows the intended interpretation as close as possible.

y in F, then y is called Pareto-optimal in F.

An interpretation of a possible equilibrium in this framework focusses on the characteristics of trade: A young consumer in period t may purchase b^i units of the asset with the proceeds of her net sales of period t's goods, which are worth $q_t(f^{0i} - y^{0i})$ units of account (cf. (6)). In period $t+1$, if the world still exists, she uses the returns of her asset portfolio (b^i units of account), in order to pay for her excess demand for goods which are available in period $t+1$, which costs $q_{t+1}(y^{1i} - f^{1i})$ (cf. (7)). In the other state of nature, after the world's end, the savings are lost since the asset return is zero. It is unnecessary to specify a budget constraint for this state of nature since the consumer has no income and is not allowed to consume positive quantities. When analyzing the consumer's savings decision, observe that the utility function v^i represents both the consumer's time preference and her behavior towards risk. The way she trades off the different states of nature against one another—for example, by using subjective probabilities—is incorporated in the functional form of v^i. Preferences which satisfy the expected utility hypothesis are included as a special case. For example, assume that consumer $i \in I_t$, $t \geq 1$, has a von-Neumann-Morgenstern utility function $w^i : R_+^{L_t} \times R_+^{L_{t+1}} \to R$ defined on consumption bundles available with certainty and that she believes that the world survives until period $t+1$ with probability $\pi \in (0, 1)$. Then, her expected utility is $v^i(y^i) = \pi w^i(y^{0i}, y^{1i}) + (1 - \pi) w^i(y^{0i}, 0)$.[6]

3. A MAP BETWEEN BOTH TYPES OF ECONOMIES

We now introduce a map T: $\mathcal{F} \to \mathcal{E}$ in order to show that both models are essentially the same. This map is defined in several steps which describe how to find the image economy $T(F) \in \mathcal{E}$ for any given economy $F \in \mathcal{F}$. First, the numbers of commodities and consumers in some economy F are mapped to the respective numbers in the image economy $T(F)$ by the maps K_t: $\{1, 2, \dots\} \to \{1, 2, \dots\}$, for all $t = 1, 2, \dots$, and H_t: $\{1, 2, \dots\} \to \{1, 2, \dots\}$ for all $t = 0, 1, 2, \dots$ $K_t(.)$ and $H_t(.)$ only depend on the number of commodities and consumers in period t. These maps are defined as the identity map for all t, i.e., in every period the economy $T(F)$ entails the same numbers of commodities and consumers as the economy F. The notation $H_t(I_t)$ and $K_t(L_t)$ is used both for the number and the set of consumers resp. commodities in $T(F)$. We also define maps between the names of the commodities and the consumers in both economies denoted by $k(l)$ and $h(i)$. Since in any period t, there are as many commodities resp. consumers in the economy $T(F)$ as in the economy F, for all $k \in K_t(L_t)$ (resp. $h \in H_t(I_t)$), there is exactly one $l \in L_t$ (resp. $i \in I_t$) such that $k = k(l)$ (resp. $h = h(i)$).

In a second step, the consumption sets of all consumers in the economy $T(F)$ are defined. For a consumer who belongs to the oldest generation in $T(F)$, the consumption set is the same as the consumption set of the corresponding consumer in F, i.e. if

[6] Quite clearly, this special case in which the survival probability of the world is time-invariant has motivated the title of the present paper. Since π is strictly less than 1, the probability that the world ends at some finite time is 1, or equivalently, the probability that it never ends is zero.

$h=h(i)$ and $h \in H_0(I_0)$, then $X^h=X^{h(i)}=Y^i$. For a consumer in $T(F)$ who belongs to a later generation $t \geq 1$, the consumption set is obtained by projecting the consumption set of the corresponding consumer in F on its first $L_t + L_{t+1}$ components. Formally, for all $h \in H_t(I_t), t \geq 1$, if $h = h(i)$, we define:

$$X^h = X^{h(i)} = \{x \in R_+^{L_t} \times R_+^{L_{t+1}} | (x, \underbrace{0,...,0}_{L_{t+1} \text{ zeroes}}) \in Y^i\}. \tag{11}$$

Finally, the utility functions and the endowments of consumers in $T(F)$ are obtained as follows. For the oldest consumers, these objects are again the same as in the economy F, i.e. if $h=h(i)$ and $h \in H_0(I_0)$, then $e^h=e^{h(i)}=f^i$ and $u^h(x) = u^{h(i)}(x) = v^i(x)$ for all $x \in X^h$. The endowment $e^h \in X^h$ of a consumer of generation $t \geq 1$ consists of the first L_t+L_{t+1} components of the corresponding consumer's endowment, i.e. for all $h \in H_t(I_t)$ with $t \geq 1$, $e^h = e^{h(i)} = (f^{0i}, f^{1i})$ if $h=h(i)$. The utility function $u^h : X^h \to R$ of such a consumer is defined by

$$u^h(x) = u^{h(i)}(x) = v^i(x, \underbrace{0,...,0}_{L_{t+1} \text{ zeroes}}) \tag{12}$$

for all $x \in X^h$. This completes the description of the image economy $T(F)$. In every step, the map T either is equal to the identity map, or consists of dropping some components of commodity bundles which have to be zero. Since this can be done in exactly one way, there is exactly one image economy $T(F)$ to each $F \in \mathcal{F}$.

We only sketch the inverse of the map T which associates an economy $T^{-1}(E) \in \mathcal{F}$ to each economy in $E \in \mathcal{E}$. Commodities and households in the SOG-economy again appear in the same numbers at each date, and are identified with each other by $i(h)$ and $l(k)$. Also the consumption sets and the utility functions of the oldest consumers in the economy $T^{-1}(E)$ are obtained from those of E by the identity map. In order to find the consumption sets of consumers born in $t \geq 1$, one has to inverse the operation of dropping $K_{t+1}=L_{t+1}$ zeroes. Hence, one has to append $K_{t+1}=L_{t+1}$ zeroes. Since for a given economy $E \in \mathcal{E}$, this number is given for every t, there is only one way to do this. Finally, the utility function of the individual $i(h)$ in the SOG-economy is found by defing $v^{i(h)}$ through (12). Thus, at all steps, also the inverse map T^{-1} associates exactly one OG-economy to each SOG-economy. Hence, T is one-to-one.

Given the map T, we can also construct a one-to-one map between allocations and price systems in the two economies F and $T(F)$. For any allocation $(y, b) = (\{(y^i)_{i \in I_t}\}_{t=0}^{\infty}, \{(b^i)_{i \in I_t}\}_{t=1}^{\infty})$ and any price system $q = \{q_t\}_{t=0}^{\infty}$ in F, the allocation $(x(y), a(b))$ and the price system $p(q)$ in $T(F)$ are defined as follows: for all $h \in H_0(I_0)$, if $h=h(i)$, $x^h(y)=y^i$ and for all $t=1, 2, \ldots$: $p_t(q)=q_t$ and for all $h \in H_t(I_t)$, if $h=h(i)$, $x^h(y)=(y^{0i}, y^{1i})$, $a^h(b)=b^i$. The theorems state that the sets of equilibrium allocations and the sets of Pareto-optimal allocations in both economies are equivalent. The proofs of both theorems are given in the appendix.

THEOREM 1. *For all* $F = (\{L_t\}_{t=1}^{\infty}, \{I_t\}_{t=0}^{\infty}, \{(Y^i, v^i, f^i)_{i \in I_t})\}_{t=0}^{\infty}) \in \mathcal{F}$, *for all* $\beta \in R^{I_0}$, *for all* (y, b) *and all* q: (y, b, q) *is a competitive equilibrium of* (F, β), *if and only if*

$(x(y), a(b), p(q))$ is a competitive equilibrium of $(T(F), \alpha)$, where for all $h \in H_0(I_0)$, $\alpha^h = \beta^i$ if $h = h(i)$.

THEOREM 2. *For all $F = (\{L_t\}_{t=1}^{\infty}, \{I_t\}_{t=0}^{\infty}, \{(Y^i, v^i, f^i)_{i \in I_t})\}_{t=0}^{\infty}) \in \mathcal{F}$, and for all y, y is Pareto-optimal in F if and only if $x(y)$ is Pareto-optimal in $T(F)$.*

To illustrate the meaning of the map T, we explain what it *does not*. A different way to associate a standard OG economy to an SOG-economy with a possible end of the world would be to simply ignore the end of the world. Think of an economy $F \in \mathcal{F}$ where the world survives in each period with a constant probability π, and where preferences satisfy the expected utility hypothesis. Then, the utility function v^i is obtained as the expectation of some von-Neumann-Morgenstern utility function w^i :

$$v^i(y^{0i}, y^{1i}, 0) = \pi w^i(y^{0i}, y^{1i}) + (1 - \pi) w^i(y^{0i}, 0). \tag{13}$$

Now w^i is a utility function which belongs to an economy in $\mathcal{E}$. To associate it to F would amount to setting the survival probability π equal to one, that is, to ignore the possible end of the world. This is not intended with the map T. Thus w^i is not the utility function $u^{h(i)}$ of the individual $h(i)$ in the image economy $T(F)$. That utility is computed exactly like v^i from (13) where one just writes x for y and h for i. Since there is no more end of the world, this formula cannot be interpreted as an expected utility index in $T(F)$. Its mathematical form, however, is the same as in the SOG-economy.

Conversely, one could think of going from a standard OG-economy $E \in \mathcal{E}$ to an economy in $\mathcal{F}$ by simply introducing the possible end of the world with a probability $1 - \pi$. To do this, one might start from a utility function u^h for some consumer h in the standard OG-economy E and compute an expected utility

$$U^h(x^{0h}, x^{1h}, 0) = \pi u^h(x^{0h}, x^{1h}) + (1 - \pi) u^h(x^{0h}, 0).$$

The resulting function U^h is not, however, the utility $v^{i(h)}$ of the consumer $i(h)$ in the image economy $T^{-1}(E)$. According to (12), $v^{i(h)}$ is still given by the original function u^h but it now has to be interpreted differently. It now also describes the consumer's preferences with respect to the risk of the world's ending, not only the intertemporal tradeoff.

Formally, the map T does nothing but change the notation. It moves every symbol one letter backwards in alphabetical order and drops a couple of zeroes from all consumption bundles of generations born from period 1 onwards. We would therefore like to be honest about the mathematical triviality of the results. The point in this construction is that this change of notation is possible, showing that the two sets of economies have exactly the same mathematical structure.

From the point of view of economic interpretation, however, it seems that the two models describe rather different economic environments. Models in $\mathcal{E}$ only deal with an intertemporal allocation problem, while models in $\mathcal{F}$ are additionally subject to uncertainty. The results show that the same formal apparatus can be employed to analyze both issues. Moreover, we conclude that the stochastic structure of models in $\mathcal{F}$ is degenerate, since they can be reduced to models without uncertainty. Finally, an economy in $\mathcal{E}$ has a complete market system. Hence, the theorems also show that it is possible to analyze an SOG-economy in $\mathcal{F}$ as an economy with a complete market system.

4. INTEREST RATES, EFFICIENCY, AND A POSSIBLE END OF THE WORLD

This section employs the theorems derived above in order to examine the changes in the savings behavior and the equilibrium interest rate caused by the introduction of a possible end of the world into an OG-model. This serves to illustrate the respective effects on the existence of stationary competitive equilibria which are not Pareto-optimal. The most simple stationary OG-model $E_0 \in \mathcal{E}$ constitutes the starting point. Upon introduction of the world's end, we obtain an SOG-economy $F_0 \in \mathcal{F}$. We apply the map T to F_0 in order to obtain the equivalent OG-economy $T(F_0) \in \mathcal{E}$. Recall that the economy $T(F_0)$ is *not* the original economy E_0, but rather an economy structured as E_0, with a possible end of the world which is implicitly incorporated in the consumers' preferences. We determine the interest rates of stationary equilibria of $T(F_0)$ and provide a necessary and sufficient condition for the inefficiency of such an equilibrium. In a last step, this condition is compared to the condition under which a stationary, competitive equilibrium in the original economy E_0 is inefficient. The use of $T(F_0)$ instead of F_0 is justified in order to carry out this comparison, since, by Theorem 1, there is always a corresponding equilibrium in F_0 and, by Theorem 2, it is inefficient under the same condition.

4.1 *A Simple Samuelsonian Economy*

Consider the OG-economy $E_0 \in \mathcal{E}$ in which there is one consumer and one commodity per period, no government intervention and stationary consumption sets, endowments and preferences. For all consumers born in periods 1 or later, these are $X = R^2_+$, $e = (e^0, e^1) \in R^2_+$ and $w : R^2_+ \to R$. The oldest consumer's consumption set is R_+, the endowment is $e^1 \in R_+$, and her utility function is simply her consumption in period 1. The function w is assumed to be strictly concave, monotonic and twice continuously differentiable. For $k=0$, 1 and any $x = (x^0, x^1) \in R^2_+$, let $w_k(x^0, x^1) = \partial w(x^0, x^1)/\partial x^k$ and let $S(x) = w_0(x)/w_1(x)$ denote the marginal rate of substitution between old age consumption and youth consumption. Restricting the attention to interior endowment bundles, recall that the autarchic, stationary competitive equilibrium of this economy is inefficient, if and only if $S(e) < 1$.

Now, assume that this model provides an inapproriate description of reality, since it ignores the possibility that the world may end. Instead, the correct model is the economy $F_0 \in \mathcal{F}$ in which for every $t = 1, 2, \ldots$, conditional on the fact that the world is still there in period t, the world will also exist in period $t+1$ with probability $\pi \in (0, 1)$. In F_0, the preferences of every consumer except the oldest one are described by the utility function $v : R^2_+ \times \{0\} \to R$ which is defined by

$$v(x^0, x^1, 0) = \pi w(x^0, x^1) + (1 - \pi) w(x^0, 0) \tag{14}$$

for all $(x^0, x^1) \in R^2_+$. As in E_0, the oldest consumer just cares about her old age consumption. If one applies the map T to this model, one obtains the same endowments as in the original economy E_0. However, in $T(F_0)$, the preferences of consumers born from the first period onwards are given by the utility function $u : R^2_+ \to R$ defined by $u(x^0, x^1) = v(x^0, x^1, 0)$ for all $(x^0, x^1) \in R^2_+$.

It is easy to show that (strict) concavity of w implies (strict) concavity of u. Hence, in the economy $T(F_0)$, autarchy is again a competitive equilibrium which is supported by prices satisfying $p_t/p_{t+1} = Q(e^0, e^1)$, where for all $(x^0, x^1) \in R^2_+$, $Q(x^0, x^1) = u_0(x^0, x^1)/u_1(x^0, x^1)$ denotes the marginal rate of substitution between old age consumption and youth consumption in the economy $T(F_0)$. Using (14) and the definition of u, we have

$$Q(x^0, x^1) = S(x^0, x^1) + \frac{(1-\pi)}{\pi} \frac{w_0(x^0, 0)}{w_1(x^0, x^1)}. \tag{15}$$

Equation (15) shows how the savings decisions differ between the economies E_0 and $T(F_0)$. Clearly, if a consumer becomes aware of the possible end of the world, she requires an interest factor which exceeds the former by the amount $[(1-\pi)w_0(x^0, 0)]/[\pi w_1(x^0, x^1)]$ in order to take an identical savings decision. This reflects the disutility caused by a potential loss of the savings in case the world ends before the consumer grows old. The autarchic equilibrium is inefficient, if and only if $Q(e^0, e^1) < 1$, i.e., if and only if

$$S(e^0, e^1) < 1 - \frac{(1-\pi)}{\pi} \frac{w_0(e^0, 0)}{w_1(e^0, e^1)}. \tag{16}$$

Because the utility function is monotonic, the second term on the r.h.S. of (16) is nonnegative. Hence whenever $S(e^0, e^1) \geq 1$, the autarchic equilibrium in the SOG-economy with a possible end of the world is efficient. Introducing a possible end of the world never introduces inefficiency into an otherwise efficient economy. On the contrary, a possible end of the world may reduce or even eliminate dynamic inefficiency. This is shown by some examples in the following subsections.

4.2 Additively Separable Utility Functions

To specialize the Samuelsonian model further, assume that w is a sum of time-invariant utilities per period, i.e., $w(x^0, x^1) = a(x^0) + a(x^1)$, where $a' > 0, a'' < 0$, and $a(0) = 0$.[7] Here, we have $S(x^0, x^1) = a'(x^0)/a'(x^1)$. The expected utility in the economy with a possible end of the world is

$$u(x^0, x^1) = v(x^0, x^1, 0) \tag{17}$$

$$= \pi[a(x^0) + a(x^1)] + (1-\pi)[a(x^0) + a(0)] \tag{18}$$

[7] If this assumption is replaced by $a(0) = -\infty$, the SOG-economy becomes trivial. As can be seen from (18), for $\pi < 1$, the expected utility is $u(x^0, x^1) = -\infty$, regardless of the consumer's choice. Hence, this specification is not very useful in the context of a possible end of the world. Notice, however, that Theorems 1 and 2 also hold for this specification: The image of such an SOG-economy under T is an OG-economy in which the utility function is a constant.

$$= a(x^0) + \pi a(x^1). \tag{19}$$

Notice that the world's survival probability π enters the utility function in exactly the same way as a discount factor. The derivative $w_0(x^0, x^1) = a'(x^0)$ is independent of x^1 and, therefore, (15) can be simplified to $Q(x^0, x^1) = S(x^0, x^1)/\pi$. In the economy F_0, the consumer's savings are both a medium of intertemporal transfer of purchasing power and a bet on the world's survival. If this is a fair bet, a prize with a present value of $1/\pi$ DM is paid out for every DM invested in case of the world's survival. Since this prize is paid out in the next period, it has to be multiplied with the interest factor yielding a gross rate of return on savings of S/π.

Assume now that there is a total endowment of one unit of the consumption good in every period. In the OG-model, the autarchic equilibrium is inefficient whenever the distribution of this endowment is such that $a'(e^0)/a'(e^1) < 1$, or $e^0 > 1/2 > e^1$. The stationary allocation $(1/2, 1/2)$ is known as the golden-rule allocation of such an economy. Refocussing on an economy in which the world ends with probability $1-\pi$ after each period, consider the pair $(\tilde{x}^0, \tilde{x}^1)$ which maximizes the utility function u under the feasibility constraint $x^0 + x^1 = 1$. This pair is unique since u is strictly concave, and $\pi < 1$ implies $\tilde{x}^0 > 1/2$. If $(\tilde{x}^0, \tilde{x}^1)$ is an interior solution, it is characterized by $a'(\tilde{x}^0)/a'(\tilde{x}^1) = \pi$ (which is equivalent to $Q(\tilde{x}^0, \tilde{x}^1) = 1$) and $\tilde{x}^0 + \tilde{x}^1 = 1$. This pair describes the golden rule allocation of any economy with a utility function u and a total endowment of one unit per period. Figure 2 displays the construction of the golden rule allocation of both economies E_0 and $T(F_0)$ for the case of an interior solution.

In the economy $T(F_0)$, the autarchic equilibrium is inefficient as well, if the initial endowment is sufficiently concentrated in the hands of the young generation. To be precise, whenever $e^0 > \tilde{x}^0$ (and hence, $e^1 = 1 - e^0 < 1 - \tilde{x}^0 = \tilde{x}^1$), it is Pareto-improving to move to the allocation $(\tilde{x}^0, \tilde{x}^1)$ from period one onwards. To verify this claim, observe that the oldest consumer obtains $\tilde{x}^1$ instead of e^1 with certainty, which makes her strictly better off. Correspondingly, the young consumer of the first period obtains only $\tilde{x}^0$ instead of e^0 with certainty. In the next period, this consumer will get $\tilde{x}^1$ if the world still exists then and nothing if it is terminated meanwhile. Thus, in the event that the world ends after period one, she will regret whatever she has done for the old generation, and would rather have kept her initial endowment. On the other hand, if the world survives until period 2, she consumes more than the initial endowment. Ex-ante the reallocation is beneficial for her if her expected utility increases. Now observe that the pair $(\tilde{x}^0, \tilde{x}^1)$ uniquely maximizes the expected utility function u under the constraint $x^0 + x^1 \leq 1$. Since the initial endowment also satisfies this constraint, and since it is not equal to the golden rule allocation, it yields a strictly lower expected utility than the golden-rule pair $(\tilde{x}^0, \tilde{x}^1)$, i.e.

$$u(\tilde{x}^0, \tilde{x}^1) > u(e^0, e^1). \tag{20}$$

Hence, generation 1 is made better off by the reallocation. Finally, the utility of a generation $t \geq 2$ is unaffected if the world ends before period t since it is not born. If the

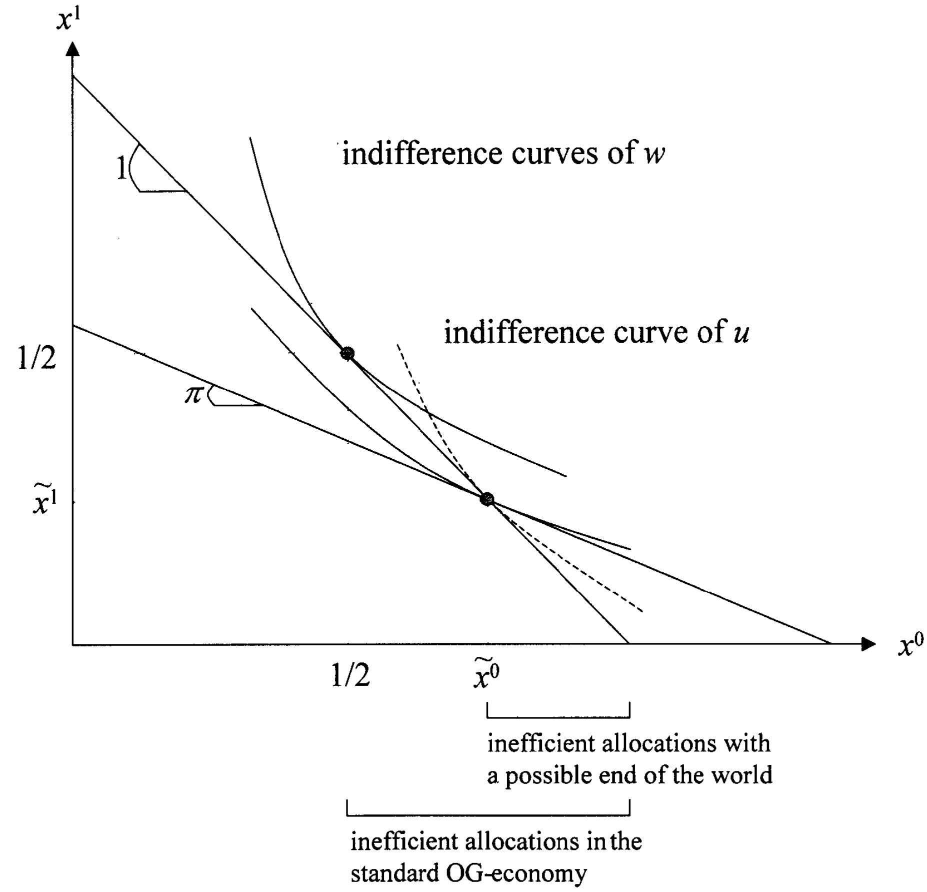

Fig. 2. Inefficient autarchic equilibria in a stationary OG-model with and without a possible end of the world.

world still exists at the beginning of period t, generation t is treated exactly as generation 1. Hence, under the golden-rule allocation, all future generations are also better off than they were with the initial endowment. This shows that the golden-rule allocation is Pareto-superior to autarchy.

As can be seen from figure 2, the interval of endowments e^0 below the golden rule quantity is smaller than in the original OG-economy E_0. Hence, introducing a possible end of the world into the model in a sense reduces the range of endowment distributions which lead to the "Samuelsonian" case of an inefficient autarchic equilibrium. Nevertheless, it does *not* in general rule out this possibility. To conclude, we consider a corner solution $(\tilde{x}^0, \tilde{x}^1) = (1, 0)$ for the golden rule pair which occurs if $\pi \leq a'(1)/a'(0)$. In this case it is impossible to experience an endowment $e^0 > \tilde{x}^0$. Therefore, an inefficient stationary equilibrium does not exist, if a possible end of the world is introduced into such a model.

4.3 Numerical Examples

Assume first $a(x) = \sqrt{x}$. This yields the CES-utility function $w(x^0, x^1) = \sqrt{x^0} + \sqrt{x^1}$, with $S(x^0, x^1) = \sqrt{x^1/x^0}$. If the world survives with probability π, the expected utility is $u(x^0, x^1) = \sqrt{x^0} + \pi\sqrt{x^1}$. The respective marginal rate of substitution is given by $Q(x^0, x^1) = (1/\pi)\sqrt{x^1/x^0}$. Clearly, $\tilde{x}^0 = 1/(1+\pi^2), \tilde{x}^1 = \pi^2/(1+\pi^2)$ and $u(\tilde{x}^0, \tilde{x}^1) = \sqrt{1+\pi^2}$. Thus, if $\pi = 0.75$, it follows $\tilde{x}^0 = 0.64, \tilde{x}^1 = 0.36$ and $u(\tilde{x}^0, \tilde{x}^1) = 1.25$. For initial endowments $e^0 = 0.75 > 0.64$ and $e^1 = 0.25$, the expected utility equals $\sqrt{e^0} + \pi\sqrt{e^1} = \sqrt{3/4} + 3/4\sqrt{1/4} = 1/2(\sqrt{3} + 3/4)$. Using $\sqrt{3} < 7/4$, it follows that $\sqrt{e^0} + \pi\sqrt{e^1} < 1.25$.

As a second example, assume $a(x) = \ln(x+1)$ which yields $u(x^0, x^1) = \ln(x^0+1) + \pi\ln(x^1+1)$, and $Q(x^0, x^1) = (x^1+1)/[\pi(x^0+1)]$. If $\pi > 1/2$, the golden rule allocation in the economy $T(F_0)$ is $(\tilde{x}^0, \tilde{x}^1) = ((2-\pi)/(1+\pi), (2\pi-1)/(1+\pi))$. Despite a possible end of the world, the autarchic equilibrium is inefficient, if $e^0 > (2-\pi)/(1+\pi)$. Notice that this expression decreases in π. Hence, the range of endowment distributions yielding an inefficient autarchic equilibrium becomes smaller as the world's end becomes more likely. If $\pi \leq 1/2$, the golden rule pair is the corner solution $(\tilde{x}^0, \tilde{x}^1) = (1, 0)$, and this range is empty. In this model, introducing a possible end of the world restores efficiency of the autarchic equilibrium for all endowment distributions, if the probability of the world's end is sufficiently high.

4.4 *Time-dependent Survival Probability*

The previous subsections were confined to a very simple stochastic structure. The probability that the world survives was constant in every period. In this subsection, this is generalized by means of two additional examples. So assume now that at date $t = 1, 2, \ldots$, if the world still exists, it will survive until $t+1$ with probability π_t. This transforms the utility function of generation t so as to yield $u^t(x^{0t}, x^{1t}) = a(x^{0t}) + \pi_t a(x^{1t})$. In this subsection we do not vary the endowments. This allows to simplify notation by writing $S = a'(e^0)/a'(e^1)$ for the marginal rate of substitution at the endowment point in the economy without a possible end of the world. After introducing the possible end of the world, autarchy is an equilibrium if interest factors satisfy

$$p_t/p_{t+1} = S/\pi_t \tag{21}$$

in all periods $t = 1, 2, \ldots$

The sequence $\{\pi_t\}_{t=1}^{\infty}$ in principle may take any form. However, it is most instructive to look first at a minor generalization of the model used so far. Assume that the world survives with certainty until some given, fixed period θ. Afterwards it ends with a constant positive probability after each period. Formally, $\pi_t = 1$ for all $t < \theta$, and $\pi_t = \pi$ for $t \geq \theta$. The utility function of generations until $\theta - 1$ remains unchanged after introducing the end of the world, since these households are sure to live their old age. Generations born at date θ or later have to face the possibility that the world ends before they die. Therefore, their second period utility is multiplied by the constant π. Correspondingly, for periods $t = 1, 2, \ldots, \theta - 1$, the equilibrium interest factor is S as in the model without an end of the world. From period $t = \theta$ onwards, it increases to S/π.

Also here, the equilibrium is inefficient if $S/\pi < 1$. A Pareto-improvement is obtained if one keeps the autarchic allocation in all periods until $\theta - 1$, and then replaces endowments by the consumption bundle $(\tilde{x}^0, \tilde{x}^1)$. Thus, one moves to the golden rule allocation corresponding to the economy with a possible end of the world only from period θ onwards.

In order to find a sufficient condition for efficiency, one must use a criterion which is appropriate for non-stationary economies. Such a criterion is provided by Balasko and Shell (1980, proposition 5.3, p. 294).[8] In our simple model with bounded endowments, this result says that the autarchic equilibrium is efficient if present value prices p_t tend to zero as time goes to infinity. Using (21), this means for an arbitrary sequence of π_t:

$$p_1 \lim_{t\to\infty} \prod_{s=1}^{t-1} \left(\frac{\pi_s}{S}\right) = 0. \tag{22}$$

If π_t is one until period $\theta - 1$ and π afterwards, this reduces to

$$\frac{p_1}{S^{\theta-1}} \lim_{t\to\infty} \left(\frac{\pi}{S}\right)^{t-\theta} = 0. \tag{23}$$

Regardless of θ, this is satisfied if and only if $\pi < S = a'(e^0)/a'(e^1)$. This is the same condition for efficiency as was derived for the model where the world's survival is uncertain from the beginning. It is not important at which date the end of the world is possible for the first time. Efficiency only requires that one day, it becomes sufficiently probable.

If π decreases to zero, the model converges to one where the world has a known end after date θ. From the first theorem of welfare economics, such a finite Arrow-Debreu model has only efficient equilibria. The present example shows that efficiency is restored well before this limiting case is reached.

We turn now to the second variant of the model with time-dependent survival probabilities. Assume that in the far future, after every period it is almost sure that the world ends immediately. Formally, let the sequence $\{\pi_t\}_{t=1}^{\infty}$ tend to zero as time approaches infinity. This implies that for any π arbitrarily close to zero, there is some θ such that $\pi_t < \pi$ for all $t \geq \theta$. Hence, the l.h.s. of (22) is less than the l.h.s. of (23). By choosing π smaller than S, one sees that the Balasko-Shell criterion is satisfied in this specification. Hence, if the survival probability of the world converges to zero over time, inefficiency is ruled out. In fact, it is not necessary that this probability converges all the way down to zero. If it converges to some $\pi < S$, then again, from some period θ onwards, all interest factors S/π_t must be greater than one implying (22).

Just as in the previous example, one concludes that in order to ensure efficiency, the end of the world need not come soon. As is usual in OG-economies, it is important how interest rates behave as time tends to infinity. Efficiency is obtained if the possible end of the world raises interest rates in the long run.

4.5 *The End of the World May Be Irrelevant*

Returning to a constant survival probability, we now present a second specification

[8] Cf. also Okuno and Zilcha (1980), theorem 2, p. 802.

for the utility function in the Samuelsonian economy. Assume that old age consumption is an essential commodity. That is, $w(x^0, 0) = 0$ for all x^0. (The Cobb-Doulas utility function $w(x^0, x^1) = x^0 x^1$ provides an example.) Then, $u(x^0, x^1) = \pi w(x^0, x^1)$ and (15) reduces to $Q(x^0, x^1) = S(x^0, x^1)$. The equilibrium in the model with a possible end of the world is inefficient whenever the original equilibrium is. Accounting for finiteness of the world reduces utility, but has no effect on the consumer's behavior. For the consumer, everything is vacuous if the world ends. Therefore, she does not care about the world's end except that thinking of it makes her a little (to be precise, $1/\pi$ times) "less happy".

5. CONCLUSION

The present study has demonstrated the formal equivalence between OG-models, which with certainty extend to the infinite future, and SOG-models, in which the world ends after every period with some probability. For the application of OG-models, this implies that it is not necessary to model the possible end of the world explicitly, although a world which is infinite with certainty seems to be unrealistic. An OG-model can always be interpreted as an SOG-model with a possible end of the world after each period. However, one has to bear in mind that the preferences of the consumers in the SOG-model represent both intertemporal substitution and the way the consumers evaluate the risk of the world's end. Thus, an OG-model in general is equivalent to an SOG-model with a possible end of the world where consumers are more patient than in the OG-economy, but find future goods less valuable because it is uncertain whether they can be consumed at all.

Some special cases have shown that accounting for a possible end of the world typically increases equilibrium interest rates. This implies that dynamically inefficient equilibria are less likely, if the world ends with positive probability after each period. Yet, dynamic inefficiency cannot be excluded on theoretical grounds, since agents may be very patient, or endowments may be rather concentrated in youth, such that the fear of losing ones's savings does not increase the interest rate sufficiently (*i.e.*, beyond the growth rate of the economy). Empirically, however, dynamic inefficiency seems to be less relevant if one thinks of the possibility that the world ends.

APPENDIX

Since feasibility is required both for an equilibrium and a Pareto-optimal allocation, we start by showing that feasible allocations in both economies are equivalent.

LEMMA 1. *For all* $F = (\{L_t\}_{t=1}^{\infty}, \{I_t\}_{t=0}^{\infty}, \{(Y^i, v^i, f^i)_{i \in I_t})\}_{t=0}^{\infty}) \in \mathcal{F}$, *for all* $\beta \in R^{I_0}$, *for all* (y, b): (y, b) *is a feasible allocation of* (F, β) *if and only if (iff)* $(x(y), a(b))$ *is a feasible allocation of* $(T(F), \alpha)$, *where for all* $h \in H_0(I_0), \alpha^h = \beta^i$ if $h = h(i)$.

Proof. According to (3), feasibility in the economy $T(F)$ implies

$$\sum_{h\in H_t(I_t)}(x^{0h}(y) - e^{0h}) + \sum_{h\in H_{t-1}(I_{t-1})}(x^{1h}(y) - e^{1h}) = 0.$$

Using the one-to-one map between individuals in both economies, this can be rewritten as

$$\sum_{h(i)\in H_t(I_t)}(x^{0h(i)}(y) - e^{0h(i)}) + \sum_{h(i)\in H_{t-1}(I_{t-1})}(x^{1h(i)}(y) - e^{1h(i)}) = 0,$$

or, since $H_t(I_t) = I_t$, as $\sum_{i\in I_t}(x^{0h(i)}(y) - e^{0h(i)}) + \sum_{i\in I_{t-1}}(x^{1h(i)}(y) - e^{1h(i)}) = 0$. Inserting the definition of $x(y)$ yields $\sum_{i\in I_t}(y^{0i} - f^{0i}) + \sum_{i\in I_{t-1}}(y^{1i} - f^{1i}) = 0$, which restates (8). Hence, (3) is true in the economy $T(F)$, iff (8) is true in the economy F. According to (4), feasibility in the economy $T(F)$ also requires $\sum_{h\in H_t(I_t)} a^h(b) = \sum_{h\in H_0(I_0)} \alpha^h$ for all $t = 1, 2, \ldots$ Using $\alpha^{h(i)} = \beta^i$ for all $i \in I_0$, the same construction of arguments as used above reveals that this equality is equivalent to $\sum_{i\in I_t} a^{h(i)}(b) = \sum_{h(i)\in H_0(I_0)} \alpha^{h(i)}$ or $\sum_{i\in I_t} b^i = \sum_{i\in I_0} \beta^i$. Hence, (4) is satisfied in $T(F)$, iff (9) is satisfied in F. Q.E.D.

PROOF OF THEOREM 1. From Lemma 1, we only have to deal with optimality. First, consider all generations except the oldest one in both economies and choose an arbitrary $t \geq 1$ and any pair (i, h) of a consumer i in F and a consumer h in $T(F)$ such that $i \in I_t, h \in H_t(I_t)$ and $h = h(i)$. We have to show that, for any such pair, $(x^h(y), a^h(b))$ maximizes u^h over the budget set $\phi^h(p(q))$, iff (y^i, b^i) maximizes v^i over the budget set $\chi^i(q)$. We proceed in two steps, showing first that (y^i, b^i) is in i's budget set, iff $(x^h(y), a^h(b))$ is in h's budget set. In the second step, it will then be shown that there is no other pair of a consumption bundle and a portfolio in i's budget set yielding a higher utility than the presumed equilibrium consumption bundle, iff the same is true for consumer h.

Inserting $f^{0i} = e^{0h}, f^{1i} = e^{1h}, q = p$ and the definition of $(x^h(y), a^h(b))$ into (1) and (2) restates (6) and (7). Together with (11) this implies that $(x^h(y), a^h(b)) \in \phi^h(p(q))$, iff $(y^i, b^i) \in \chi^i(q)$. This completes the first step.

For the second step, it has to be shown that the following two assertions are equivalent: *Assertion* 1: "There is no (z^0, z^1, c) satisfying

$$(z^0, z^1, 0, c) \in Y^i \times R \tag{24}$$

$$v^i(z^0, z^1, 0) > v^i(y^i) \tag{25}$$

$$q_t(z^0 - f^{0i}) + c = 0 \tag{26}$$

$$q_{t+1}(z^1 - f^{1i}) = c\text{''} \tag{27}$$

and *Assertion* 2: "There is no (z^0, z^1, c) satisfying

$$(z^0, z^1, c) \in X^h \times R \tag{28}$$

$$u^h(z^0, z^1) > u^h(x^h(y)) \tag{29}$$

$$p_t(q)(z^0 - e^{0h}) + c = 0 \tag{30}$$

$$p_{t+1}(q)(z^1 - e^{1h}) = c''. \tag{31}$$

By the definition of X^h, (24) is equivalent to (28), the definitions of $x^h(y)$ and of u^h show that (25) is equivalent to (29), and the definitions of $p(q)$ and of e^h show that (26) and (27) are equivalent to (30) and (31), respectively. Hence, the two assertions are equivalent which completes the proof for any pair of consumers from both economies who belong to a generation $t \geq 1$. The proof for a pair of consumers $i \in I_0$ and $h \in H_0(I_0), h = h(i)$ follows along the same lines. Q.E.D.

PROOF OF THEOREM 2. Inserting the definitions of $x(y)$ and $x(\overline{y})$ into (10) and using the definition of u^h yields (5), where $h = h(i)$. Since for all $t = 0, 1, 2, \ldots$, all $h \in H_t(I_t)$ satisfy $h = h(i)$ for exactly one $i \in I_t$, (10) holds for all consumers in F iff (5) holds for all consumers in $T(F)$. Similarly, there is a consumer i in F for whom (10) is true with strict inequality iff there is a consumer h in $T(F)$ such that (5) holds for her with strict inequality. From Lemma 1, we know that $\overline{y}$ is feasible in F, iff $x(\overline{y})$ is feasible in $T(F)$. Thus, we conclude that $\overline{y}$ Pareto-dominates y in F, iff $x(\overline{y})$ Pareto-dominates $x(y)$ in $T(F)$. Since there is a one-to-one map between allocations in F and $T(F)$, there is no allocation which Pareto-dominates y in F, iff there is no allocation which Pareto-dominates $x(y)$ in $T(F)$. Q.E.D.

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