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STRATEGY-PROOF AND EFFICIENT EXCHANGE ON A FINITE SET OF COBB–DOUGLAS UTILITY FUNCTIONS

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Abstract: We consider the possibility of designing strategy-proof and efficient social choice functions in two-agent, two-good, pure exchange economies. First, we show the nonexistence of strategy-proof, efficient, and symmetric social choice functions on an arbitrary domain with the following three Cobb–Douglas utility functions: (i) $u_\alpha(x_A, x_B) = (x_A)^{2/3}(x_B)^{1/3}$, (ii) $u_\beta(x_A, x_B) = (x_A)^{1/2}(x_B)^{1/2}$, and (iii) $u_\gamma(x_A, x_B) = (x_A)^{1/3}(x_B)^{2/3}$. Second, we conjecture that any strategy-proof and efficient social choice function is dictatorial on an arbitrary subset of Cobb–Douglas utility functions with the above (i)–(iii).

JEL Classification Numbers: C72, D71, D82.
Key words: Strategy-proofness, exchange economy.

1. INTRODUCTION

We consider the possibility of designing strategy-proof and efficient social choice functions in pure exchange economies. Hurwicz (1972) showed the nonexistence of strategy-proof, efficient, and individually rational social choice functions in two-agent, two-good, pure exchange economies where both agents have continuous, quasi-concave, and increasing utility functions. Dasgupta, Hammond, and Maskin (1979) replaced individual rationality in Hurwicz's result with non-dictatorship although they admitted discontinuous utility functions. Improving on these results, Zhou (1991) proved that any strategy-proof and efficient social choice function is dictatorial in two-agent, m-good ($m \geq 2$), pure exchange economies where both agents have continuous, strictly quasi-concave, and strictly increasing utility functions.

It is well known whether the set of strategy-proof and efficient social choice functions is rich depends on the domain (the set of utility functions or preferences) on which social choice functions are defined. In social choice environments, the

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restrictive domain of single peaked preferences leads to positive results (Moulin, 1980), whereas the unrestricted domain of preferences leads to negative results (Gibbard, 1973; Satterthwaite, 1975). For the provision of pure public goods, the Clarke–Groves mechanisms (Clarke, 1971; Groves, 1973) are strategy-proof on the set of quasi-linear utility functions, but they are not strategy-proof on the set of continuous, strictly quasi-concave, and strictly increasing utility functions.

These comparisons suggest the possibility of designing some desirable social choice functions which are strategy-proof and efficient on some smaller set of utility functions. However, Schummer (1997) proved that any strategy-proof and efficient social choice function is dictatorial in two-agent, m-good ($m \geq 2$), pure exchange economies where both agents have continuous, strictly quasi-concave in the interior, increasing, and homothetic utility functions.

In this paper we restrict the domain of social choice functions further. We examine strategy-proof and efficient social choice functions in two-agent, two-good, pure exchange economies where both agents have Cobb–Douglas utility functions. First, we show the nonexistence of strategy-proof, efficient, and symmetric social choice functions on an arbitrary domain with the following three Cobb–Douglas utility functions: (i) $u_\alpha(x_A, x_B) = (x_A)^{2/3}(x_B)^{1/3}$, (ii) $u_\beta(x_A, x_B) = (x_A)^{1/2}(x_B)^{1/2}$, and (iii) $u_\gamma(x_A, x_B) = (x_A)^{1/3}(x_B)^{2/3}$.[1] Second, we conjecture that any strategy-proof and efficient social choice function is dictatorial on an arbitrary subset of Cobb–Douglas utility functions with the above (i)–(iii). We conclude that the usual richness condition on the set of utility functions is not essential for the impossibility of strategy-proof and efficient social choice functions in pure exchange economies.[2]

2. THE MODEL

We consider 2×2 pure exchange economies. Let $N = \{1, 2\}$ be the set of agents. There are two private goods: A and B. The aggregate endowment of goods available for these economies is normalized to be $(e_A, e_B) = (1, 1)$.[3] The set of (balanced) allocations is given by $A = \{x = (x^1, x^2) = ((x_A^1, x_B^1), (x_A^2, x_B^2)) \in R_+^2 \times R_+^2 \mid x_A^1 + x_A^2 = 1$ and $x_B^1 + x_B^2 = 1\}$, where x^i denotes the allocation of two goods given to agent i, and x_A^i and x_B^i denote agent i's allocation of the goods A and B respectively.

Agent i has a preference which can be represented by a utility function u^i: $R_+^2 \to R$. U denotes the set of utility functions which each agent possibly has. A utility function profile is denoted by $u = (u^1, u^2) \in U^2$. For any $u \in U^2$, $C(u)$ denotes the contract curve, the set of Pareto efficient allocations, for u.

[1] These Cobb–Douglas utility functions are not contained in the domain of Zhou (1991). However, we can establish the same conclusions with the following three utility functions: (i) $\bar{u}_\alpha(x_A, x_B) = (x_A + 1)^{2/3}(x_B + 1)^{1/3}$, (ii) $\bar{u}_\beta(x_A, x_B) = (x_A + 1)^{1/2}(x_B + 1)^{1/2}$, and (iii) $\bar{u}_\gamma(x_A, x_B) = (x_A + 1)^{1/3}(x_B + 1)^{2/3}$, which are contained in the Zhou's domain.

[2] In the problem of allocating an indivisible good, Ohseto (1996) proved the nonexistence of strategy-proof and efficient social choice functions on a finite set of quasi-linear utility functions.

[3] Allocations and utility functions are defined on the basis of the normalized quantities of goods.

A *social choice function* is a function from utility functions into allocations, f: $U^2 \to A$. $f^i(u)$ denotes the allocation of agent i at u, and $f^i_{\mathrm{A}}(u)$ and $f^i_{\mathrm{B}}(u)$ represent the allocation of agent i at u with respect to the goods A and B respectively. A social choice function f is *strategy-proof* if for any (u^1, u^2), and $(\bar{u}^1, \bar{u}^2) \in U^2$, $u^1(f^1(u^1, u^2)) \geq u^1(f^1(\bar{u}^1, u^2))$ and $u^2(f^2(u^1, u^2)) \geq u^2(f^2(u^1, \bar{u}^2))$. A social choice function f is *efficient* if for any $u \in U^2$, $f(u) \in C(u)$. A social choice function f is *symmetric* if for any $(u^1, u^2) \in U^2$ with $u^1 \equiv u^2$, $u^1(f^1(u^1, u^2)) = u^2(f^2(u^1, u^2))$. A social choice function f is *dictatorial* if either $f^1(u) = (1, 1)$ for any $u \in U^2$, or $f^2(u) = (1, 1)$ for any $u \in U^2$.

3. RESULTS

In this section we try to prove that any strategy-proof and efficient social choice function is dictatorial on an arbitrary subset of Cobb–Douglas utility functions with at least the following three elements: (i) $u_\alpha(x_{\mathrm{A}}, x_{\mathrm{B}}) = (x_{\mathrm{A}})^{2/3}(x_{\mathrm{B}})^{1/3}$, (ii) $u_\beta(x_{\mathrm{A}}, x_{\mathrm{B}}) = (x_{\mathrm{A}})^{1/2}(x_{\mathrm{B}})^{1/2}$, and (iii) $u_\gamma(x_{\mathrm{A}}, x_{\mathrm{B}}) = (x_{\mathrm{A}})^{1/3}(x_{\mathrm{B}})^{2/3}$. The set of these elementary utility functions (i)–(iii) is denoted by U_e.[4] The set of Cobb–Douglas utility functions is given by $U_c = \{u(x_{\mathrm{A}}, x_{\mathrm{B}}) = (x_{\mathrm{A}})^t(x_{\mathrm{B}})^{1-t} \mid t$ is a real number such that $0 < t < 1\}$. Since our discussions are ordinal, we may simply write u_α, u_β, and u_γ as $u_\alpha(x_{\mathrm{A}}, x_{\mathrm{B}}) = (x_{\mathrm{A}})^2 x_{\mathrm{B}}$, $u_\beta(x_{\mathrm{A}}, x_{\mathrm{B}}) = x_{\mathrm{A}} x_{\mathrm{B}}$, and $u_\gamma(x_{\mathrm{A}}, x_{\mathrm{B}}) = x_{\mathrm{A}}(x_{\mathrm{B}})^2$.

We will derive a necessary condition for the existence of strategy-proof and efficient social choice functions on U^2, where $U_e \subset U$. (Fig. 1 helps the understanding of the following argument.)

Suppose that a social choice function f is strategy-proof and efficient on U^2, where $U_e \subset U$. It is clear that $C(u^1_\alpha, u^2_\alpha) = C(u^1_\gamma, u^2_\gamma) = \{x \in A \mid x^1_{\mathrm{A}} = x^1_{\mathrm{B}}\}$. By efficiency, we let $f(u^1_\alpha, u^2_\alpha) = ((a, a), (1-a, 1-a))$ and $f(u^1_\gamma, u^2_\gamma) = ((b, b), (1-b, 1-b))$ for some $0 \leq a, b \leq 1$. Without loss of generality, we consider the case of $a \geq b$. It follows from efficiency that $f(u^1_\beta, u^2_\alpha) \in C(u^1_\beta, u^2_\alpha) = \{x \in A \mid x^1_{\mathrm{B}} = 2x^1_{\mathrm{A}}/(x^1_{\mathrm{A}} + 1)\}$. Let $(p_{\mathrm{A}}, p_{\mathrm{B}}) \in R^2_+$ be the intersection of $x^1_{\mathrm{B}} = 2x^1_{\mathrm{A}}/(x^1_{\mathrm{A}} + 1)$ and $x^1_{\mathrm{A}} x^1_{\mathrm{B}} = a^2$. A simple computation yields: $p_{\mathrm{A}}(a) = \{(a^4 + 8a^2)^{1/2} + a^2\}/4$ and $p_{\mathrm{B}}(a) = \{(a^4 + 8a^2)^{1/2} - a^2\}/2$. By strategy-proofness, it must hold that $f^1_{\mathrm{A}}(u^1_\beta, u^2_\alpha) \geq p_{\mathrm{A}}(a)$ and $f^1_{\mathrm{B}}(u^1_\beta, u^2_\alpha) \geq p_{\mathrm{B}}(a)$. Otherwise agent 1 can manipulate f at (u^1_β, u^2_α) via u^1_α. It follows from efficiency that $f(u^1_\gamma, u^2_\alpha) \in C(u^1_\gamma, u^2_\alpha) = \{x \in A \mid x^1_{\mathrm{B}} = 4x^1_{\mathrm{A}}/(3x^1_{\mathrm{A}} + 1)\}$. Let $(q_{\mathrm{A}}, q_{\mathrm{B}}) \in R^2_+$ be the intersection of $x^1_{\mathrm{B}} = 4x^1_{\mathrm{A}}/(3x^1_{\mathrm{A}} + 1)$ and $x^1_{\mathrm{A}}(x^1_{\mathrm{B}})^2 = p_{\mathrm{A}}(a)\{p_{\mathrm{B}}(a)\}^2$. Let $p(a) = p_{\mathrm{A}}(a)\{p_{\mathrm{B}}(a)\}^2 = a^3\{(a^2+8)^{1/2} - a\}/2$. Solving $\{q_{\mathrm{B}}(a)\}^3 + 3p(a)q_{\mathrm{B}}(a) - 4p(a) = 0$ by Cardano's formulas,[5] we obtain $q_{\mathrm{B}}(a) = \{2p(a) + p(a)[p(a) + 4]^{1/2}\}^{1/3} + \{2p(a) - p(a)[p(a) + 4]^{1/2}\}^{1/3}$. By strategy-proofness, it must hold that $f^1_{\mathrm{B}}(u^1_\gamma, u^2_\alpha) \geq q_{\mathrm{B}}(a)$. Otherwise agent 1 can manipulate f at (u^1_γ, u^2_α) via u^1_β. It follows from efficiency that $f(u^1_\gamma, u^2_\beta) \in C(u^1_\gamma, u^2_\beta) = \{x \in A \mid x^2_{\mathrm{B}} = x^2_{\mathrm{A}}/(-x^2_{\mathrm{A}} + 2)\}$. Let $(r_{\mathrm{A}}, r_{\mathrm{B}}) \in R^2_+$ be the intersection of $x^2_{\mathrm{B}} = x^2_{\mathrm{A}}/(-x^2_{\mathrm{A}} + 2)$ and $x^2_{\mathrm{A}} x^2_{\mathrm{B}} = (1-b)^2$ A simple computation yields:

[4] An alternative choice of three Cobb–Douglas utility functions makes a mathematical analysis more difficult.

[5] See Borwein and Erdelyi (1995), pp. 3–4 for Cardano's formulas.

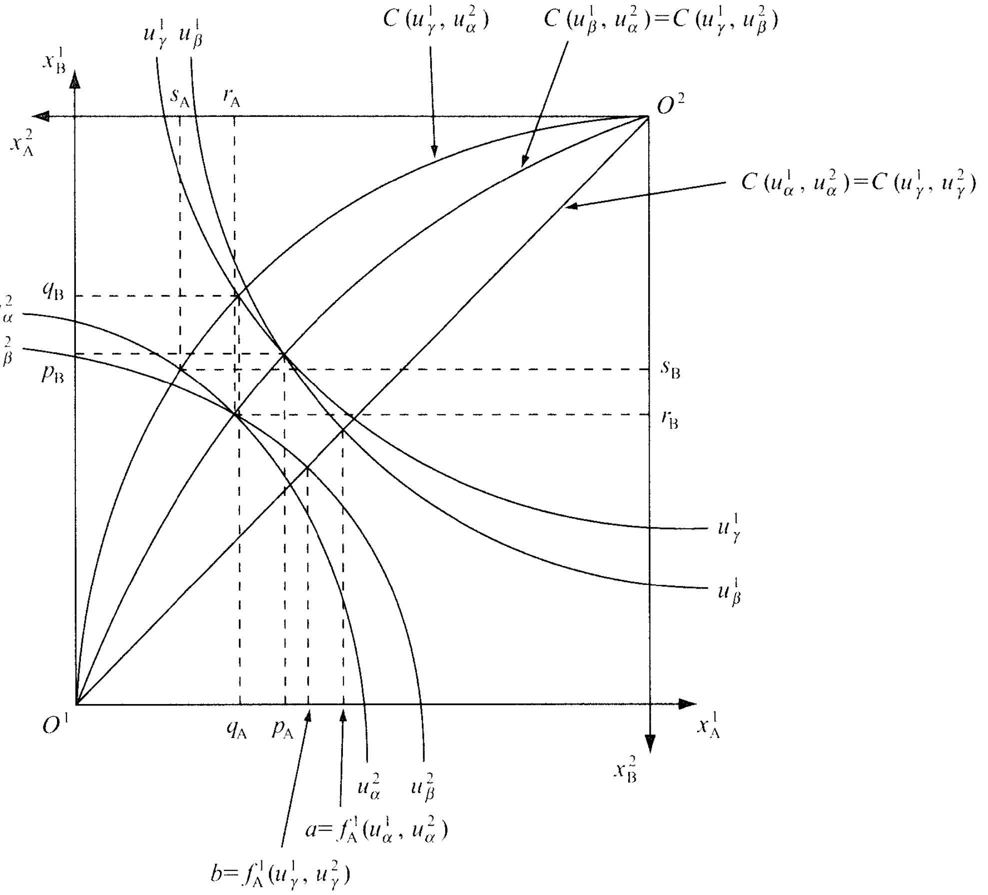

Fig. 1. A necessary condition of strategy-proof and efficient social choice functions.

$r_A(b)=\{[(1-b)^4+8(1-b)^2]^{1/2}-(1-b)^2\}/2$ and $r_B(b)=\{[(1-b)^4+8(1-b)^2]^{1/2}+(1-b)^2\}/4$. By strategy-proofness, it must hold that $f_A^2(u_\gamma^1, u_\beta^2)\geq r_A(b)$ and $f_B^2(u_\gamma^1, u_\beta^2)\geq r_B(b)$. Otherwise agent 2 can manipulate f at (u_γ^1, u_β^2) via u_γ^2. It follows from efficiency that $f(u_\gamma^1, u_\alpha^2)\in C(u_\gamma^1, u_\alpha^2)=\{x\in A \mid x_B^2=x_A^2/(-3x_A^2+4)\}$. Let $(s_A, s_B)\in R_+^2$ be the intersection of $x_B^2=x_A^2/(-3x_A^2+4)$ and $(x_A^2)^2x_B^2=\{r_A(b)\}^2r_B(b)$. Let $r(b)=\{r_A(b)\}^2r_B(b)=(1-b)^3\{[(1-b)^2+8]^{1/2}-(1-b)\}/2$. Solving $\{s_A(b)\}^3+3r(b)s_A(b)-4r(b)=0$ by Cardano's formula, we obtain $s_A(b)=\{2r(b)+r(b)[r(b)+4]^{1/2}\}^{1/3}+\{2r(b)-r(b)[r(b)+4]^{1/2}\}^{1/3}$. $s_B(b)$ can be computed by $s_B(b)=s_A(b)/(-3s_A(b)+4)$. By strategy-proofness, it must hold that $f_B^2(u_\gamma^1, u_\alpha^2)\geq s_B(b)$. Otherwise agent 2 can manipulate f at (u_γ^1, u_α^2) via u_β^2. Let $Z'(a,b)=q_B(a)+s_B(b)-1=\{[q_B(a)-4/3][s_A(b)-4/3]-4/9\}/\{s_A(b)-4/3\}$. Since $f_B^1(u_\gamma^1, u_\alpha^2)+f_B^2(u_\gamma^1, u_\alpha^2)=1$, it must hold that $Z'(a,b)\leq 0$. Let $Z''(a,b)=\{q_B(a)-4/3\}\{s_A(b)-4/3\}-4/9$. Since $s_A(b)-4/3<0$ for any $0\leq b\leq 1$, it holds that $Z''(a,b)\geq 0$. Notice that $Z''(a,b)$ is increasing in b since $s_A(b)$ is decreasing in b, and $0\leq q_B(a), s_A(b)\leq 1$. Hence, by letting $Z(a)=Z''(a,a)$, we have $Z(a)\geq Z''(a,b)\geq 0$. Summing up these arguments, we present the following lemma.

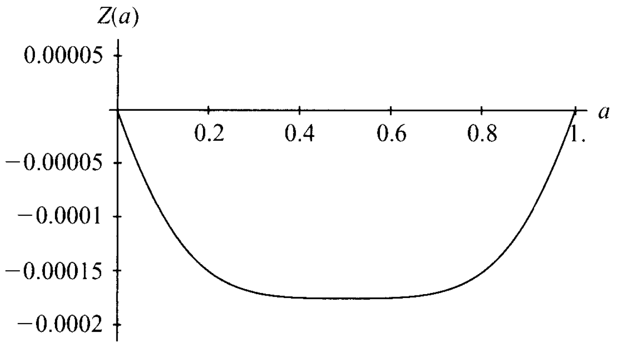

Fig. 2. The graph of $Z(a)$ for $0<a<1$.

LEMMA. *If a social choice function f is strategy-proof and efficient on U^2, where $U_e \subset U$, then it holds that $Z(a) \geq 0$, where $0 \leq a = f^1_A(u^1_\alpha, u^2_\alpha) \leq 1$.*

This lemma represents a necessary condition of strategy-proof and efficient social choice functions by some property of the function $Z(a)$. Hence, we can establish some interesting results on strategy-proof and efficient social choice functions by examining the function $Z(a)$.

THEOREM 1. *There exists no strategy-proof, efficient, and symmetric social choice function on any U^2 such that $U_e \subset U$.*[6]

Proof. Suppose that there exists some strategy-proof, efficient, and symmetric social choice function f on some U^2 such that $U_e \subset U$. By efficiency and symmetry, $f(u^1_\alpha, u^2_\alpha) = ((1/2, 1/2), (1/2, 1/2))$. This implies $a = f^1_A(u^1_\alpha, u^2_\alpha) = 1/2$. A computation yields $Z(1/2) \cong -0.000179$. This contradicts Lemma. Q.E.D.

It is an interesting question whether symmetry in the above theorem can be replaced by a weaker condition, such as non-dictatorship. For this purpose, we need to know some further properties of the function $Z(a)$. It is easy to show that (i) $Z(0) = Z(1) = 0$, and (ii) $Z(a) = Z(1-a)$ for any $0 \leq a \leq 1$. Since the function $Z(a)$ is simply a composite of polynomial functions and (square and cube) root functions, it seems that Mathematica draws the accurate shape of the graph of $Z(a)$ (see Fig. 2). Although we have not yet provided a mathematical proof, the following conjecture is strongly supported by calculations by Mathematica (see Table 1).

CONJECTURE. $Z(a) < 0$ *for any* $0 < a < 1$.

THEOREM 2. *If Conjecture is true, then for any U^2 such that $U_e \subset U \subset U_c$, any strategy-proof and efficient social choice function on U^2 is dictatorial.*

Proof. Choose any strategy-proof and efficient social choice function f on U^2. It follows from Lemma and Conjecture that $a = 0$ or $a = 1$. Thus, it must hold

[6] There always exist strategy-proof, efficient, and symmetric social choice functions on any U^2, where U contains exactly two Cobb–Douglas utility functions. This exercise is left to the readers.

Table 1. The values of $Z(a)$.

a	$Z(a)$
0	0
0.00001	-1.33985×10^{-8}
0.0001	-1.33953×10^{-7}
0.001	-1.33633×10^{-6}
0.01	-1.30465×10^{-5}
0.1	-1.01871×10^{-4}
0.5	-1.79071×10^{-4}

that either $f(u_\alpha^1, u_\alpha^2) = ((0,0),(1,1))$ or $f(u_\alpha^1, u_\alpha^2) = ((1,1),(0,0))$. Without loss of generality, we consider the case of $f(u_\alpha^1, u_\alpha^2) = ((1,1),(0,0))$. For any $u^1 \in U$, it holds that $f(u^1, u_\alpha^2) = ((1,1),(0,0))$. Otherwise agent 1 can manipulate f at (u^1, u_α^2) via u_α^1. Notice that for any $(u^1, u^2) \in U^2 \subset U_c^2$, $C(u^1, u^2) \cap \{x \in A \,|\, x$ is a boundary point of $A\} = \{((0,0),(1,1)),\ ((1,1),(0,0))\}$. For any $(u^1, u^2) \in U^2$, it holds that $f(u^1, u^2) = ((1,1),(0,0))$. Otherwise agent 2 can manipulate f at (u^1, u_α^2) via u^2. Therefore, agent 1 is a dictator of f on U^2 in this case. Q.E.D.

Some readers may not believe my conjecture. The truth may be that Conjecture is not true, and there exists some $0 < a < 1$ such that $Z(a) \geq 0$. However, it is likely that such a exists sufficiently near 0 or 1, if any. This observation implies that either $f(u_\alpha^1, u_\alpha^2) \cong ((0,0),(1,1))$ or $f(u_\alpha^1, u_\alpha^2) \cong ((1,1),(0,0))$. This is still an impossibility result, that is, symmetric utility functions result in very asymmetric allocations.

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