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GROSS SUBSTITUTABILITY AND THE LAWS OF COMPARATIVE STATICS: A SIMULTANEOUS DEMAND SHIFT

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Abstract: This paper extends the well-known Mosak's theorem on determinant in order to assess the comparative static price effect of a simultaneous shift in demand in the Walrasian economy in which all goods are gross substitutes. Contrary to what may be expected on the basis of the traditional result in which demand shifting from one good to another raises the latter price relative to any other price, the paper demonstrates that if the shift is from one good to two or more goods, the latter price may fall relative to *the price of some* other goods whose demand is not directly impacted by the shift. Therefore, the third Hicksian law of price change fails in this setting. The first two laws due to Hicks, on the other hand, are shown to hold.

JEL Classification: D00

1. INTRODUCTION

The assumption of gross substitutability of consumption goods has played a central role in the development of the modern theory of competitive equilibrium. The properties of equilibrium in terms of uniqueness, comparative statics and stability are well known when goods are gross substitutes.[1] In terms of the comparative statics entailing a shift in demand (typically in tastes) from one good to another, it is known that the following three laws due to Hicks (1939) apply: the price of the good to which demand shifts increases (the first law); the prices of all other goods increase (the second law); the price of the good absorbing the demand shift increases relatively the most (the third law).[2]

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[1] For an insightful review of the major works in this area, see McKenzie (1987).

[2] The laws are stated under a demand shift from the numeraire good to another good. (See Mukherji, 1975, p. 41.) If the demand shift is from a non-numeraire good to another non-numeraire good, the three laws also apply as stated above, except that in the second law, the qualification, "relative to the price of the good from which demand shifts" must be added, since the price change in terms of the numeraire good may be negative for some of the goods that are not directly impacted by the demand shift.

There does not exist an extensive literature, however, that addresses the question of how a shift in demand from one good to two or more goods affects the prices of individual goods.[3] Thus it is not clear, in the traditional Walrasian economy in which all goods are gross substitutes, as to whether the prices of goods to which demand shifts will each rise relative to the prices of other goods that are not impacted directly by the demand shift. It turns out, as demonstrated in this paper, that a shift of demand from one *non-numeraire* good (say, q) to two other *non-numeraire* goods (say, r and s) will: (a) increase the price of both r and s relative to the price of q; and (b) increase the price of every other good relative to the price of q. Hence, the first two Hicksian laws hold with minor modifications regarding the reference price. However, it also turns out that such a shift may lower the price of either r or s in terms of an arbitrarily chosen numeraire good, and do so by more than the price of any other good in the system save q. Hence, the third Hicksian law fails in general when the demand shift involves several goods.

2. THE WALRASIAN SYSTEM

Let there be $n+1$ goods, $0, 1, \cdots, n$, with prices denoted by $P_0, P_1, \cdots, P_n$. Denote the excess demand function for the i-th good as $f_i\ (P_0, P_1, \cdots, P_n; \alpha)$, which is assumed to be differentiable and homogeneous of degree zero in the $n+1$ prices. We assume that all prices are positive in equilibrium. Choosing good 0 as numeraire, we specify the equilibrium as

$$f_i(1, p_1, \cdots, p_n; \alpha)=0 \qquad (i=0, 1, \cdots, n) \tag{1}$$

where $p_j \equiv P_j/P_0\ (j=1, \cdots, n)$.

Define commodity units in eq. (1) such that $p_j=1\ (j=1, \cdots, n)$ at initial equilibrium. All goods are assumed to be gross substitutes satisfying the condition,

$$(S) \qquad f_{ij}>0 \quad \text{for all } i \text{ and } j \quad (i \neq j)$$

where $f_{ij} \equiv \partial f_i(1, p_1, \cdots, p_n; \alpha)/\partial p_j$. The Walras Law states that the sum of the value of each excess demand is zero,

$\sum_{i=0}^{n} P_i f_i = 0$. Hence, at initial equilibrium,

$$(W) \qquad \sum_{i=1}^{n} f_{ij} = -f_{0j} < 0 \qquad (j=1, \cdots, n)\,.$$

It may be noted that (S) and (W) together imply $f_{ii}<0$, precluding the Giffen Paradox. Finally, the homogeneity of f_i in the $n+1$ prices implies by the Euler equation:

[3] Notable exceptions are Mosak (1944, pp. 47–49) and Mukherji (1975, p. 46).

$$(H) \qquad \sum_{j=1}^{n} f_{ij} = -f_{i0} < 0 \qquad (i=1, \cdots, n).$$

Differentiating eq. (1) for $i=1, \cdots, n$ with respect to the parameter, and solving the system,

$$\mathrm{d}p_i/\mathrm{d}\alpha = -\sum_{j=1}^{n} (\partial f_j/\partial\alpha)(D_{ji}/D) \qquad (i=1, \cdots, n) \tag{2}$$

where D is the determinant of matrix F whose typical element is f_{ij} $(i, j=1, \cdots, n)$, and D_{ji} is the cofactor of the j-th row and the i-th column of F.[4]

It is known that the following conditions hold for all i and j $(i, j=1, \cdots, n)$ under (H), (W) and (S), respectively:

(i) $|D_{ii}|>|D_{ij}|$ $(i \neq j)$

(ii) $|D_{ii}|>|D_{ji}|$ $(i \neq j)$

(iii) $D_{ij}/D<0$.

Conditions (i) and (ii) state that in absolute value each principal cofactor of F dominates every other cofactor in the same row (due to H), as well as in the same column (due to W).[5] Condition (iii), due to Mosak (1944, pp. 49–51), states that the sign of every $(n-1)$-order cofactor is the opposite of the sign of D.

In addition to the above three conditions, it turns out that we can derive one other condition stated in terms of a set of $(n-1)$-order cofactors that proves to be useful in the comparative statics analysis of a simultaneous demand shift. Consider the following set of cofactors of F:

$$\begin{array}{ccccccc}
D_{11} & \cdots & D_{1j} & \cdots & D_{1k} & \cdots & D_{1n} \\
 & & \cdots & \cdots & & & \\
D_{k1} & \cdots & D_{kj} & \cdots & D_{kk} & \cdots & D_{kn} \\
 & & \cdots & \cdots & & & \\
D_{i1} & \cdots & D_{ij} & \cdots & D_{ik} & \cdots & D_{in} \\
 & & \cdots & \cdots & & & \\
D_{n1} & \cdots & D_{nj} & \cdots & D_{nk} & \cdots & D_{nn}
\end{array}$$

where i, j and k $(i \neq j \neq k \neq i)$ are any arbitrary set of row and column numbers of F. Focusing on the set of four cofactors, D_{kk}, D_{ij}, D_{kj}, D_{ik}, in which D_{kk} is the only principal cofactor, we obtain the following theorem.

THEOREM (Condition iv). *Consider any $(n-1)$-order principal cofactor of F, say D_{kk} $(k=1, \cdots, n)$. Then, for any arbitrary row i $(i \neq k)$ and column j $(j \neq k)$ of F, the sign of $[(D_{kk}+D_{ij})-(D_{ik}+D_{kj})]$ is the opposite of the sign of D.*

Proof. We wish to show that $(D_{kk}+D_{ij}-D_{ik}-D_{kj})/D<0$ for all k, i, j $(k \neq i \neq j \neq k)$. Write for any choice of k, i and j,

[4] D_{ji} is defined as $(-1)^{j+i}$ times the determinant formed by deleting the j-th row and the i-th column of F.

[5] For proof, see Mundell (1965), p. 353.

$$|D_{kk}|=|D_{ik}|+a \tag{3-a}$$

$$|D_{kk}|=|D_{kj}|+b \tag{3-b}$$

where a and b are each a strictly positive constant due to the conditions (i) and (ii). Since all cofactors of the same order have an identical sign on account of the condition (iii), we find

$$|D_{ik}|\cdot|D_{kj}|=D_{ik}D_{kj}=(D_{kk})^2-(a+b)|D_{kk}|+ab\,. \tag{4}$$

By Mosak's Theorem, we know that the condition (iii) holds for a system containing any set of $n-1$ goods. Thus, deleting the k-th good, for example, we have $D_{kk.ij}/D_{kk}<0$, where $D_{kk.ij}$ is the cofactor associated with the i-th row, j-th column element after deleting all k-th row and k-th column entries from F. But since $D_{kk}/D<0$, we must have

$$D\cdot D_{kk.ij}>0\,. \tag{5}$$

Since $D\cdot D_{kk.ij}=D_{ij}\cdot D_{kk}-D_{ik}\cdot D_{kj}$ (see Mosak, p. 51), substitution of eq. (4) into (5) yields

$$D_{ij}D_{kk}>(D_{kk})^2-(a+b)|D_{kk}|+ab\,. \tag{6}$$

Dividing both sides of ineq. (6) by $|D_{kk}|$, we obtain

$$(D_{ij}D_{kk})/|D_{kk}|=|D_{ij}|>|D_{kk}|-(a+b)+(ab/|D_{kk}|)\,. \tag{7}$$

Adding $|D_{kk}|$ on both sides of ineq. (7), and noting $|D_{kk}|+|D_{ij}|=|D_{kk}+D_{ij}|$, and also that from (3-a) and (3-b), $2|D_{kk}|-(a+b)=|D_{ik}|+|D_{kj}|$, we find

$$|D_{kk}+D_{ij}|>|D_{ik}+D_{kj}|+(ab/|D_{kk}|)\,. \tag{8}$$

Therefore, $|D_{kk}+D_{ij}|>|D_{ik}+D_{kj}|$, and it follows from (iii) that

$$(D_{kk}+D_{ij}-D_{ik}-D_{kj})/D<0\,. \tag{QED}$$

3. THE COMPARATIVE STATICS OF A SIMULTANEOUS DEMAND SHIFT

Consider a demand shift from any *non-numeraire* good (say, q) to two other *non-numeraire* goods (say, r and s). Then, for each dollar's worth of reduced demand for q, an equivalent value of new demand for r and s is created such that $\partial f_q/\partial\alpha+\delta f_r/\partial\alpha+\partial f_s/\partial\alpha=0$. Without loss of generality, let

$$\partial f_r/\partial\alpha+\partial f_s/\partial\alpha=-\partial f_q/\partial\alpha=1\,,\quad \text{where}\quad \partial f_r/\partial\alpha>0 \text{ and } \partial f_s/\partial\alpha>0\,.$$

Define $\Phi_r\equiv\partial f_r/\partial\alpha$ and $\Phi_s\equiv\partial f_s/\partial\alpha$. We obtain from eq. (2):

$$\mathrm{d}p_q/\mathrm{d}\alpha=\Phi_r(D_{qq}-D_{rq})/D+\Phi_s(D_{qq}-D_{sq})/D \tag{9}$$

$$\mathrm{d}p_r/d\alpha=-(D_{rr}-D_{qr})/D+\Phi_s(D_{rr}-D_{sr})/D \tag{10}$$

$$\mathrm{d}p_s/\mathrm{d}\alpha=-(D_{ss}-D_{qs})/D+\Phi_r(D_{ss}-D_{rs})/D\,. \tag{11}$$

It follows from the conditions (i)–(iii) that $dp_q/d\alpha < 0$ in eq. (9), so the price of q necessarily falls. However, the sign of $dp_r/d\alpha$ is ambiguous, since the two additive terms on the right-hand side (RHS) of eq. (10) have opposite signs, as is $dp_s/d\alpha$ in eq. (11). Hence, this leaves the possibility that the price of either r or s may fall.

It turns out, however, that any price fall entailing either good r or s is still less than that of q. To see this, we compare the price change on r with that of q:

$$dp_q/d\alpha - dp_r/d\alpha = \Phi_s(D_{qq} + D_{sr} - D_{sq} - D_{qr})/D + \Phi_r(D_{qq} - D_{rq} + D_{rr} - D_{qr})/D\,. \tag{12}$$

But $(D_{qq} + D_{sr} - D_{sq} - D_{qr})/D < 0$ due to our condition (iv); in addition, $(D_{qq} - D_{rq})/D < 0$, and $(D_{rr} - D_{qr})/D < 0$, on account of the conditions (i)–(iii). Hence, $dp_q/d\alpha - dp_r/d\alpha < 0$.

Thus, the first Hicksian law holds with the modification that the price of the good to which demand shifts rises relative to the price of the good from which demand shifts, but not necessarily in terms of the arbitrarily chosen numeraire good. But what about the second law? Considering the price change $dp_m/d\alpha$ for any other good m ($m \neq q, r, s$) whose demand is not impacted directly by the shift, and comparing it with the price change for q, we obtain

$$dp_q/d\alpha - dp_m/d\alpha = \Phi_s[(D_{qq} + D_{sm} - D_{sq} - D_{qm})/D] + \Phi_r[(D_{qq} + D_{rm} - D_{rq} - D_{qm})/D]\,. \tag{13}$$

We know from the condition (iv) that each of the two bracketed expressions in eq. (13) is negative in sign. Hence, $dp_q/d\alpha - dp_m/d\alpha < 0$, and the second law holds with the modification that the prices of all other goods will each rise relative to the price of the good from which demand shifts.

Consider now the third Hicksian law. Can the price of a good absorbing the demand shift ever *fall* relative to other goods whose demand is not directly affected, thus violating the third law? Comparing the price change $dp_r/d\alpha$ with $dp_m/d\alpha$ ($m \neq q, r$), we obtain after simplification,

$$dp_r/d\alpha - dp_m/d\alpha = [-(D_{rr} + D_{qm} - D_{qr} - D_{rm})/D] + \Phi_s[(D_{rr} + D_{sm} - D_{sr} - D_{rm})/D]\,. \tag{14}$$

From the condition (iv) we find that the two bracketed terms on the RHS of (14) have opposite signs. It can be verified that the first term coincides with the differential price effect of a *single* demand shift from good q to good r, affecting the prices of goods r and m: i.e., $[dp_r/d\alpha_{q \Rightarrow r} - dp_m/d\alpha_{q \Rightarrow r}]$, where the subscript of α, $q \Rightarrow r$ signifies that the demand shift from good q is absorbed solely by good r. Likewise, the second bracketed term in (14) coincides with the negative of the differential price effect on goods r and m: i.e., $-[dp_r/d\alpha_{s \Rightarrow r} - dp_m/d\alpha_{s \Rightarrow r}]$, which obtains in a *single* demand shift from good s to good r. It is possible for the latter price effect, weighted by the demand shift share (Φ_s) of good s, to dominate the first so as to make $dp_r/d\alpha$ less than $dp_m/d\alpha$ in (14). Hence, when there is a simultaneous demand shift from one good to two or more goods, the price of a good absorbing the demand shift can indeed fall relative to any other good in the system save q, and the third Hicksian law fails.[6]

To illustrate, consider a 5-good Walrasian economy, in which good 0 serves as numeraire. Let the matrix $[\partial f_i/\partial p_j]$ $(i, j = 1, \cdots, 4)$ be given by

$$\begin{bmatrix} -0.8 & +0.1 & +0.2 & +0.4 \\ +0.3 & -1.0 & +0.2 & +0.1 \\ +0.1 & +0.3 & -0.9 & +0.3 \\ +0.1 & +0.2 & +0.4 & -1.0 \end{bmatrix}$$

Suppose that demand shifts from the fourth to the first and the second good simultaneously $(4 \Rightarrow 1, 2)$ such that good 2 absorbs 90% of the shift $(\Phi_2 = 0.9)$ and good 1 the remaining 10% $(\Phi_1 = 0.1)$. It can be verified that the solution is given by

$$dp_1/d\alpha_{4\Rightarrow 1,2} = -0.279$$

$$dp_2/d\alpha_{4\Rightarrow 1,2} = +0.702$$

$$dp_3/d\alpha_{4\Rightarrow 1,2} = -0.107$$

$$dp_4/d\alpha_{4\Rightarrow 1,2} = -0.931\ .$$

Hence, in this example, the *increased* demand for good 1 has depressed its price in terms of the numeraire good, and this price fall is greater than that of good 3 whose demand is not directly affected by the shift.

4. CONCLUDING REMARKS

The world is rife with instances of a demand shift impacting directly on many goods simultaneously. For example, if a major health scare shocks the apples market, will the subsequent shift in demand toward substitutes such as oranges and grapes ever lead to a *decrease* in the price of some of the substitutes?[7] The theoretical possibility that the price of some of the close substitutes to which demand shifts may fall, and do so even in relation to another good which does not directly absorb any of the demand shift, cannot be ruled out. This surprising possibility, and the concomitant failure of the third Hicksian law to hold when demand shifts to two or more goods, perhaps serves as a sobering reminder that

[6] It is still the case, however, that the prices of the goods to which demand shifts cannot all fall. To see this, suppose that demand shifts from good q to the first k other goods in the system, $q \Rightarrow 1, \cdots, k$. Define a composite good C containing all such k goods so that one unit of C consists of Φ_i unit each of good i $(i = 1, \cdots, k)$. Then, $\sum_{i=1}^{k} dp_i/d\alpha_{q\Rightarrow 1,\cdots,k} = dp_c/d\alpha_{q\Rightarrow c}$. But $dp_c/d\alpha_{q\Rightarrow c} > 0$ by the first Hicksian law. Hence, $dp_i/d\alpha_{q\Rightarrow 1,\cdots,k}$ cannot all be negative.

[7] Alar, a trade name for daminozide, used to be widely used by apple growers in the U.S. to improve fruit appearance for unsuspecting consumers. The Columbia Broadcasting System (CBS) aired a report in its popular "60 Minutes" program in 1989, linking the chemical to an increased risk for cancer, especially among children. Many consumers reacted by refusing to buy apples, whether they had been treated by Alar or not, and apples sales plummeted.

a counter-intuitive economic anomaly can exist even in the traditional and restrictive Walrasian economy in which all goods are gross substitutes.

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