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A CHARACTERIZATION OF THE SUSTAINABLE MONOPOLY UNDER SHARKEY'S MODEL

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Abstract: This paper provides a characterization of the existence of a monopolistic sustainable price under Sharkey's simplified model from a view point of cooperative games.

1. INTRODUCTION

From a view point of cooperative games Faulhaber (1975) analyzed the cross-subdization problem, which often occurs in telecommunications industry as a phenomenon of remarkable differences of toll rates between long-distance and local-exchange sectors, regarded unfair since deficits of the local sector is covered by profits of the long-distance sector. Panzar and Willig (1977) reinterpreted and extended it to a setting of the existing monopolist who should provide goods or services which meet necessities of the society, realized in the market demand, and potential entrants seeking for hit-and-run profits without innovative incentives when multi-goods are produced and the production technique shows economies of scope.[1] They showed that cost complimentarity ensures the existence of a monopolistic sustainable price, a price at which the monopolist can survive against potential threats of entrants. Sharkey (1981) simplified their model and derived a necessary and sufficient condition for its existence by using techniques of coopeative game theory. Basically, he considered the usual anti-core which is constrained by an exogenously given constraint which relates the cost condition with the demand condition. We charcterize the sustainable monopoly of Sharkey's model by contriving so called the contracted anti-core.

Section 2 introduce the anti-core, a solution concept of TU (transferrable utility) cooperative games. The concept of the sustainable monopoly appeared in Sharkey's model is introduced and implications of an equivalent condition for a price vector to be monopolistic sustainable is considered in section 3. Finally, we provide a characterization of the sustainable monopoly by considering the contracted anti-core.

[1] Examples are telephone, gas and electricity services.

2. THE ANTI-CORE

We briefly introduce the anti-core, a solution concept of TU (transferrable utility) cooperative game theory which is deeply related with the concept of sustainability.[2] $N=\{1, 2, \cdots, n\}$ is a set of players and $S \in \mathscr{P}(N)$, a nonempty subset of N is called a *coalition* where $\mathscr{P}(N)$ is a power set of N excluding the empty set $\varnothing$. The *characteristic function* $F: \mathscr{P}(N) \rightarrow R$ associates each coalition with a real number. Assume that $F(\varnothing)=0$. A pair (N, F) is called a *game*. It is called *subadditive* if $S, T \in \mathscr{P}(N)$ and $S \cap T = \varnothing$ implies $F(S)+F(T) \geq F(S \cup T)$ and *concave* if $F(S \cup T)+F(S \cap T) \leq F(S)+F(T)$ for any $S, T \in \mathscr{P}(N)$ or equivalently, $F(T \cup \{i\}) - F(T) \leq F(S \cup \{i\}) - F(S)$ for any $S, T \in \mathscr{P}(N)$ such that $S \subseteq T$ and any $i \in N \backslash T$.[3,4] This property is often called the *snowballing effect or bandwagon effect* since the larger is a coalition which player i joins, the smaller its marginal cost increase is. The *anti-core* is defined by $C(N, F)=\{x \in R^N \mid x(N)=F(N)$ *and* $x(S) \leq F(S)$ *for any* $S \in \mathscr{P}(N)\}$ where a vector x satisfying Pareto optimality is called a *cost share vector*. Concavity guarantees the existence of the anti-core. (See Shapley (1971).)[5]

3. THE SUSTAINABLE MONOPOLY

We reinterpret $N=\{1, 2, \cdots, n\}$ as a set of goods or services, which we call the *grand collection*, and $S \in \mathscr{P}(N)$, a nonempty *sub-collection* of N.[6] y_i is the quantity of good i and p_i, its price. Also, $\boldsymbol{y}=(y_1, \cdots, y_n)$ and $\boldsymbol{p}=(p_1, \cdots, p_n)$ respectively denote n-dimensional supply and price vectors. $d_i=d_i(\boldsymbol{p})$ is the demand function of the ith good and $C(\boldsymbol{y})$ is the cost function. Following Sharkey (1981), we impose some restrictions on both demand and cost functions as follows: First, each good is independent, i.e., each demand function depends upon its own price only and is denoted by $d_i=d_i(p_i)$, $i \in N$. Also, the cost function is explicitly assumed as follows:

$$C(\boldsymbol{y}_S)=F(S)+\sum_{i \in S} c_i y_i, \qquad S \in \mathscr{P}(N) \tag{1}$$

[2] In the literature of game theory the core, a set of stable payoffs is a more frequently used jargon than the anti-core, which deals with stability of cost shares among players. But, we prefer the anti-core, a dual concept of the core since such topics as the cross-subdization problem, sustainability and minimum cost spanning tree games, maily focusing on issues of telecommunications industry are essentially involved with cost allocations.

[3] If inequalities are reversed, it is respectively called *superadditive* and *convex*.

[4] $N \backslash T=\{i \in N \mid i \notin T\}$.

[5] Precisely speaking, he proved the nonemptiness of the core if a game is convex where the core is defined by $\tilde{C}(N, F)=\{x \in R^N \mid x(N)=F(N)$ *and* $x(S) \geq F(S)$ *for any* $S \in \mathscr{P}(N)\}$. But, it can be easily confirmed that if the characteristic function F satisfies concavity, its dual characteristic function, $\bar{F}(S)=F(N)-F(N \backslash S)$, $S \in \mathscr{P}(N)$ is convex and then $C(N, F)=\tilde{C}(N, \bar{F})$ and hence, Shapley's result still holds.

[6] Each $S \in \mathscr{P}(N)$ can be regarded as an entity responsible for producing and managing associated goods.

Note that $\boldsymbol{y}_S$ is an n-dimensional vector defined in such a way that if $i \in S$, $y_i > 0$ and otherwise, $y_i = 0$ where $\boldsymbol{p}_S$ is a corresponding vector.[7] Also, c_i denotes the constant marginal cost of good i and $F: \mathscr{P}(N) \to R$ is the *common cost function*. $F(S)$ denotes the fixed cost incurred when sub-collection S is jointly produced. This cost function plays a role of the characteristic function under Sharkey's scheme.

A price vector $\boldsymbol{p}^m \in R^N$ is called *monopolistic sustainable* if the following conditions are satisfied:

$$\begin{aligned} &\text{i)} \quad y_i^m = d_i(p_i^m) \quad \textit{for any} \quad i \in N, \\ &\text{ii)} \quad \sum_{i \in N} (p_i^m - c_i) y_i^m \geq F(N), \qquad (2) \\ &\text{iii)} \quad \sum_{i \in S} (p_i^e - c_i) y_i^e < F(S) \quad \textit{for any} \ S \in \mathscr{P}(N),\ p_i^e < p_i^m,\ \textit{and}\ y_i^e \leq d_i(p_i^m),\ i \in S \end{aligned}$$

where y_i^m and p_i^m denote the price and quntity of good i of the incumbent monopoly, respectively and y_i^e, p_i^e, for potential entrants.[8]

A monopolistic sustainable price is such a price under which the monopolist supplies each of every good so as to balance the demand for each market with nonnegative profit, while any potential entrants cannot expect nonnegative profit even if any sub-collection of goods can be produced with lower prices than the monopolist's and under-supplied comparing with market demands. We shall call both conditions i) and ii), the *universal service requirement* and the *viability condition* of the incumbent, respectively and iii), the *no-cream-skimming condition* of potential entrants.

Some remarks are referred. First, every potential entrant has the equal access to the same technology of the monopolist, i.e., it is implicitly assumed that potential entrants would not provide innovative or advanced technologies. Secondly, asymmetric Nash–Bertrand behaviours are assumed between the monopolist and potential entrants, which means that the latter have a higher degree of freedom to choose quantities and prices as strategic variables than the former, regulated under the universal service requirement by a regulating authority, who cannot respond swiftly to the potential entrants' reactions.

We next introduce Sharkey's result on an equivalent condition for the monopoly to be sustainable. Let $r_i(p_i) = (p_i - c_i) d_i(p_i)$, $i \in N$ be the excess revenue function of good i. Also, $\tilde{r}_i = \max_{p_i \in (c_i, \infty)} (p_i - c_i) d_i(p_i)$ denotes its maximum value where $\tilde{\boldsymbol{r}} = (\tilde{r}_1, \tilde{r}_2, \cdots, \tilde{r}_n)$ is a corresponding vector. Assume that each demand function d_i, $i \in N$ is continuous on (c_i, ∞) and $\tilde{r}_i$ is well-defined.

THEOREM 1 (Sharkey (1981)). *If each good is independent and the cost function assumes the form of* (1), *a necessary and sufficient condition for a price to be*

[7] For notational simplicity let $\boldsymbol{y}_N = \boldsymbol{y}$ and $\boldsymbol{p}_N = \boldsymbol{p}$.

[8] Note that N corresponds to the monopoly and each $S \in \mathscr{P}(N)$ to a potential entrant.

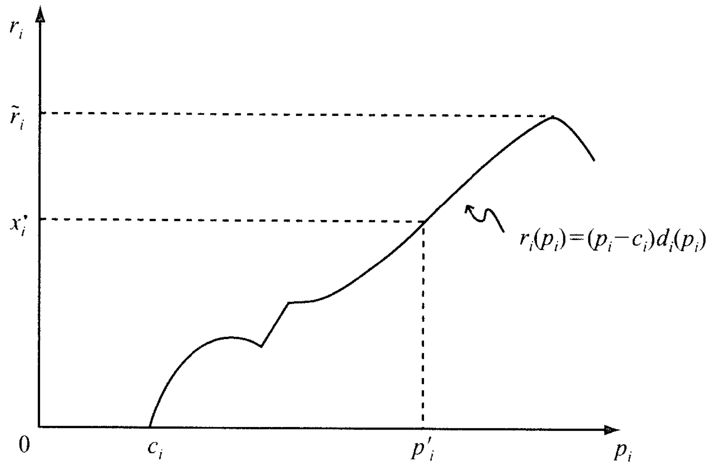

Fig. 1. Correspondences between stable cost shares and prices.

monopolistic sustainable is the existnece of a vector $\boldsymbol{x} \in R^N$ *which satisfies the following conditions.*

$$\begin{array}{lll} \text{i)} & x(N)=F(N)\,, & \\ \text{ii)} & x(S)\leq F(S) \quad \textit{for any} \quad S\in\mathscr{P}(N)\,, & (3) \\ \text{iii)} & \boldsymbol{x}\leq\tilde{\boldsymbol{r}}\,. & \end{array}$$

We discuss implications of Theorem 1 in line with his proof. Setting $\boldsymbol{x}=\boldsymbol{r}$, necessity is evident.[9] As for sufficiency, note that the first two conditions are equivalent to $\boldsymbol{x} \in C(N, F)$, i.e., the vector $\boldsymbol{x}$ is an element of the anti-core. But, its existence does not imply that of a sustainable price vector since it is merely an n-dimensional vector passed some tests for stability and does not related to the excess revenue function and hence, to monopolistic sustainable prices. But, under aforementioned assumptions condition iii) gurantees the existence of a corresponding sustainable price vector via intermediate value theorem. As illustrated in the diagram below, we can find a price of every good associated with the value of each coordinate of the vector $\boldsymbol{x}$.[10]

Condition iii) associates demand conditions of markets with stable allocation of cost shares solely dependent on the cost condition. Also, costs are allocated to consumers who are not explicit in the model, but incarnated in the demand functions.[11]

[9] r_i, $i \in N$ is the revenue net the variable cost of good i, which we called the excess revenue. But, it can be also interpreted as the cost share of good i in some sub-collection S to cover the common cost $F(S)$.

[10] There may be multiple corresponding prices. Then, we choose the smallest one as in Sharkey's proof (1981). Or, the assumption of monotonicity of the excess revenue function can exclude non-uniqueness. Also, note that independence among goods is important for the scheme to be workable.

[11] Consumers are explicit in Demange and Henriet's model (1991). Their scheme is based on an NTU (non-transferrable utility) cooperative game whereas utility functions are additive over goods consumed and homogenous across consumers in Sharkey's model.

4. A CHARACTERIZATION

Before characterizing a condition for the existence of a sustainable price, we shall introduce a convenient tool suitable for analyzing it under the scheme of TU cooperative games. Suppose that a vector $\tilde{x} \in R^N$ is given exogenously, and consider the anti-core of which each cost share vector x is constrained by the condition of '$x \leq \tilde{x}$' given as follows:

$$C(N, F)|_{\tilde{x}} = \{x \in X(N, F) \mid x \leq \tilde{x} \text{ and } x(S) \leq F(S) \text{ for any } S \in \mathscr{P}(N)\} \tag{4}$$

We call it the *$\tilde{x}$-contracted anti-core*. It is the usual anti-core to which the constraint that the cost share of each player should not exceed the amount prescribed in the vector $\tilde{x}$ simultaneously is added. The $\tilde{x}$-contracted anti-core can be characterized by the anti-core of a game with the characteristic function newly derived by implementing the constraint into the characteristic function per se if it satisfies a certain regular condition. Such a characteristic function is defined as follows:

$$F_{\tilde{x}}(S) = \min_{T \subseteq S} \{F(T) + \tilde{x}(S \setminus T)\}, \qquad S \in \mathscr{P}(N) \tag{5}$$

It is called the *$\tilde{x}$-contracted characteristic function*. The cost of $F_{\tilde{x}}(S)$ is obtained by choosing the cheapest among feasible configurations of any sub-coalition formation within coalition S where cost shares at $\tilde{x}$ are allocated to the remaining players. We call a pair $(N, F_{\tilde{x}})$ an *$\tilde{x}$-contracted game*. Note that $F_{\tilde{x}}(S) \leq F(S)$ for any coalition S and $F_{\tilde{x}}(\phi) = 0$.

PROPOSITION 2. *Given are a game (N, F) and an exogenous vector $\tilde{x} \in R^N$. If the vector satisfies the condition of '$\tilde{x}(S) \geq \bar{F}(S)$ for any $S \in \mathscr{P}(N)$', then*

$$C(N, F)|_{\tilde{x}} = C(N, F_{\tilde{x}}) \tag{6}$$

where $\bar{F}(S) = F(N) - F(N \setminus S)$, $S \in \mathscr{P}(N)$ and $C(N, F_{\tilde{x}})$ is the anti-core of the $\tilde{x}$-contracted game $(N, F_{\tilde{x}})$.

Proof. First, note that the condition on the vector implies $F(N \setminus S) + \tilde{x}(S) \geq F(N)$ for any $S \in \mathscr{P}(N)$ and hence $F_{\tilde{x}}(N) \geq F(N)$. Since $F(N) \geq F_{\tilde{x}}(N)$ from the definition of $F_{\tilde{x}}$, it follows that $F_{\tilde{x}}(N) = F(N)$. Suppose that $x \in C(N, F)|_{\tilde{x}}$. Then, since $x(S \setminus T) \leq F(S \setminus T)$ for any coalitions S and T such that $T \subseteq S$, we have $x(S \setminus T) + x(T) \leq F(S \setminus T) + \tilde{x}(T)$, i.e., $x(S) \leq F_{\tilde{x}}(S)$, from which it follow that $x \in C(N, F_{\tilde{x}})$. Conversely, let $x \in F(N, F_{\tilde{x}})$. Then, from $F_{\tilde{x}}(S) \leq F(S)$ for any $S \in \mathscr{P}(N)$, it immediately follows that $x \in C(N, F)|_{\tilde{x}}$. □

$\bar{F}(S)$ represents the *opportunity payoff* which coalition S can receive when it withholds from the grand coalition N. Hence, the condition given in the above proposition can be interpreted in such a way that for any coalition the corresponding total sum evaulated at the exogenous (payoff) vector should exceed

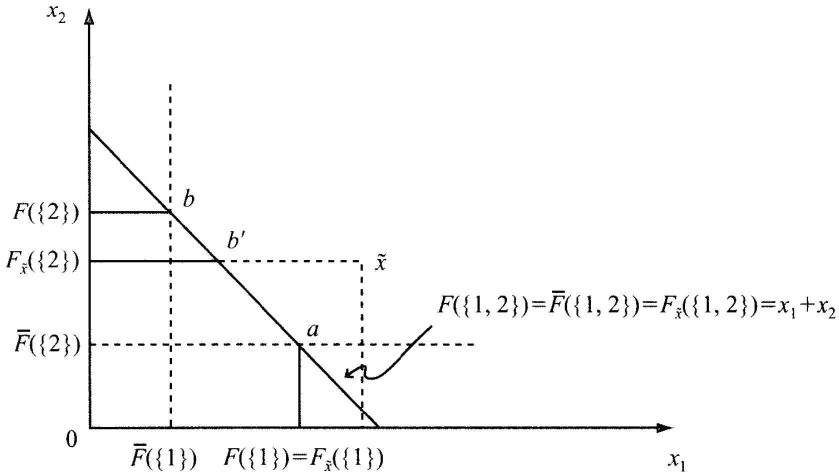

Fig. 2. An example of the contracted anticore with two players.

its opportunity payoff. We name the condition as the '*grand coalition preferability*' (for short, GCP). Note that this condition implies $F_{\tilde{x}}(N)=F(N)$, i.e., it technically guarantees Pareto optimality of the $\tilde{x}$-contracted anticore when expressed in terms of the $\tilde{x}$-contracted characteristic function.

The geometric intuition of the above proposition will be clarified in Figure 2 depicting a simple example of a two-player case. By calculating the cost of of each coalition, we get

$$F_{\tilde{x}}(\phi)=0\,,$$
$$F_{\tilde{x}}(\{1\})=\min\{F(\{1\}),\tilde{x}_1\}=F(\{1\})\,,$$
$$F_{\tilde{x}}(\{2\})=\min\{F(\{2\}),\tilde{x}_2\}=\tilde{x}_2\,,$$
$$F_{\tilde{x}}(\{1,2\})=\min\{F(\{1,2\}),F(\{1\})+\tilde{x}_2,\tilde{x}_1+F(\{2\}),\tilde{x}_1+\tilde{x}_2\}=F(\{1,2\})\,.$$

The line segment **ab** shows the anticore of a game $(\{1,2\},F)$ and **ab**$'$ is the $\tilde{x}$-contracted anticore, $C(\{1,2\},F_{\tilde{x}})$.[12]

From Proposition 2 we get the following characterization of the existence of a monopolistic sustainable price of Sharkey's model.[13]

THEOREM 3. *Suppose that all goods are independent and the cost function is of the form* (1). *An equivalent condition for the existence of a monopolistic sustainable price is that there exists a vecotr* $\boldsymbol{x}$ *satisfying the conditions as follows*:

$$\text{i)}\qquad \boldsymbol{x}(N)=F_{\tilde{x}}(N)\,,$$
$$\text{ii)}\qquad \boldsymbol{x}(T)\le F_{\tilde{x}}(T)\quad \textit{for any}\quad T\in\mathscr{P}(N)\,,\tag{7}$$

[12] Note that the vector $\tilde{x}$ in the diagram is given so as to satisfy the condition of $\tilde{x}_1>F(\{1\})$, $\tilde{x}_2>F(\{2\})$, $\tilde{x}_1+\tilde{x}_2>F(\{1,2\})$.

[13] Sharkey called subadditivity *economies of scope* and convexity, *cost complimentarity*. He proved that concavity of the common cost function gurantees the existence of a monopolistic sustainable price and implicitly used a logic based on the dual concept of $\tilde{x}$-contracted anticore in his proof.

iii) $$\tilde{r}(T) \geq \tilde{F}(T) \quad for\ any \quad T \in \mathscr{P}(N)$$

where $F_{\tilde{r}}$ is the $\tilde{r}$-contracted common cost function and $\tilde{r}$ is the maximum excess revenue vector.

Proof. Note that by Theorem 1, an equivalent condition for a price vector to be monopolistic sustainable is that $x \in C(N, F)|_{\tilde{r}}$. Hence, Proposition 2 immediately implies the desired result. □

We finally refer related works. Lee (1993) extended the sustainable monopoly to that of oligopolies from a view point of the contracted anticore with coalition structure, a partition of N, another solution concept TU cooperative games and characterized it so as to confirm whether an oligopolistic structure is intrinsic to the cost condition provided that a game is concave. NTU approaches of sustainable oligopolies and its logical structure can be found in Demange and Henriet (1991), Demange (1993) and Greenberg and Weber (1986, 1993).

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